Cylinder Rolling inside Another Rolling Cylinder

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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1 Problem

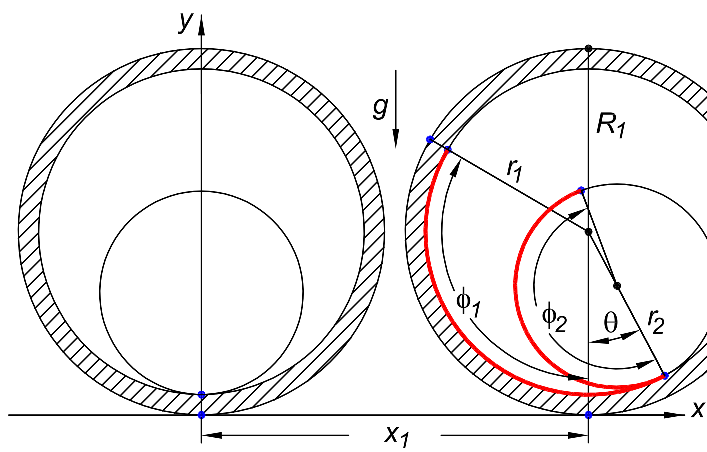
Discuss the motion of a cylinder that rolls without slipping inside another cylinder, when the latter rolls without slipping on a horizontal plane.

2 Solution

This problem is a variant of the case of one cylinder rolling on the outside of another rolling cylinder [1]. Special cases involving cylindrical shells or a solid inner cylinder are considered in ex. 2, p. 372 of [4], and in sec. 8.5, p. 111 of [5].

When one cylinder is directly above the other, we define the line of contact of the outer cylinder, 1, with the horizontal plane to be the z-axis, at $x = y = 0$. Then, the condition of rolling without slipping for the outer cylinder, of outer radius R_1 is that when it has rolled (positive) distance x_1, the initial line of contact has rotated through angle $\phi_1 = x_1/R_1$, clockwise with respect to the vertical, as shown in the figure below. This rolling constraint can be written as,

$$x_1 = R_1\phi_1. \tag{1}$$



Meanwhile, if the inner cylinder, 2, rolls such that the line of centers (in the x-y plane) makes angle θ (positive counterclockwise) to the vertical, then the initial point of contact of the upper cylinder has rotated through angle ϕ_2, measured clockwise from the line of centers, such that for rolling without slipping the arc lengths are equal between the initial points of contact of the two cylinders and the new point of contact. This second rolling constraint can be written in terms of the inner radius r_1 of the outer cylinder, and the (outer) radius r_2 of the inner cylinder, as,

$$r_2\phi_2 = r_1(\phi_1 + \theta)\,, \qquad \phi_2 - \theta = \frac{r_1}{r_2}\phi_1 + \frac{r_1 - r_2}{r_2}\theta = \frac{r_1\phi_1 + r\theta}{r_2} \qquad \text{with} \qquad r \equiv r_1 - r_2. \tag{2}$$

where $\phi_2 - \theta$ is the angle of the initial point of contact of cylinder 2 to the vertical.

Of course, the center of cylinder 1 is at $y_1 = R_1$, and so long as the two cylinders are touching, their axes are separated by distance $r = r_1 - r_2$. Altogether there are 4 constraints on the 6 degree of freedom (of two-dimensional motion) of the system, such that there are only two independent degrees of freedom, which we take to be the angles ϕ_1 and θ.

Energy $E = T + V$ is conserved, and since neither the kinetic energy T nor the potential energy V (taken to be zero when $\theta = \theta_0$),

$$V = m_2 g r(\cos\theta_0 - \cos\theta), \tag{3}$$

depend on coordinate ϕ_1 there will be another conserved quantity, the canonical momentum,

$$p_{\phi_1} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_1} = \frac{\partial T}{\partial \dot{\phi}_1}\,. \tag{4}$$

where $\mathcal{L} = T - V$ is the Lagrangian of the system. However, p_{ϕ_1} is not a single angular momentum.[1]

Since there are two conserved quantities and two degrees of freedom, there is no need to evaluate Lagrange's equations of motion to determine the motion, so long as the cylinders remain in contact and roll without slipping.

The kinetic energy of cylinder 1, whose axis is at (x_1, R_1) is,

$$T_1 = \frac{m_1 \dot{x}_1^2}{2} + \frac{I_1\,\dot{\phi}_1}{2} = \frac{1+k_1}{2} m_1 R_1^2\,\dot{\phi}_1^2, \tag{5}$$

using the rolling constraint (1) and the expression $I_1 = k_1 m_1 R_1^2$ for the moment of inertia I_1 in terms of parameter k_1 and the mass m_1.

The kinetic energy of cylinder 2, whose axis is at (x_2, y_2), is, using $I_2 = k_2 m_2 r_2^2$,

$$T_2 = \frac{m_2(\dot{x}_2^2 + \dot{y}_2^2)}{2} + \frac{I_2(\dot{\phi}_2 - \dot{\theta})^2}{2} = \frac{m_2(\dot{x}_2^2 + \dot{y}_2^2)}{2} + \frac{k_2 m_2 r_2^2(\dot{\phi}_2 - \dot{\theta})^2}{2}\,, \tag{6}$$

noting that the separation of kinetic energy into energy of the center-of-mass motion plus energy of rotation about the center of mass requires the angular velocity to be measured with respect to a fixed direction in an inertial frame. Then, recalling eqs. (1)-(2), we have,

$$x_2 = x_1 + r\sin\theta, \qquad \dot{x}_2 = R_1\,\dot{\phi}_1 + r\cos\theta\,\dot{\theta}, \tag{7}$$

$$y_2 = r_1 - r\cos\theta, \qquad \dot{y}_2 = \qquad +r\sin\theta\,\dot{\theta}, \tag{8}$$

$$\dot{\phi}_2 - \dot{\theta} = \frac{r_1\,\dot{\phi}_1 + r\,\dot{\theta}}{r_2}\,, \tag{9}$$

and the kinetic energy of cylinder 2 can be written as,

$$\begin{aligned} T_2 &= \frac{m_2}{2}[R_1^2\,\dot{\phi}_1^2 + 2R_1 r\cos\theta\,\dot{\phi}_1\,\dot{\theta} + r^2\,\dot{\theta}^2] \\ &\quad + \frac{k_2 m_2}{2}[r_1^2\,\dot{\phi}_1^2 + 2r_1 r\,\dot{\phi}_1\,\dot{\theta} + r^2\,\dot{\theta}^2] \\ &= \frac{R_1^2 + k_2 r_1^2}{2} m_2\,\dot{\phi}_1^2 + r(R_1\cos\theta + k_2 r_1) m_2\,\dot{\phi}_1\,\dot{\theta} + \frac{1+k_2}{2} m_2 r^2 \dot{\theta}^2. \end{aligned} \tag{10}$$

[1]An example of a system in which there exists a constant of the motion involving angular velocity and moments of inertia, but which is not a single angular momentum, has been given in [2]. See also [3].

The total kinetic energy $T_1 + T_2$ is,

$$T = \frac{[(1+k_1)m_1 + m_2]R_1^2 + k_2 m_2 r_1^2}{2}\dot{\phi}_1^2 + (R_1\cos\theta + k_2 r_1)m_2 r\,\dot{\phi}_1\,\dot{\theta} + \frac{1+k_2}{2}m_2 r^2\,\dot{\theta}^2, \quad (11)$$

and the conserved canonical momentum is,

$$p_{\phi_1} = \frac{\partial T}{\partial\dot{\phi}_1} = \{[(1+k_1)m_1 + m_2]R_1^2 + k_2 m_2 r_1^2\}\dot{\phi}_1 + (R_1\cos\theta + k_2 r_1)m_2 r\dot{\theta}. \quad (12)$$

The total horizontal momentum of the system is, using the rolling constraint (1),

$$P_x = (m_1+m_2)\dot{x}_1 + m_2 r\cos\theta\,\dot{\theta} = (m_1+m_2)R_1\dot{\phi}_1 + m_2 r\cos\theta\,\dot{\theta}, \quad (13)$$

while the angular momentum of the cylinder 1 about its axis is,

$$L_1 = k_1 m_1 R_1^2\dot{\phi}_1, \quad (14)$$

and that of cylinder 2 about its axis is, using the constraint (2),

$$L_2 = k_2 m_2 r_2^2(\dot{\phi}_2 - \dot{\theta}) = k_2 m_2 r_2(R_1\dot{\phi}_1 + r\dot{\theta}). \quad (15)$$

Hence, the conserved canonical momentum (12) can be written as,

$$p_{\phi_1} = R_1 P_x + L_1 + \frac{r_1}{r_2}L_2. \quad (16)$$

Equation (12) for the constant p_{ϕ_1} can be rewritten as,

$$\begin{aligned}
\dot{\phi}_1 &= \omega_0 - \frac{(R_1\cos\theta + k_2 r_1)m_2 r}{[(1+k_1)m_1 + m_2]R_1^2 + k_2 m_2 r_1^2}\dot{\theta} = \omega_0 - Ar(R_1\cos\theta + k_2 r_1)\,\dot{\theta}, && (17)\\
\ddot{\phi}_1 &= -Ar\left[(R_1\cos\theta + k_2 r_1)\,\ddot{\theta} - R_1\sin\theta\,\dot{\theta}^2\right], && (18)\\
\text{where}\quad A &= \frac{m_2}{[(1+k_1)m_1 + m_2]R_1^2 + k_2 m_2 r_1^2}\,. && (19)
\end{aligned}$$

Equation (17) integrates to give, for $\theta_0(t=0) = 0$,

$$\phi_1 = \omega_0 t - Ar(R_1\sin\theta + k_2 r_1\,\theta). \quad (20)$$

The total energy $E = T + V$ can now be rewritten (for nonzero θ_0) as[2,3]

$$\frac{2E}{m_2 r^2} = \frac{\omega_0^2}{Ar^2} + \left[1 + k_2 - A(R_1\cos\theta + k_2 r_1)^2\right]\dot{\theta}^2 + 2\frac{g}{r}(\cos\theta_0 - \cos\theta) = \frac{\omega_0^2}{Ar^2}\,. \quad (21)$$

[2]The result (21) agrees with eq. (8.5.7) of [5], noting that the notation there corresponds to $M = m_1$, $m = m_2$, $k_1 = k_2 = 1$, $a = r_2$, $b = R_1 = r_1$, $c = b - a = r_1 - r_2$, $\varphi = \theta$, $\alpha = \theta_0$ and $\omega_0 = 0$.

[3]A variant is considered in sec. 66(ii) of [6] in which the outer, thin cylinder rotates freely about a fixed axis. For this, we set $x_1 = 0$, $\omega_0 = 0$, $r_1 = R_1$, $r = R_1 - r_2$, $k_1 = 1$ and $k_2 = 1/2$ in the above. Then, the energy equation (21) becomes, $2E/m_2 r^2 = \dot{\theta}^2(3m_1 + 2)/(2m_1 + m_2) + 2gr(\cos\theta_0 - \cos\theta) = 0$, as in [6].

2.1 Steady Rolling

A particular solution is that θ is constant, say θ_0 with $|\theta_0| < \pi/2$, while $\phi_1 = \omega_0 t$, in which case $\phi_2 = r_1(\omega_0 t + \theta_0)/r_2$ according to the rolling constraint (2). Here, the two cylinders roll together, with the center of cylinder 2 at fixed angle θ_0 to the vertical with respect to the center of cylinder 1.

The angular velocity in the lab frame of the outer cylinder is $\omega_1 = \dot{\phi}_1 = \omega_0$, while that the inner cylinder is $\omega_2 = \dot{\phi}_2 - \dot{\theta} = r_1\,\omega_0/r_2$ $(> \omega_1)$.

2.2 Small Oscillations of the Inner Cylinder

We recall that the energy equation for a simple pendulum of length l, which oscillates about $\theta_0 = 0$ at angular frequency $\omega = \sqrt{g/l}$ (for small oscillations), is $\dot{\theta}^2 = 2g(\cos\theta - 1)/l$. Hence, we infer from eq. (21) that the small oscillations of the two cylinders are about $\theta_0 = 0$,[4] with angular frequency,[5,6,7]

$$\omega^2 = \frac{g}{r[1 + k_2 - A(R_1 + k_2 r_1)^2]}\,. \tag{25}$$

In addition, the entire system can be moving in the x-direction with average velocity $v_x = \omega_0 R_1$, while the outer cylinder rotates with average angular velocity ω_0.

[4] If the inner cylinder can slide on the outer cylinder, there exist oscillatory solutions for nonzero θ_0 [7].

[5] Since $A(R_1 + k_2 r_1)^2 \le 1$ according to eq. (19), ω^2 cannot be negative.

[6] Animations of the case where cylinder 1 has a fixed axis are available at
http://demonstrations.wolfram.com/SolidCylinderRollingInATurnableHollowCylinder/
http://demonstrations.wolfram.com/DiskRollingInsideARotatingRing/

[7] Another method to deduce the angular frequency ω of small oscillations of the system about the stable solution is to write the small oscillation in angle θ as,

$$\theta(t) = \theta_0 + \epsilon \sin\omega t, \qquad \dot{\theta} = \epsilon\omega\cos\theta, \tag{22}$$

and consider the constant energy E to second order in ϵ, requiring that the terms in $\epsilon^2\cos^2\omega t$ sum to zero.

For this we need the relation,

$$\begin{aligned} \cos\theta &\approx \cos\theta_0\left(1 - \frac{\epsilon^2\sin^2\omega t}{2}\right) - \epsilon\sin\theta_0\sin\omega t \\ &= \cos\theta_0\left(1 - \frac{\epsilon^2(1 - \cos^2\omega t)}{2}\right) - \epsilon\sin\theta_0\sin\omega t, \end{aligned} \tag{23}$$

Then, the energy (21) of the system is given approximately by,

$$\begin{aligned} \frac{2E}{m_2 r^2} \approx\ & \frac{\omega_0^2}{Ar^2} + \epsilon^2\omega^2\left[1 + k_2 - A(R_1\cos\theta + k_2 r_1)^2\right]\cos^2\omega t \\ & + \epsilon^2\frac{g}{r}\cos\theta_0(1 - \cos^2\omega t) + 2\epsilon\frac{g}{r}\sin\theta_0\sin\omega t. \end{aligned} \tag{24}$$

The term in $\epsilon\sin\omega t$ must be zero, which implies that the oscillations can only be about $\theta_0 = 0$, as anticipated above. That is, the formal solution for steady motion with nonzero, constant θ_0 is unstable unless $\theta_0 = 0$.

Finally, setting the terms in $\cos^2\omega t$ to zero, we find the angular frequency ω of small oscillations to be that given in eq. (25).

In the limit that $m_1 \gg m_2$ the outer cylinder is not perturbed by the oscillation of the inner cylinder, $A \to 0$, $\phi_1 \to \omega_0 t$, and,

$$\omega^2 \to \frac{g}{r(1+k_2)} \qquad (m_1 \gg m_2), \tag{26}$$

as can readily be verified by a more elementary analysis.

2.3 Angle of Separation

The above analysis holds only so long as the two cylinders remain in contact, and the normal force N_{12} between the cylinders is nonzero. For a method that does not use the forces to find the angle θ_s at which the cylinders separate,[8] we go to the accelerated frame of the lower sphere, in which there appears to be an effective acceleration due to "gravity" of,

$$\mathbf{g}_{\text{eff}} = -\ddot{x}_1\,\hat{\mathbf{x}} - g\,\hat{\mathbf{y}} = -R_1\ddot{\phi}_1\,\hat{\mathbf{x}} - g\,\hat{\mathbf{y}}. \tag{27}$$

Cylinder 2 loses contact with cylinder 1 when the component of $\mathbf{g}_{\text{eff}}$ along the line of centers, $\hat{\mathbf{r}} = (-\sin\theta, \cos\theta)$, of the cylinders equals the instantaneous radial acceleration, $r\dot{\theta}^2$. That is, separation occurs at angle θ_s where,

$$\begin{aligned} r\dot{\theta}_s^2 &= \hat{\mathbf{r}}\cdot\mathbf{g}_{\text{eff}} = -g\cos\theta_s + R_1\sin\theta_s\,\ddot{\phi}_1 \\ &= -g\cos\theta_s - rR_1A\sin\theta_s\left[(R_1\cos\theta_s + r_1k_2)\,\ddot{\theta}_s - R_1\sin\theta_s\,\dot{\theta}_s^2\right], \end{aligned} \tag{28}$$

using eq. (18).

2.4 Looping the Loop

Motion is possible in which the inner cylinder "loops the loop", reaching $\theta = 180°$, provided,

$$\dot{\theta}_{\text{top}}^2 \ge \frac{g}{r}, \tag{29}$$

when the point of contact of the inner cylinder is at the top of the outer one.

For motion with $\theta_0 = 0$, the energy relation (21), together with eq. (25), tell us that,

$$\left[1 + k_2 - A(R_1 + k_2r_1)^2\right]\dot{\theta}_0^2 = \frac{\dot{\theta}_0^2}{\omega^2}\frac{g}{r} = \left[1 + k_2 - A(k_2r_1 - R_1)^2\right]\dot{\theta}_{\text{top}}^2 + \frac{4g}{r}. \tag{30}$$

The condition (29) for looping the loop is then,

$$\dot{\theta}_0^2 \ge \left[5 + k_2 - A(R_1 - k_2r_1)^2\right]\omega^2 = \frac{5 + k_2 - A(R_1 - k_2r_1)^2}{1 + k_2 - A(R_1 + k_2r_1)^2}\frac{g}{r}. \tag{31}$$

In the limit that $m_1 \gg m_2$, for which $A \to 0$, this becomes,

$$\dot{\theta}_0^2 \ge (5 + k_2)\omega^2 = \frac{5 + k_2}{1 + k_2}\frac{g}{r} \qquad (m_1 \gg m_2), \tag{32}$$

so that for $k_2 = 0$, corresponding to a point mass sliding inside the outer cylinder,

$$\dot{\theta}_0^2 \ge \frac{5g}{r} \qquad (m_1 \gg m_2,\ k_2 = 0). \tag{33}$$

[8]For a discussion of the angle of separation based on forces, see [1].

References

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