Can $1^z = 2$ for Complex z?

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In the equation $1^z = 2$, we can follow Euler to write $1 = e^{2\pi i}$, and take the logarithm of both sides of the equation, $1^z = (e^{2\pi i})^z = e^{2\pi i z} = 2$, yielding $2\pi i z = \ln 2$, and $z = -i\ln(2)/2\pi$. Of course, we also expect that $1^z = 1$ is valid for this z.

Since $1^{1/2} = \pm 1$, 1^x can have multiple values for real x. Hence, we should not exclude multiple values of 1^z for complex z.

However, we note a somewhat related "paradox", given by Clausen in 1827,[1] that supposing $(e^x)^y = e^{xy}$ holds for complex numbers x and y leads to the conclusion that $1 = e^{-4\pi^2 n^2}$ for any integer n.[2]

A simpler version of Clausen's "paradox" was noted (Comment 3, Nov. 17, 2020) at https://math.stackexchange.com/questions/3911762/again-on-clausen-paradox
If we raise the equation $e^{2\pi i} = 1$ to the power i we have $(e^{2\pi i})^i = e^{-2\pi} = 1^i = 1$, supposing that 1 to any power is just 1.

A lesson here is that we should avoid using $(e^x)^y = e^{xy}$ when considering 1^z. As explained at https://en.wikipedia.org/wiki/Exponentiation#Failure_of_power_and_logarithm_identities, we should instead use $(e^x)^y = e^{y\ln e^x}$, being careful to note that for a complex number $z = r\,e^{i\theta}$ with r and θ real, $\ln z = \ln r + i(\theta + 2n\pi)$.[3]

This leads us to reconsider the process of complex exponentiation.[4] For real numbers a, b, c and d, we can write,

$$(a+ib) = (a^2+b^2)^{1/2}\,e^{i\arg(a+ib)} = (a^2+b^2)^{1/2}\,e^{i\theta}, \text{where } \arg(a+ib) = \tan^{-1}(b/a) = \theta, \quad (1)$$

$$\begin{aligned}(a+ib)^{c+id} &= \left((a^2+b^2)^{1/2}\,e^{i\theta}\right)^{c+id} = (a^2+b^2)^{(c+id)/2}\,e^{(c+id)\ln e^{i\theta}} \\ &= (a^2+b^2)^{c/2}(a^2+b^2)^{id/2}\,e^{ic(\theta+2n\pi)}\,e^{-d(\theta+2n\pi)} \\ &= (a^2+b^2)^{c/2}\,e^{id\ln(a^2+b^2)/2}\,e^{ic(\theta+2n\pi)}\,e^{-d(\theta+2n\pi)} \\ &= (a^2+b^2)^{c/2}\,e^{-d(\theta+2n\pi)}\left\{\cos\left[d\ln(a^2+b^2)/2 + c(\theta+2n\pi)\right]\right. \\ &\qquad \left. + i\sin\left[d\ln(a^2+b^2)/2 + c(\theta+2n\pi)\right]\right\}. \quad (2)\end{aligned}$$

For $a = 1$, $b = 0$, $c = 0$ and $d = -\ln(2)/2\pi$, we have,[5,6]

$$1^{-i\ln(2)/2\pi} = 1\cdot e^{\ln(2)(\theta+2n\pi)/2\pi}\cos(0). \quad (3)$$

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[1] T. Clausen, *Aufgabe*, J. Reine Angew. Math. **2**, 286 (1827), http://kirkmcd.princeton.edu/examples/mechanics/clausen_jram_2_286_27.pdf

[2] Clausen's argument was that since we can write $1 = e^{2n\pi i}$ for any integer n, we can also write $e = e^{1+2n\pi i}$, and then $e = (e^{1+2n\pi i})^{1+2n\pi i} = e^{(1+2n\pi i)^2} = e^{1+4n\pi i-4n^2\pi^2} = e^{1+4n\pi i}\,e^{-4n^2\pi^2} = e\cdot e^{-4n^2\pi^2}$, and finally, $1 = e^{-4n^2\pi^2}$, which is "absurd" (both in German and in English).

[3] Thus, $(e^{2\pi i})^i = e^{i\ln e^{2\pi i}} = e^{i\cdot 2n\pi i} = e^{-2n\pi} = 1^i = 1$, which is valid only for for $n = 0$.

[4] This is reviewed at, for example, https://mathworld.wolfram.com/ComplexExponentiation.html and at the Wikipedia Exponentiation page, but using $(e^x)^y = e^{xy}$.

[5] For $a = 1$, $b = 0$, $c = 0$ and $d = 1$, we have $1^i = e^{-2n\pi}$, which now agrees with the result $(e^{2\pi i})^i = e^{-2n\pi}$ in footnote 3 for any n.

[6] For $a = 0$, $b = 1$, $c = 0$ and $d = 1$, we have $i^i = e^{-\pi/2-2n\pi}$, as discussed at http://kirkmcd.princeton.edu/examples/itoi.pdf

Now, arg(1) can be $2m\pi$ for any integer m, so eq. (3) can take the form,

$$e^{N\ln(2)} = 2^N, \tag{4}$$

for any integer $N = m + n$. In particular, we can have $1^{-i\ln(2)/2\pi} = 2$. On the other hand, we can take $N = 0$, yielding $1^{-i\ln(2)/2\pi} = 1$.

In general, there is a countably infinite set of solutions to $1^z = w$ for any nonzero complex number w, including $1^z = 1$. We may wish to say that 1 is the principal value of 1^z, but we should be aware of the existence of other values.

This puzzler was posed to the author by Derek Abbott. It has been popularized at Can $1^x = 2$? *(Jan. 7, 2021)*, https://www.youtube.com/watch?v=9wJ9YBwHXGI
Discussion of it appeared in 2012 at
https://math.stackexchange.com/questions/233184/is-1-raised-to-any-complex-power-equal-to-1#: