Cylinder Rolling on Another Rolling Cylinder

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1 Problem

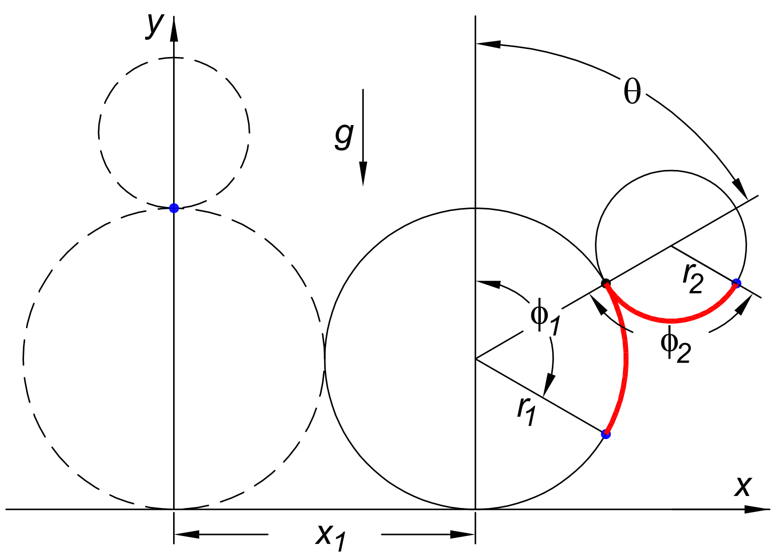
Discuss the motion of a cylinder that rolls without slipping on another cylinder, when the latter rolls without slipping on a horizontal plane. The cylinders have axial moments of inertia $I_i = k_i m_i r_i^2$ where m_i are the masses and r_i are the radii of rolling.[1]

2 Solution

This problem was suggested by Bradley Klee. See also sec. 16.2, p. 233, Vol. 1, of [1], and prob. 2.9, p. 24 of [2]. For the related case of one cylinder rolling inside another, see [3].

When one cylinder is directly above the other, we define the line of contact of the lower cylinder, 1, with the horizontal plane to be the z-axis, at $x = y = 0$. Then, the condition of rolling without slipping for the lower cylinder is that when it has rolled (positive) distance x_1, the initial line of contact has rotated through angle $\phi_1 = x_1/r_1$, clockwise with respect to the vertical, as shown in the figure below. This rolling constraint can be written as

$$x_1 = r_1 \phi_1. \tag{1}$$



Meanwhile, if the upper cylinder, 2, rolls such that the line of centers (in the x-y plane) makes angle θ (positive clockwise) to the vertical, then the initial point of contact of the upper cylinder has rotated through angle ϕ_2, measured counterclockwise from the line of centers, such that for rolling without slipping the arc lengths are equal between the initial

[1] One of the two dimensionless positive constants k_i can be greater than 1 for a “cylinder” in the form of a bobbin that rolls on a narrow cylinder or track.

points of contact of the two cylinders and the new point of contact. This second rolling constraint can be written as,

$$r_2\phi_2 = r_1(\phi_1 - \theta)\,, \qquad \phi_2 - \theta = \frac{r_1}{r_2}\phi_1 - \frac{r_1 + r_2}{r_2}\theta = \frac{r_1\phi_1 - r\theta}{r_2} \qquad \text{with} \qquad r \equiv r_1 + r_2. \tag{2}$$

where $\phi_2 - \theta$ is the angle of the initial point of contact of cylinder 2 to the vertical.

Of course, the center of cylinder 1 is at $y_1 = r_1$, and so long as the two cylinders are touching, their axes are separated by distance $r = r_1 + r_2$. Altogether there are 4 constraints on the 6 degree of freedom (of two-dimensional motion) of the system, such that there are only two independent degrees of freedom, which we take to be the angles ϕ_1 and θ.

Energy $E = T + V$ is conserved, and since neither the kinetic energy T nor the potential energy V (taken to be zero when $\theta = \theta_0$),

$$V = -m_2 g r(\cos\theta_0 - \cos\theta), \tag{3}$$

depend on coordinate ϕ_1 there will be another conserved quantity, the canonical momentum,

$$p_{\phi_1} = \frac{\partial \mathcal{L}}{\partial \dot\phi_1} = \frac{\partial T}{\partial \dot\phi_1}\,. \tag{4}$$

where $\mathcal{L} = T - V$ is the Lagrangian of the system. However, p_{ϕ_1} is not a single angular momentum.[2]

Since there are two conserved quantities and two degrees of freedom, there is no need to evaluate Lagrange's equations of motion to determine the motion, so long as the cylinders remain in contact and roll without slipping.[3]

The kinetic energy of cylinder 1, whose axis is at (x_1, r_1), is,

$$T_1 = \frac{m_1\dot x_1^2}{2} + \frac{I_1\dot\phi_1}{2} = \frac{1 + k_1}{2} m_1 r_1^2 \dot\phi_1^2, \tag{5}$$

using the rolling constraint (1) and the expression $I_1 = k_1 m_1 r_1^2$ for the moment of inertia I_1 in terms of parameter k_1.

The kinetic energy of cylinder 2, whose axis is at (x_2, y_2), is, using $I_2 = k_2 m_2 r_2^2$,

$$T_2 = \frac{m_2(\dot x_2^2 + \dot y_2^2)}{2} + \frac{I_2(\dot\phi_2 - \dot\theta)^2}{2} = \frac{m_2(\dot x_2^2 + \dot y_2^2)}{2} + \frac{k_2 m_2 r_2^2(\dot\phi_2 - \dot\theta)^2}{2}\,, \tag{6}$$

noting that the separation of kinetic energy into energy of the center-of-mass motion plus energy of rotation about the center of mass requires the angular velocity to be measured

[2]An example of a system in which there exists a constant of the motion involving angular velocity and moments of inertia, but which is not a single angular momentum, has been given in [4]. See also [5].

[3]For the implausible case of n cylinders, one on top of another, there are $3n$ degrees of freedom, n constraints of touching, and n rolling constraints, leaving n independent degrees of freedom. Energy is conserved, and if we take the n independent coordinates to be angle ϕ_1 and the $n-1$ angles $\theta_{i,i+1}$ of the lines of centers of adjacent cylinders, then the energy depends on the $\theta_{i,i+1}$ but not ϕ_1. Hence, there is one conserved canonical momentum. For $n > 2$ it is necessary to use some of Lagrangre's equations of motion to solve for the motion.

with respect to a fixed direction in an inertial frame. Then, recalling eqs. (1)-(2), we have,

$$x_2 = x_1 + r\sin\theta, \qquad \dot{x}_2 = r_1\dot{\phi}_1 + r\cos\theta\,\dot{\theta}, \tag{7}$$

$$y_2 = r_1 + r\cos\theta, \qquad \dot{y}_2 = \qquad -r\sin\theta\,\dot{\theta}, \tag{8}$$

$$\dot{\phi}_2 - \dot{\theta} = \frac{r_1\dot{\phi}_1 - r\dot{\theta}}{r_2}, \tag{9}$$

and the kinetic energy of cylinder 2 can be written as,

$$\begin{aligned} T_2 &= \frac{m_2}{2}[r_1^2\dot{\phi}_1^2 + 2r_1 r\cos\theta\,\dot{\phi}_1\dot{\theta} + r^2\dot{\theta}^2] \\ &\quad + \frac{k_2 m_2}{2}[r_1^2\dot{\phi}_1^2 - 2r_1 r\dot{\phi}_1\dot{\theta} + r^2\dot{\theta}^2] \\ &= \frac{1+k_2}{2}m_2 r_1^2\dot{\phi}_1^2 + (\cos\theta - k_2)m_2 r_1 r\dot{\phi}_1\dot{\theta} + \frac{1+k_2}{2}m_2 r^2\dot{\theta}^2. \end{aligned} \tag{10}$$

The total kinetic energy $T_1 + T_2$ is,

$$T = \frac{(1+k_1)m_1 + (1+k_2)m_2}{2}r_1^2\dot{\phi}_1^2 + (\cos\theta - k_2)m_2 r_1 r\dot{\phi}_1\dot{\theta} + \frac{1+k_2}{2}m_2 r^2\dot{\theta}^2, \tag{11}$$

and the conserved canonical momentum is,

$$p_{\phi_1} = \frac{\partial T}{\partial\dot{\phi}_1} = [(1+k_1)m_1 + (1+k_2)m_2]r_1^2\dot{\phi}_1 + (\cos\theta - k_2)m_2 r_1 r\dot{\theta} = \text{constant}. \tag{12}$$

The total horizontal momentum of the system is, using the rolling constraint (1),

$$P_x = (m_1 + m_2)\dot{x}_1 + m_2 r\cos\theta\,\dot{\theta} = (m_1 + m_2)r_1\dot{\phi}_1 + m_2 r\cos\theta\,\dot{\theta}, \tag{13}$$

while the angular momentum of the cylinder 1 about its axis is,

$$L_1 = k_1 m_1 r_1^2\dot{\phi}_1, \tag{14}$$

and that of cylinder 2 about its axis is, using the constraint (2),

$$L_2 = k_2 m_2 r_2^2(\dot{\phi}_2 - \dot{\theta}) = k_2 m_2 r_2(r_1\dot{\phi}_1 - r\dot{\theta}). \tag{15}$$

Hence, the conserved canonical momentum (12) can be written as,

$$p_{\phi_1} = r_1 P_x + L_1 + \frac{r_1}{r_2}L_2. \tag{16}$$

Equation (12) for the constant p_{ϕ_1} can be rewritten as,

$$\dot{\phi}_1 = \omega_0 - \frac{(\cos\theta - k_2)m_2 r}{[(1+k_1)m_1 + (1+k_2)m_2]r_1}\dot{\theta} = \omega_0 - \frac{Ar}{r_1}(\cos\theta - k_2)\,\dot{\theta}, \tag{17}$$

$$\ddot{\phi}_1 = -\frac{Ar}{r_1}\left[(\cos\theta - k_2)\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2\right], \tag{18}$$

$$\text{where} \quad A = \frac{m_2}{(1+k_1)m_1 + (1+k_2)m_2}. \tag{19}$$

Equation (17) integrates to give, for $\theta_0(t=0)=0$,

$$\phi_1 = \omega_0 t - \frac{Ar}{r_1}(\sin\theta - k_2\,\theta). \tag{20}$$

A particular solution is that θ is constant, say θ_0 with $|\theta_0| < \pi/2$, while $\phi = \omega_0 t$, in which case $\phi_2 = r_1(\omega_0 t - \theta_0)/r_2$ according to the rolling constraint (2). Here, the two cylinders roll together, with cylinder 2 at fixed angle θ_0, but this motion is unstable.[4]

For $k_2 < 1$ (as for typical cylinders) and motion that starts with $\omega_0 = 0$ and $x_{1,0} = \phi_{1,0} = \theta_0 = 0$, after a small perturbation, the motion leads to angles ϕ_1 and θ with opposite signs until $\sin\theta = k_2\theta$ after which the signs are the same (if the cylinders remain in contact). Similarly, the angular velocities $\dot{\phi}$ and $\dot{\theta}$ begin with opposite signs, but the signs become the same when $\cos\theta = k_2$.[5] For a bobbin-like cylinder with $k_2 > 1$, angles ϕ_1 and θ (and angular velocities $\dot{\phi}_1$ and $\dot{\theta}$) always have the same signs. *The figure on p. 1 corresponds to $k_2 > 1$, in which the system has positive x-momentum, although it started from rest.*

From the rolling constraint (2) we now have (for motion starting from rest),

$$\phi_2 = \frac{r_1}{r_2}(\phi_1 - \theta) = \frac{r_1}{r_2}\omega_0 - \frac{Ar}{r_2}(\sin\theta - k_2\,\theta) - \frac{r_1}{r_2}\theta. \tag{21}$$

For $k_2 < 1$, angles ϕ_1 and ϕ_2 have the same signs at small times, both opposite to that of θ. For $k_2 > 1$ the sign of ϕ_2 can be the same as that of θ, but only for a subset of the possible values for the other parameters of the system.

The constant energy $E = T + V$ can now be expressed as a function only of θ and $\dot{\theta}$, with the form,

$$\frac{E}{m_2 r^2} = 0 = [1 + k_2 - A(\cos\theta - k_2)^2]\frac{\dot{\theta}^2}{2} - \frac{g}{r}(1 - \cos\theta), \tag{22}$$

for motion that starts from with $\theta = 0 = \phi_1 = \phi_2$, and with $\omega_0 = 0$.[6]

2.1 Time Dependence

Thus far, we have obtained analytic expressions for angles ϕ_1 and ϕ_2 in terms of angle θ, and from these analytic expressions for x_1, x_2 and y_2 can also be obtained as a function of θ. However, we do not know the time dependence $\theta(t)$, from which the time dependence of all other quantities could be inferred.

By differentiating the energy equation (22), we obtain a second-order time-differential equation for θ,

$$\ddot{\theta} = \frac{g/r - A(\cos\theta - k_2)\dot{\theta}^2}{1 + k_2 - A(\cos\theta - k_2)^2}\sin\theta = \frac{1 + k_2 - A(\cos\theta - k_2)(3 - 2\cos\theta)}{[1 + k_2 - A(\cos\theta - k_2)^2]^2}\frac{g}{r}\sin\theta. \tag{23}$$

[4] If the upper cylinder is a "supercylinder", making elastic bounces off the horizontal surface, during which bounces the point of contact of the cylinder comes to rest, the motion of the upper cylinder is a series of pairs of "hops", with or without net horizontal motion [6, 7].

[5] For a much simpler example in which a constrained cylinder begins rotation in one sense and later reverses, see [8].

[6] The case that $m_1 = m_2$, $r_1 = r_2 = a = r/2$ and $k_1 = k_2 = 1/2$ is considered in ex. 33, p. 492 of [9]. It follows from eq. (22) that $\dot{\theta}^2 = 12(1 - \cos\theta)/a(17 + 4\cos\theta - 4\cos^2\theta)$.

For the special case that the upper cylinder is a hollow shell, $k_2 = 1$, the equation of motion for small θ simplifies to,

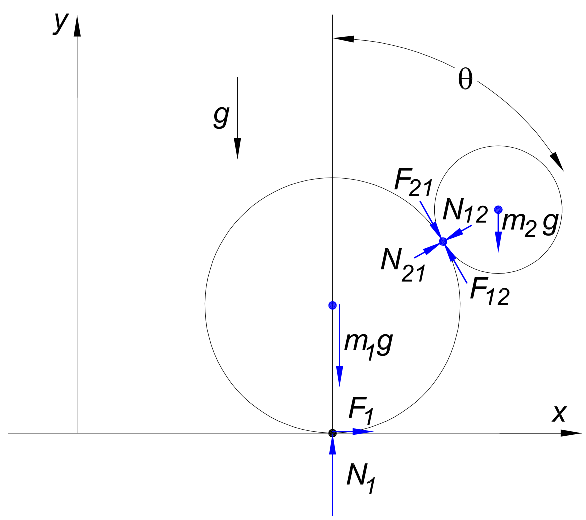
$$\ddot{\theta} \approx \frac{g}{2r}\theta, \qquad (k_2 = 1,\ \theta \ll 1). \tag{24}$$

which is the (Mathieu) equation for an inverted pendulum (of length $l = 2r$), for which solutions are tabulated in, for example, [12].

Numerical methods must be used to deduce $t(\theta)$ via either eqs. (23) or (24). Strictly, infinite time is required to reach any finite value of θ if the system starts from rest, so discussions of such examples usually consider a small, nonzero initial angle or angular velocity. While $\theta(t)$ is a monotonic function for the present example, if the axis of the lower cylinder were subject to a periodic horizontal force in the x- (or y-) direction, the system could exhibit stability at $\theta = 0$, as discussed, for example, in sec. 30 of [13].

2.2 Constraint Forces

The various forces on the two rolling cylinders are illustrated in the figure below. Here, we deduce these forces via Newton's equations of motion, plus the knowledge of the motion obtained above via a variant of Lagrange's method.[7]



2.2.1 Forces at the Horizontal Surface

The system of two cylinders, whose center of mass is at,

$$x_{\text{cm}} = \frac{(m_1 + m_2)x_1 + m_2 r \sin\theta}{m_1 + m_2}, \qquad y_{\text{cm}} = \frac{(m_1 + m_2)r_1 + m_2 r \cos\theta}{m_1 + m_2}, \tag{25}$$

[7]Lagrange's method was devised to deduce the equations of motion of a system without consideration of constraint forces that do no work. The method can be extended to include such forces by use of appropriate additional coordinates in the Lagrangian, and representing the effects of constraints in terms with Lagrange multipliers. See, for example, sec. 2.4 of [10] and sec. 19 of [11], as well as the Appendix.

is subject to the external force $F_1\,\hat{\mathbf{x}} + [N_1 - (m_1+m_2)g]\,\hat{\mathbf{y}}$, so the equation of motion of the center of mass are,

$$
\begin{aligned}
F_1 &= (m_1+m_2)\ddot{x}_{\rm cm} = (m_1+m_2)\ddot{x}_1 + m_2 r\left(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2\right) \\
&= (m_1+m_2) r_1\ddot{\phi}_1 + m_2 r\left(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2\right), \qquad (26) \\
N_1 &= (m_1+m_2)g + (m_1+m_2)\ddot{y}_{\rm cm} = (m_1+m_2)g - m_2 r\left(\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2\right), \qquad (27)
\end{aligned}
$$

using the rolling constraint (1). Then, using eqs. (18), (22) and (23) we obtain F_1 and N_1 as functions of angle θ.

A single cylinder that rolls without slipping on a horizontal plane has constant horizontal speed, and hence the force of friction is zero at the line of contact between the cylinder and plane.

In the present example the horizontal speeds of the two cylinder are not constant, and the force of friction F_1, eq. (26), due to the plane is not zero, such that the x-momentum of the system is not constant (as in the figure above).

2.2.2 Friction between the Cylinders

The force of friction, $F_{21} = -F_{12}$ on cylinder 2 due to cylinder 1, can be determined from the angular acceleration of cylinder 2, using a torque equation and the rolling constraint (2),

$$
F_{21} = \frac{I_2}{r_2}(\ddot{\phi}_2 - \ddot{\theta}) = k_2 m_2 r_2(\ddot{\phi}_2 - \ddot{\theta}) = k_2 m_2(r_1\ddot{\phi}_1 - r\ddot{\theta}). \qquad (28)
$$

Then, the friction force F_1 at the horizontal surface can also be determined from the angular acceleration of cylinder 1, using the torque equation,

$$
(F_1 + F_{12})r_1 = (F_1 - F_{21})r_1 = -I_1\ddot{\phi}_1 = -k_1 m_1 r_1^2\ddot{\phi}_1, \qquad (29)
$$

such that,

$$
F_1 = -(k_1 m_1 + k_2 m_2) r_1\ddot{\phi}_1 + k_2 m_2 r\ddot{\theta}. \qquad (30)
$$

This is consistent with eq. (26) in view of the relation (18).

These nonzero frictional forces imply that linear momentum P_x and angular momenta L_1 and L_2 are not conserved in this example, although there is a conserved quantity (16).

2.2.3 Normal Force between the Cylinders

The normal force $N_{12} = -N_{21}$ of cylinder 2 on cylinder 1 can be determined two ways, by consideration of the x- or y-components of the forces on cylinder 1 (or equivalently, on cylinder 2).

The vertical force components on cylinder 1 sum to zero, which implies that,

$$
\begin{aligned}
N_{12}\cos\theta &= N_1 - m_1 g + F_{12}\sin\theta \\
&= m_2\left[g - r\left(\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2\right) + k_2(r_1\ddot{\phi}_1 - r\ddot{\theta})\sin\theta\right] \qquad (31)
\end{aligned}
$$

using eqs. (27) and (28). Likewise, The horizontal force components on cylinder 1 sum to $m_1\ddot{x}_1$, which implies that,

$$\begin{aligned} N_{12}\sin\theta &= F_1 - m_1\ddot{x}_1 - F_{12}\cos\theta \\ &= m_2\left[r_1\ddot{\phi}_1 + r\left(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2\right) - k_2(r_1\ddot{\phi}_1 - r\ddot{\theta})\cos\theta\right], \end{aligned} \tag{32}$$

using eqs. (26) and (28). Then,

$$N_{12} = N_{12}\cos^2\theta + N_{12}\sin^2\theta = g\cos\theta + r_1\sin\theta\,\dot{\phi}_1 - r\,\dot{\theta}^2. \tag{33}$$

When N_{12} goes to zero, the cylinders separate.

2.3 Angle of Separation

The above analysis holds only so long as the two cylinders remain in contact, and the normal force N_{12} between the cylinders is nonzero, *i.e.*, when,

$$r\,\dot{\theta}^2 = g\cos\theta + r_1\sin\theta\,\dot{\phi}_1. \tag{34}$$

For a method that does not use the forces to find the angle θ_s at which the cylinders separate, we go to the accelerated frame of the lower cylinder, in which there appears to be an effective acceleration due to "gravity" of,

$$\mathbf{g}_{\rm eff} = -\ddot{x}_1\,\hat{\mathbf{x}} - g\,\hat{\mathbf{y}} = -r_1\ddot{\phi}_1\,\hat{\mathbf{x}} - g\,\hat{\mathbf{y}}. \tag{35}$$

Cylinder 2 loses contact with cylinder 1 when the component of $\mathbf{g}_{\rm eff}$ along the line of centers, $\hat{\mathbf{r}} = -(\sin\theta, \cos\theta)$, of the cylinders equals the instantaneous radial acceleration, $r\dot{\theta}^2$. That is, separation occurs at angle θ_s where,[8]

$$r\dot{\theta}_s^2 = \hat{\mathbf{r}}\cdot\mathbf{g}_{\rm eff} = g\cos\theta_s + r_1\sin\theta_s\,\ddot{\phi}_1 = g\cos\theta_s - rA\sin\theta_s\left[(\cos\theta_s - k_2)\,\ddot{\theta}_s - \sin\theta_s\,\dot{\theta}_s^2\right], \tag{37}$$

using eq. (18). This confirms eq. (34).

Even for the special case of identical cylinders, $m_1 = m_2$, $r_1 = r_2 = r/2$ and $k_1 = k_2$, the expression (37) remains intricate.

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[8]When the lower cylinder is fixed, $\mathbf{g}_{\rm eff} = \mathbf{g}$, and eq. (37) reduces to $r\dot{\theta}_s^2 = g\cos\theta_s$. The energy expression (22) simplifies to $(1+k_2)r\dot{\theta}^2/2 = g(1-\cos\theta)$, for motion that starts with $\theta = 0 = \phi_1 = \phi_2$. Separation occurs when $r\dot{\theta}_s^2 = g\cos\theta_s$, such that,

$$\cos\theta_s = \frac{2}{3+k_2}. \tag{36}$$

In the limit that the upper cylinder is a line/point, $k_2 \to 0$ and $\cos\theta_s \to 2/3$, as in the well known "freshman physics" problem of a bug sliding off a log. For a solid cylinder, $k_2 = 1/2$ and $\cos\theta_s = 4/7$, for a solid sphere $k_2 = 2/5$ and $\cos\theta_s = 10/17$, *etc.*

3 Variants

Thus far we have assumed that both cylinders roll without slipping. Variants include the three cases in which it is assumed instead that there is no friction at one or both lines of contact, and the cases where either one or two of the coordinates x_1, ϕ_1 and ϕ_2 are held fixed with either no friction anywhere or rolling without slipping where rolling is possible. Here, we consider only the first of these examples.

In all cases the potential energy v is given by eq. (3) and the kinetic energy T by a variant of eq. (11). We only consider systems that start from rest with cylinder 2 directly above cylinder 1.

3.1 No Friction Anywhere

In the case of no friction anywhere, the cylinders do not rotate.[9] Energy is conserved, and the conserved total horizontal momentum is always zero. The system has two degrees of freedom, which we take to be x_1, the coordinate of the center of the lower cylinder, and the angle θ of the line of centers between the two cylinders.

The total kinetic energy can be obtained from eq. (11) by setting k_1 and k_2 to zero, and replacing factors of $r_1\dot{\phi}_1$ by $\dot{x}_1$ (undoing the rolling constraint (1), so to speak),

$$T = \frac{m_1 + m_2}{2}\dot{x}_1^2 + m_2 r \cos\theta\, \dot{x}_1 \dot{\theta} + \frac{m_2}{2} r^2 \dot{\theta}^2. \tag{38}$$

The conserved canonical momentum is, for motion starting from rest with $\phi_1 = \phi_2 = \theta = 0$,

$$p_{x_1} = \frac{\partial T}{\partial \dot{x}_1} = (m_1 + m_2)\dot{x}_1 + m_2 r \cos\theta\, \dot{\theta} = P_x = 0, \tag{39}$$

which is just the total horizontal momentum.[10] Using eq. (39) to eliminate $\dot{x}_1$ from the kinetic energy, we obtain the total energy as,

$$\frac{E}{m_2 r^2} = 0 = \frac{m_1 + m_2 \sin^2\theta}{m_1 + m_2} \frac{\dot{\theta}^2}{2} - \frac{g}{r}(1 - \cos\theta). \tag{40}$$

To find the angle θ_s at which the cylinders separate, we again go to the accelerated frame of the lower cylinder, in which there appears to be an effective acceleration due to "gravity",

$$\mathbf{g}_{\text{eff}} = -\ddot{x}_1\, \hat{\mathbf{x}} - g\, \hat{\mathbf{y}} = \frac{m_2 r}{m_1 + m_2} \left(\cos\theta\, \ddot{\theta} - \sin\theta\, \dot{\theta}^2 \right) \hat{\mathbf{x}} - g\, \hat{\mathbf{y}}, \tag{41}$$

where $\dot{\theta}$ and $\ddot{\theta}$ can be deduced in terms of θ from eq. (40).

Cylinder 2 loses contact with cylinder 1 when the component of $\mathbf{g}_{\text{eff}}$ along the line of centers, $\hat{\mathbf{r}} = -(\sin\theta, \cos\theta)$, of the cylinders equals the instantaneous radial acceleration, $r\dot{\theta}^2$. That is, separation occurs at angle θ_s where,

$$r\dot{\theta}_s^2 = \hat{\mathbf{r}} \cdot \mathbf{g}_{\text{eff}} = g\cos\theta_s - \frac{m_2 r \sin\theta_s}{m_1 + m_2} \left(\cos\theta_s\, \ddot{\theta}_s - \sin\theta_s\, \dot{\theta}_s^2 \right). \tag{42}$$

[9]The case of two spheres with no friction was discussed on p. 260 of [17].

[10]This example includes other conserved/zero generalized momenta such as the z-component of the linear momentum, and the angular momentum about the x- and y-axes.

After considerable effort, one can verify that eqs. (40) and (42) combine to give,

$$m_2 \cos\theta_s^2 = (m_1 + m_2)(3\cos\theta_s - 2), \tag{43}$$

as noted in ex. 6, p. 121 of [15].

When the lower cylinder is fixed, we can set $k_2 = 0$ and the result (36) again becomes $\cos\theta_s = 2/3$ as for a point mass sliding on a cylinder/sphere.

3.2 No Friction at the Horizontal Plane

In the case of no friction at the horizontal plane, but rolling without slipping of cylinder 2 on cylinder 1, both cylinders rotate, being torqued by the friction along the line of contact of the cylinders. Energy is conserved, and the conserved total horizontal momentum is always zero. The system has three degrees of freedom, which we take to be x_1, ϕ_1 and θ.

The total kinetic energy can be obtained from eq. (11) by replacing factors of $r_1\dot\phi_1$ not associated with k_1 or k_2 by $\dot x_1$ (again undoing the rolling constraint),

$$T = \frac{m_1 + m_2}{2}\dot x_1^2 + \frac{k_1 m_1 + k_2 m_2}{2} r_1^2 \dot\phi_1^2 + m_2 r\cos\theta\,\dot x_1\dot\theta - k_2 m_2 r_1 r\,\dot\phi_1\dot\theta + \frac{1+k_2}{2} m_2 r^2\,\dot\theta^2. \tag{44}$$

There are now two the conserved canonical momenta,

$$p_{x_1} = \frac{\partial T}{\partial \dot x_1} = (m_1 + m_2)\dot x_1 + m_2 r\cos\theta\,\dot\theta = P_x = 0, \tag{45}$$

which is the total horizontal momentum (for motion starting from rest with $\phi_1 = \phi_2 = \theta = 0$), and,

$$p_{\phi_1} = \frac{\partial T}{\partial \dot\phi_1} = (k_1 m_1 + k_2 m_2) r_1^2\,\dot\phi_1 - k_2 m_2 r_1 r\,\dot\theta = L_1 + \frac{r_1}{r_2} L_2 = 0. \tag{46}$$

Using eqs. (45)-(46) to eliminate $\dot x_1$ and $\dot\phi_1$ from the kinetic energy, we obtain the total energy as,[11]

$$\frac{E}{m_2 r^2} = 0 = \left(1 + k_2 - \frac{m_2\cos^2\theta}{m_1 + m_2} - \frac{k_2^2 m_2}{k_1 m_1 + k_2 m_2}\right)\frac{\dot\theta^2}{2} - \frac{g}{r}(1 - \cos\theta). \tag{47}$$

To find the angle θ_s at which the cylinders separate, we again go to the accelerated frame of the lower cylinder, in which there appears to be an effective acceleration due to "gravity",

$$\mathbf{g}_{\rm eff} = -\ddot x_1\,\hat{\mathbf{x}} - g\,\hat{\mathbf{y}} = \frac{m_2 r}{m_1 + m_2}\left(\cos\theta\,\ddot\theta - \sin\theta\,\dot\theta^2\right)\hat{\mathbf{x}} - g\,\hat{\mathbf{y}}, \tag{48}$$

where $\dot\theta$ and $\ddot\theta$ can be deduced in terms of θ from eq. (47).

[11] This case is considered in ex. 32, p. 492 of [9] for $k_1 = k_2 = 1/2$, where eq. (47) takes the form $(3m_1 + 2m_2\sin^2\theta)\dot\theta^2 = 4(m_1 + m_2)g(1 - \cos\theta)/r$.

Cylinder 2 loses contact with cylinder 1 when the component of $\mathbf{g}_{\rm eff}$ along the line of centers, $\hat{\mathbf{r}} = -(\sin\theta, \cos\theta)$, of the cylinders equals the instantaneous radial acceleration, $r\dot{\theta}^2$. That is, separation occurs at angle θ_s where,

$$r\dot{\theta}_s^2 = \hat{\mathbf{r}} \cdot \mathbf{g}_{\rm eff} = g\cos\theta_s - \frac{m_2 r}{m_1 + m_2}\left(\cos\theta_s\,\ddot{\theta}_s - \sin\theta_s\,\dot{\theta}_s^2\right)\sin\theta_s. \tag{49}$$

This has the same form as eq. (42), but since the energy expressions (40) and (47) are different, the value of θ_s will be different.[12]

When the lower cylinder is fixed, we again have $\cos\theta_s = 2/(3 + k_2)$ as in eq. (36).

3.3 No Friction between the Cylinders

In the case of no friction between the cylinders, but cylinder 1 rolls without slipping on the horizontal plane, only cylinder 1 rotates, being torqued by the friction at the horizontal surface. Energy is conserved, but the total horizontal momentum is not. The system has two degrees of freedom, which we take to be ϕ_1 and angle θ, using the rolling constraint (1) to eliminate x_1 from the energy, and the rolling constraint (2) to eliminate ϕ_2 in favor of ϕ_1 and θ.

The total kinetic energy can be obtained from eq. (11) by setting k_2 to zero,

$$T = \frac{(1 + k_1)m_1 + m_2}{2} r_1^2 \dot{\phi}_1^2 + m_2 r_1 r\cos\theta\,\dot{\phi}_1\dot{\theta} + \frac{m_2}{2} r^2\,\dot{\theta}^2, \tag{51}$$

and the conserved canonical momentum is, for motion starting from rest with $\phi_1 = \phi_2 = \theta = 0$,

$$p_{\phi_1} = \frac{\partial T}{\partial\dot{\phi}_1} = [(1 + k_1)m_1 + m_2]r_1^2\dot{\phi}_1 + m_2 r_1 r\cos\theta\,\dot{\theta} = 0. \tag{52}$$

Using eq. (52) to eliminate $\dot{\phi}_1$ from the kinetic energy, we obtain the total energy as,

$$\frac{E}{m_2 r^2} = 0 = \left(1 - \frac{m_2\cos^2\theta}{(1 + k_1)m_1 + m_2}\right)\frac{\dot{\theta}^2}{2} - \frac{g}{r}(1 - \cos\theta). \tag{53}$$

To find the angle θ_s at which the cylinders separate, we again go to the accelerated frame of the lower cylinder, in which there appears to be an effective acceleration due to "gravity",

$$\mathbf{g}_{\rm eff} = -\ddot{x}_1\,\hat{\mathbf{x}} - g\,\hat{\mathbf{y}} = -r_1\ddot{\phi}_1\,\hat{\mathbf{x}} - g\,\hat{\mathbf{y}} = \frac{m_2 r}{(1 + k_1)m_1 + m_2}\left(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2\right)\hat{\mathbf{x}} - g\,\hat{\mathbf{y}}, \tag{54}$$

[12] An amusing special case has been noted in ex. 5, p. 121 of [15]. Suppose the two cylinders are identical, $m_1 = m_2$, $r_1 = r_2 = a = r/2$, and $k_1 = k_2 = k$. Then, eq. (46) becomes $\dot{\phi}_1 = \dot{\theta}$, such that $\phi_1 = \theta$, and then by the rolling constraint (2), $\phi_2 = \phi_1 - \theta = 0$. That is, the two cylinder roll together as if they were a single rigid body – until they separate.

Routh also claims that the cylinders separate at angle θ related by $(k + 1 + \sin^2\theta)a\,\dot{\theta}^2 = 2g(1 - \cos\theta)$, where $a = r_1 = r_2 = r/2$, and we note that Routh's k^2 equals our ka^2. However, this is just the energy relation (47), which holds for any angle at which the cylinders touch. If I evaluated eq. (49) correctly,

$$2[(2 + k)\cos\theta_s - 1] = \cos\theta_s(1 - \cos\theta_s)^2(1 + \cos\theta_s). \tag{50}$$

where $\dot{\theta}$ and $\ddot{\theta}$ can be deduced in terms of θ from eq. (53).

Cylinder 2 loses contact with cylinder 1 when the component of $\mathbf{g}_{\rm eff}$ along the line of centers, $\hat{\mathbf{r}} = -(\sin\theta, \cos\theta)$, of the cylinders equals the instantaneous radial acceleration, $r\,\dot{\theta}^2$. That is, separation occurs at angle θ_s where,

$$r\,\dot{\theta}_s^2 = \hat{\mathbf{r}} \cdot \mathbf{g}_{\rm eff} = g\cos\theta_s - \frac{m_2 r \sin\theta_s}{(1+k_1)m_1 + m_2}\left(\cos\theta_s\,\ddot{\theta}_s - \sin\theta_s\,\dot{\theta}_s^2\right). \tag{55}$$

When the lower cylinder is fixed, we again have $\cos\theta_s = 2/3$ as in sec. 3.1.

A Appendix: Constraint Forces via Lagrange Multipliers

In general, two rigid bodies, such as the two cylinders of the present example, are to be described by six coordinates per body (say, the spatial coordinates of the center of mass of a body, the two angular directions of some fixed body axis, and the angle of orientation of the body about this axis), for a total of twelve coordinates. In the present example, only two of these twelve coordinates are independent, as there are ten constraints: the axes of the cylinders lie along the z-axis (4 constraints), the centers of mass of the cylinders are at $z = 0$ (2 constraints), the lower cylinder lies on the plane $y = 0$ (1 constraint), the two cylinders touch one another (1 constraint), the lower cylinder rolls without slipping on the plane $y = 0$ (1 constraint), and the upper cylinder rolls without slipping on the lower cylinder (1 constraint).

Furthermore, there is no dissipation of energy in this problem.

Given these constraints/conditions, Lagrange's method consists of computing the kinetic energy T and the potential energy V in terms of the independent coordinates (taken above to be ϕ_1 and θ). The total energy $E(\phi_1, \theta) = T + V$ is conserved, so the time derivative $dE/dt = 0$ provides one relation between ϕ_1 and θ.From the Lagrangian $\mathcal{L} = T - V$ we can, in principle, deduce the equations of motion via Lagrange's equations,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i}, \tag{56}$$

If $\partial\mathcal{L}/\partial q_i = 0$ (as for $q_i = \phi_1$ in the present example), then $\partial\mathcal{L}/\partial\dot{q}_i$ is constant (as in eq. (12) for the present example), and may be called a conserved quantity.[13] Thus, it may be (as in the present example) that there are as many conserved quantities as independent coordinates, and Lagrange's equations (56) are not needed to determine the motion.

In Lagrange's method, for examples like the present with no dissipation of energy and "simple" constraints on the coordinates, no mention is made of forces. If desired, expressions for various forces can be deduced from Newton's $\mathbf{F} = m\mathbf{a}$ with the acceleration $\mathbf{a}$ being obtained from Lagrange's equations (56). A subclass of the forces are those associated with the various constraints on the coordinates of the systems; these are the so-called constraint forces, which do no work (if no energy is dissipated).

[13] Astonishingly, a paper [14] was published claiming that this in not "well known to instructors and students of physics".

We can also deduce the constraint forces via a method in which more than the minimum number of coordinates are used, as apparently first proposed by Routh [15, 16] for holonomic constraints,[14] as a special case of a method for problems with nonholonomic constraints given by Ferrers [19]. See also [20].

In this method, the minimum number n of independent coordinates is augmented with m additional coordinates, so that the total set of coordinates is q_i, $i = 1, \ldots, n + m$, and for which the m constraint equations $f_j(q_i) = 0$, $j = 1, \ldots, m$, are known, but not explicity enforced initially. Then, we consider the $n + m$ modified Lagrange equations,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \sum_{j=1}^{m} \lambda_j \frac{\partial f_j}{\partial q_i}\,, \tag{57}$$

where the λ_j are so-called Lagrange multipliers (which have the physical significance of being the j constraint force if the dimensions of the constraint equation $f_j = 0$ are chosen appropriately).

In the present example with 12 coordinates, of which only 2 are independent, there are 10 constraint equations. Any number of these can be ignored in an implementation of eq. (57), so there are $2^{10} = 1024$ different possible variations of the analysis of the present problem.

Here, we consider the problem to be two dimensional, in which case the first six constraints are automatically satisfied. The remaining four constrains are:

1. That the lower cylinder rolls without slipping on the plane $y = 0$, eq, (1),

$$f_1 = x_1 - r_1\phi_1 = 0, \tag{58}$$

2. That the upper cylinder rolls without slipping on the lower cylinder, eq. (2),

$$f_2 = r_2\phi_2 - r_1(\phi_1 - \theta) = 0, \tag{59}$$

3. That the two cylinders touch,

$$f_3 = r - r_1 - r_2 = 0, \tag{60}$$

   where r is the distance in between the axes of the two cylinders,

4. That the lower cylinder touches the plane $y = 0$,

$$f_4 = y_1 - r_1 = 0. \tag{61}$$

That is, we consider as many as six coordinates, x_1, y_1, ϕ_1, ϕ_2, θ and r, rather than the minimal set ϕ_1, θ used in the main body of this note.

We now consider the 15 analyses based on temporarily relaxing various subsets of the constraints f_1, f_2, f_3 and f_4.

[14]The term “holonomic” was introduced by Hertz on p. 91 of [18].

A.1 Relax the Rolling Constraint on the Lower Cylinder

If we imagine that the constraint (58) on the lower cylinder is relaxed, then we need three coordinates, x_1, ϕ_1 and θ to describe the system.

Constraints (59)-(61) are still enforced, so the kinetic energy of the lower cylinder is given by the first form of eq. (5), while the kinetic energy of the upper cylinder becomes,

$$T_2 = \frac{m_2}{2}\dot{x}_1^2 + m_2\dot{x}_1 r\cos\theta\,\dot{\theta} + \frac{(1+k_2)m_2}{2}r^2\dot{\theta}^2 + \frac{k_2 m_2 r_1^2\dot{\phi}_1^2}{2} - k_2 m_2 r_1 r\dot{\phi}_1\dot{\theta}, \tag{62}$$

and the potential energy is still given by eq. (3).

The Lagrangian $\mathcal{L} = T_1 + T_2 - V$ does not depend on x_1 or ϕ_1, so it is useful to identify the canonical momenta,

$$p_{x_1} = \frac{\partial\mathcal{L}}{\partial\dot{x}_1} = (m_1+m_2)\dot{x}_1 + m_2 r\cos\theta\,\dot{\theta} = P_x, \tag{63}$$

which is the total horizontal momentum, eq. (13), of the system, and,

$$\begin{aligned} p_{\phi_1} &= \frac{\partial\mathcal{L}}{\partial\dot{\phi}_1} = (k_1m_1+k_2m_2)r_1^2\dot{\phi}_1 - k_2m_2r_1r\dot{\theta} = k_1m_1r_1^2\dot{\phi}_1 + r_1[k_2m_2(r_1\dot{\phi}_1 - k_2m_2r\dot{\theta})] \\ &= L_1 + \frac{r_1}{r_2}L_2, \end{aligned} \tag{64}$$

where L_1 and L_2 are the angular momenta, eqs. (14)-(15), of the two cylinders about their axes.

The derivatives of the constraint equation (58) are,

$$\frac{\partial f_1}{\partial x_1} = 1, \qquad \frac{\partial f_1}{\partial\phi_1} = -r_1, \qquad \frac{\partial f_1}{\partial\theta} = 0. \tag{65}$$

The extended Lagrange method for this case involves a single multiplier λ_1 associated with the rolling constraint (58), such that the three Lagrange equations are now,

$$\frac{dp_{x_1}}{dt} = \lambda_1\frac{\partial f_1}{\partial x_1} = \lambda_1, \tag{66}$$

$$\frac{dp_{\phi_1}}{dt} = \lambda_1\frac{\partial f_1}{\partial\phi_1} = -r_1\lambda_1, \tag{67}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} = \lambda_1\frac{\partial f_1}{\partial\theta} = 0. \tag{68}$$

Combining eqs. (66) and (67), we have that,

$$\frac{d}{dt}\left(p_{x_1} + \frac{p_{\phi_1}}{r_1}\right) = 0, \qquad p_{x_1} + \frac{p_{\phi_1}}{r_1} = P_x + \frac{L_1}{r_1} + \frac{L_2}{r_2} = 0, \tag{69}$$

supposing that the system starts with $x_1 = \phi_1 = \theta = 0$, which is eq. (16) divided by r_1.

The force λ_1 associated with the constraint f_1 that the lower cylinder rolls without slipping on the plane $y = 0$ is related by,

$$-\lambda_1 = \frac{1}{r_1}\frac{dp_{\phi_1}}{dt} = (k_1m_1+k_2m_2)r_1\ddot{\phi}_1 - k_2m_2r\ddot{\theta}, \tag{70}$$

which is the force F_1 found in eq. (30).

A.2 Relax the Rolling Constraint on the Upper Cylinder

If we imagine that the constraint (59) on the upper cylinder is relaxed, then we need three coordinates, ϕ_1, ϕ_2 and θ to describe the system.

Constraints (58) and (60)-(61) are still enforced, so the kinetic energy of the lower cylinder is given by the second form of eq. (5), while the kinetic energy of the upper cylinder is given by,

$$T_2 = \frac{m_2}{2}\left[r_1^2\dot{\phi}_1^2 + 2r_1 r\cos\theta\,\dot{\phi}_1\dot{\theta} + r^2\dot{\theta}^2\right] + \frac{k_2 m_2 r_2^2}{2}\left(\dot{\phi}_2^2 - 2\dot{\phi}_2\dot{\theta} + \dot{\theta}^2\right), \tag{71}$$

and the potential energy is still given by eq. (3).

The Lagrangian $\mathcal{L} = T_1 + T_2 - V$ does not depend on ϕ_1 or ϕ_2, so it is useful to identify the canonical momenta,

$$p_{\phi_1} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_1} = [(1+k_1)m_1 + m_2]r_1^2\dot{\phi}_1 + m_2 r_1 r\cos\theta\,\dot{\theta}, \tag{72}$$

and,

$$p_{\phi_2} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_2} = k_2 m_2 r_2^2(\dot{\phi}_2 - \dot{\theta}). \tag{73}$$

The derivatives of the constraint equation (59) are,

$$\frac{\partial f_2}{\partial\phi_1} = -r_1, \qquad \frac{\partial f_2}{\partial\phi_2} = r_2, \qquad \frac{\partial f_2}{\partial\theta} = r_1. \tag{74}$$

The extended Lagrange method for this case involves a single multiplier λ_2 associated with the rolling constraint (59), such that the three Lagrange equations are now,

$$\frac{dp_{\phi_1}}{dt} = \lambda_2\frac{\partial f_2}{\partial\phi_1} = -r_1\lambda_2, \tag{75}$$

$$\frac{dp_{\phi_2}}{dt} = \lambda_2\frac{\partial f_2}{\partial\phi_2} = r_2\lambda_2, \tag{76}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} = \lambda_2\frac{\partial f_2}{\partial\theta} = r_1\lambda_2. \tag{77}$$

Combining eqs. (75) and (76), we have that,

$$\frac{d}{dt}\left(\frac{p_{\phi_1}}{r_1} + \frac{p_{\phi_2}}{r_2}\right) = 0, \qquad \frac{p_{\phi_1}}{r_1} + \frac{p_{\phi_2}}{r_2} = 0 \tag{78}$$

supposing that the system starts with $x_1 = \phi_1 = \theta = 0$. After we enforce the rolling constraint (59), this becomes $P_x + L_1/r_1 + L_2/r_2 = 0$, as previously noted.

The force λ_2 associated with the constraint f_2 that the upper cylinder rolls without slipping on the lower cylinder is related by,

$$F_2 = \lambda_2 = \frac{1}{r_2}\frac{dp_{\phi_2}}{dt} = k_2 m_2 r_2(\ddot{\phi}_2 - \ddot{\theta}) = k_2 m_2(r_1\ddot{\phi}_1 - r\ddot{\theta}), \tag{79}$$

which was previously found as F_{21} in eq. (28),

A.3 Relax the Constraint that the Cylinders Touch

If we imagine that the constraint (60) between the cylinders is relaxed, then we need four coordinates, ϕ_1, ϕ_2, θ and r to describe the system.

Constraints (58) and (61) are still enforced, so the kinetic energy of the lower cylinder is given by the second form of eq. (5), while the kinetic energy of the upper cylinder is given by,

$$T_2 = \frac{m_2}{2}\left[r_1^2\dot{\phi}_1^2 + 2r_1(r\cos\theta\,\dot{\theta} + \dot{r}\sin\theta)\dot{\phi}_1 + r^2\dot{\theta}^2 + \dot{r}^2\right] + \frac{k_2 m_2 r_2^2}{2}\left(\dot{\phi}_2^2 - 2\dot{\phi}_2\dot{\theta} + \dot{\theta}^2\right), \qquad (80)$$

while the potential energy should now be written as $V = m_2 g(r\cos\theta - r_1 - r_2)$ (to be zero when cylinder 2 sits directly on top of cylinder 1).

The Lagrangian $\mathcal{L} = T_1 + T_2 - V$ does not depend on ϕ_1 or ϕ_2, so it is useful to identify the canonical momenta,

$$p_{\phi_1} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_1} = [(1+k_1)m_1 + m_2]r_1^2\dot{\phi}_1 + m_2 r_1(r\cos\theta\,\dot{\theta} + \dot{r}\sin\theta) = r_1 P_x + L_1 + m_2 r_1\dot{r}\sin\theta, \qquad (81)$$

and

$$p_{\phi_2} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_2} = k_2 m_2 r_2^2(\dot{\phi}_2 - \dot{\theta}) = L_2. \qquad (82)$$

The derivatives of the constraint equation (60) are,

$$\frac{\partial f_3}{\partial\phi_1} = 0, \qquad \frac{\partial f_3}{\partial\phi_2} = 0, \qquad \frac{\partial f_3}{\partial\theta} = 0, \qquad \frac{\partial f_3}{\partial r} = 1. \qquad (83)$$

The extended Lagrange method for this case involves a single multiplier λ_3 associated with the touching constraint (60), such that the four Lagrange equations are,

$$\frac{dp_{\phi_1}}{dt} = \lambda_3\frac{\partial f_3}{\partial\phi_1} = 0, \qquad (84)$$

$$\frac{dp_{\phi_2}}{dt} = \lambda_3\frac{\partial f_3}{\partial\phi_2} = 0, \qquad (85)$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} = \lambda_3\frac{\partial f_3}{\partial\theta} = 0, \qquad (86)$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} - \frac{\partial\mathcal{L}}{\partial r} = \lambda_3\frac{\partial f_3}{\partial r} = \lambda_3. \qquad (87)$$

The force λ_3 associated with the constraint f_3 that the upper cylinder touches the lower cylinder is related by,

$$\begin{aligned}\lambda_3 = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} - \frac{\partial\mathcal{L}}{\partial r} &= m_2 r_1\left(\sin\theta\,\ddot{\phi}_1 + \cos\theta\,\dot{\phi}_1\dot{\theta}\right) - m_2\left(r_1\cos\theta\,\dot{\theta}\dot{\phi}_1 + r\,\dot{\theta}^2\right) + m_2 g\cos\theta \\ &= m_2\left[r_1\sin\theta\,\ddot{\phi}_1 - r\,\dot{\theta}^2 + g\cos\theta\right], \qquad (88)\end{aligned}$$

on setting $\ddot{r} = 0$, as this expression makes physical sense only after constraint (60) is enforced. A case of particular interest is when this force goes to zero, at the angle θ_s of separation, which is now related by,

$$\begin{aligned} r\,\dot{\theta}_s^2 &= g\cos\theta + r_1 \sin\theta_s\,\ddot{\phi}_1 \\ &= g\cos\theta - Ar\sin\theta_s\left[(\cos\theta_s - k_2)\ddot{\theta}_s - \sin\theta_s\,\dot{\theta}_s^2\right], \end{aligned} \tag{89}$$

using eq. (17). This relation was previously found in eq. (37).

A.4 Relax the Constraint that the Cylinder 1 Touches the Plane $y = 0$

If we imagine that the constraint (61) is relaxed, then we need three coordinates, y_1, ϕ_1 and θ to describe the system.

Constraints (58)-(60) are still enforced, so the kinetic and potential energies of the system are given by eqs. (11) and (3) with the additional terms,

$$\Delta T = \frac{m_1 + m_2}{2}\dot{y}_1^2 - m_2 r\sin\theta\,\dot{y}_1\dot{\theta}, \qquad \Delta V = (m_1 + m_2)g(y_1 - r_1). \tag{90}$$

The Lagrangian $\mathcal{L} = T_1 + T_2 - V$ does not depend on ϕ_1, so it is useful to identify the canonical momentum,

$$p_{\phi_1} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_1} = [(1 + k_1)m_1 + (1 + k_2)m_2]r_1^2\dot{\phi}_1 + (\cos\theta - k_2)m_2 r_1 r\dot{\theta} = r_1 P_x + L_1 + \frac{r_1}{r_2}L_2. \tag{91}$$

The derivatives of the constraint equation (61) are,

$$\frac{\partial f_4}{\partial\phi_1} = 0, \qquad \frac{\partial f_4}{\partial\theta} = 0, \qquad \frac{\partial f_4}{\partial y_1} = 1. \tag{92}$$

The extended Lagrange method for this case involves a single multiplier λ_4 associated with the touching constraint (61), such that the four Lagrange equations are,

$$\frac{dp_{\phi_1}}{dt} = \lambda_4\frac{\partial f_3}{\partial\phi_1} = 0, \tag{93}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} = \lambda_4\frac{\partial f_4}{\partial\theta} = 0, \tag{94}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{y}_1} - \frac{\partial\mathcal{L}}{\partial y_1} = \lambda_4\frac{\partial f_4}{\partial r} = \lambda_4. \tag{95}$$

The force λ_4 associated with the constraint f_4 that the lower cylinder touches the plane $y = 0$ is related by,

$$F_4 = \lambda_4 = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{y}_1} - \frac{\partial\mathcal{L}}{\partial y_1} = (m_1 + m_2)\ddot{y}_1 - m_2 r\left(\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2\right) + (m_1 + m_2)g. \tag{96}$$

This makes physical sense only after the constraint (61) is enforced, such that $\ddot{y}_1 = 0$, and the constraint force is just the normal force upward on cylinder 1,

$$F_4 = N_1 = (m_1 + m_2)g - m_2 r\left(\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2\right) = (m_1 + m_2)g + m_2\ddot{y}_2, \tag{97}$$

which was previously found as N_1 in eq. (27).

A.5 Relax All Constraints

If we imagine that all constraints (58)-(61) are relaxed, then we consider the six coordinates x_1, y_1, ϕ_1, ϕ_2 θ and r.

The kinetic energy is now,

$$\begin{aligned} T &= \frac{m_1+m_2}{2}(\dot{x}_1^2+\dot{y}_1^2)+\frac{m_1k_2r_1^2}{2}\dot{\phi}_1^2+\frac{k_2m_2r_2^2}{2}\left(\dot{\phi}_2^2-2\dot{\phi}_2\dot{\theta}+\dot{\theta}^2\right)+\frac{m_2}{2}\left(\dot{r}^2+r^2\dot{\theta}^2\right) \\ &\quad +m_2\dot{r}(\dot{x}_1\sin\theta+\dot{y}_1\cos\theta)+m_2r\dot{\theta}(\dot{x}_1\cos\theta-\dot{y}_1\sin\theta), \end{aligned} \tag{98}$$

and the potential energy is,

$$V=-m_2gr(1-\cos\theta)+(m_1+m_2)g(y_1-r_1). \tag{99}$$

The Lagrangian $\mathcal{L}=T-V$ does not depend on coordinates x_1, ϕ_1, or ϕ_2, so we identify the canonical momenta,

$$p_{x_1} = \frac{\partial\mathcal{L}}{\partial\dot{x}_1}=(m_1+m_2)\dot{x}_1+m_2(\dot{r}\sin\theta+r\cos\theta\,\dot{\theta})=P_x, \tag{100}$$

$$p_{\phi_1} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_1}=m_1k_1r_1^2\dot{\phi}_1=L_1, \tag{101}$$

$$p_{\phi_2} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_2}=m_2k_2r_2^2(\dot{\phi}_2-\dot{\theta})=L_2, \tag{102}$$

The extended Lagrange method for this case involves four multipliers λ_1-λ_4 associated with the four constraints (58)-(61), such that the six Lagrange equations are,

$$\frac{dp_{x_1}}{dt} = \lambda_1\frac{\partial f_1}{\partial x_1}+\lambda_2\frac{\partial f_2}{\partial x_1}+\lambda_3\frac{\partial f_3}{\partial x_1}+\lambda_4\frac{\partial f_4}{\partial x_1}=\lambda_1, \tag{103}$$

$$\frac{dp_{\phi_1}}{dt} = \lambda_1\frac{\partial f_1}{\partial \phi_1}+\lambda_2\frac{\partial f_2}{\partial \phi_1}+\lambda_3\frac{\partial f_3}{\partial \phi_1}+\lambda_4\frac{\partial f_4}{\partial \phi_1}=-r_1\lambda_1-r_1\lambda_2, \tag{104}$$

$$\frac{dp_{\phi_2}}{dt} = \lambda_1\frac{\partial f_1}{\partial \phi_2}+\lambda_2\frac{\partial f_2}{\partial \phi_2}+\lambda_3\frac{\partial f_3}{\partial \phi_2}+\lambda_4\frac{\partial f_4}{\partial \phi_2}=r_2\lambda_2, \tag{105}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}}-\frac{\partial\mathcal{L}}{\partial\theta} = \lambda_1\frac{\partial f_1}{\partial \theta}+\lambda_2\frac{\partial f_2}{\partial \theta}+\lambda_3\frac{\partial f_3}{\partial \theta}+\lambda_4\frac{\partial f_4}{\partial \theta}=r_1\lambda_2, \tag{106}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}}-\frac{\partial\mathcal{L}}{\partial r} = \lambda_1\frac{\partial f_1}{\partial r}+\lambda_2\frac{\partial f_2}{\partial r}+\lambda_3\frac{\partial f_3}{\partial r}+\lambda_4\frac{\partial f_4}{\partial r}=\lambda_3, \tag{107}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{y}_1}-\frac{\partial\mathcal{L}}{\partial y_1} = \lambda_1\frac{\partial f_1}{\partial y_1}+\lambda_2\frac{\partial f_2}{\partial y_1}+\lambda_3\frac{\partial f_3}{\partial y_1}+\lambda_4\frac{\partial f_4}{\partial y_1}=\lambda_4, \tag{108}$$

using the derivatives (65), (74), (83) and (92). We can combine eqs. (103)-(105) to find,

$$\frac{d}{dt}\left(p_{x_1}+\frac{p_{\phi_1}}{r_1}+\frac{p_{\phi_2}}{r_2}\right)=\frac{d}{dt}\left(P_x+\frac{L_1}{r_1}\frac{L_2}{r_2}\right)=0, \qquad P_x+\frac{L_1}{r_1}+\frac{L_2}{r_2}=0, \tag{109}$$

for a system that starts with $\phi_1=\phi_2=\theta=0$. This form is suggestive, but its content is only understandable if one writes it out in detail, as in eq. (12), which integrates to (20). Then, we have a description of the motion in terms of a single variable, θ.

We now enforce the constraints, and evaluate the multipliers.

The force λ_1 associated with the constraint f_1 that the lower cylinder rolls without slipping on the plane $y = 0$ is related by eq. (103),

$$\lambda_1 = \frac{dp_{x_1}}{dt} = (m_1 + m_2)\ddot{x}_1 + m_2 \left(r\cos\theta\,\ddot{\theta} - r\sin\theta\,\dot{\theta}^2 \right), \tag{110}$$

after setting $\dot{r} = 0$, which is the force F_1 found in eq. (26).

The force λ_2 associated with the constraint f_2 that the upper cylinder rolls without slipping on the lower cylinder is related by eq. (105),

$$F_2 = \lambda_2 = \frac{1}{r_2}\frac{dp_{\phi_2}}{dt} = k_2 m_2 r_2(\ddot{\phi}_2 - \ddot{\theta}) = k_2 m_2(r_1\ddot{\phi}_1 - r\ddot{\theta}), \tag{111}$$

which was previously found as F_{21} in eq. (28).[15]

The force λ_3 associated with the constraint f_3 that the upper cylinder touches the lower cylinder is related by eq. (107),

$$\begin{aligned} \lambda_3 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} &= m_2(\ddot{x}_1\sin\theta + \dot{x}_1\cos\theta\,\dot{\theta}) - m_2\left(\dot{x}_1\cos\theta\,\dot{\theta} + r\,\dot{\theta}^2\right) + m_2 g\cos\theta \\ &= m_2\left(\ddot{x}_1\sin\theta - r\,\dot{\theta}^2 + g\cos\theta\right), \end{aligned} \tag{112}$$

on setting $\ddot{r} = 0$ and $\dot{y}_1 = 0$, as this expression makes physical sense only after constraints (60)-(61) are enforced.

The force λ_4 associated with the constraint f_4 that the lower cylinder touches the plane $y = 0$ is related by eq. (108),

$$F_4 = \lambda_4 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}_1} - \frac{\partial \mathcal{L}}{\partial y_1} = (m_1 + m_2)\ddot{y}_1 - m_2 r\left(\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2\right) + (m_1 + m_2)g. \tag{113}$$

This makes physical sense only after the constraint (61) is enforced, such that $\ddot{y}_1 = 0$, and the constraint force is just the normal force upward on cylinder 1,

$$F_4 = N_1 = (m_1 + m_2)g - m_2 r\left(\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2\right) = (m_1 + m_2)g + m_2\ddot{y}_2, \tag{114}$$

which was previously found as N_1 in eq. (27).

We return to the description of the motion, and note that since the Lagrangian does not depend on time, energy is conserved. After enforcing the constraints (58)-(61), and using the integral (20) of the conserved quantity (109), we arrive at the expression (22) for the (conserved) energy as a function of angle θ only. The time derivative of this expression[16] (as well as Lagrange's equations) provides a second-order differential equation for θ, which can in principle be integrated to describe the motion in detail, as discussed in sec. 2.1.

Thus, the method of relaxing constraints and adding Lagrange multipliers eventually recovers the description of the motion that was obtained more directly via the basic method of Lagrange, which utilizes only the minimum number of independent coordinates (2 in this example).

[15]The forces F_1 and F_{12} could also be determined via eqs. (104) and (106).

[16]This approach is called the principle of *vis viva* in sec. 141 of [15].

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