To Construct a Square with Edges on Any Four Points

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1 Problem

Given a set of four points on a plane, construct a square with one of the four points on each of its four edges, as shown in Fig. 1.

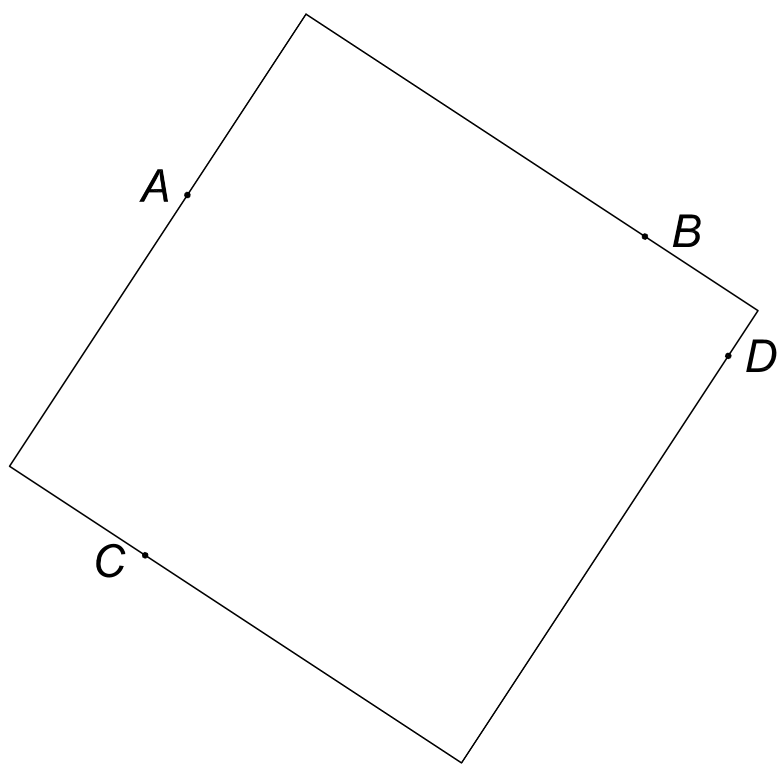

Figure 1: Construct a square with a given set of four points A, B, C and D on its edges.

2 Solution

This problem, and its solution, was suggested to us by Konstantin Shmakov.

In the figures, points are labeled by upper case letters, and angles are labeled by lower case letters.

We first prove a geometric theorem, and note two corollaries.

2.1 Theorem 0

The angle $a = \angle APB$ subtended by a chord AB of a circle with respect to any point P on the circumference on the same side of the chord as the center of the circle is one half the angle $b = \angle AOB$ subtended by the chord with respect to the center of the circle; if point P is on the other side of the chord from the center of the circle then $a = \pi - b/2$.

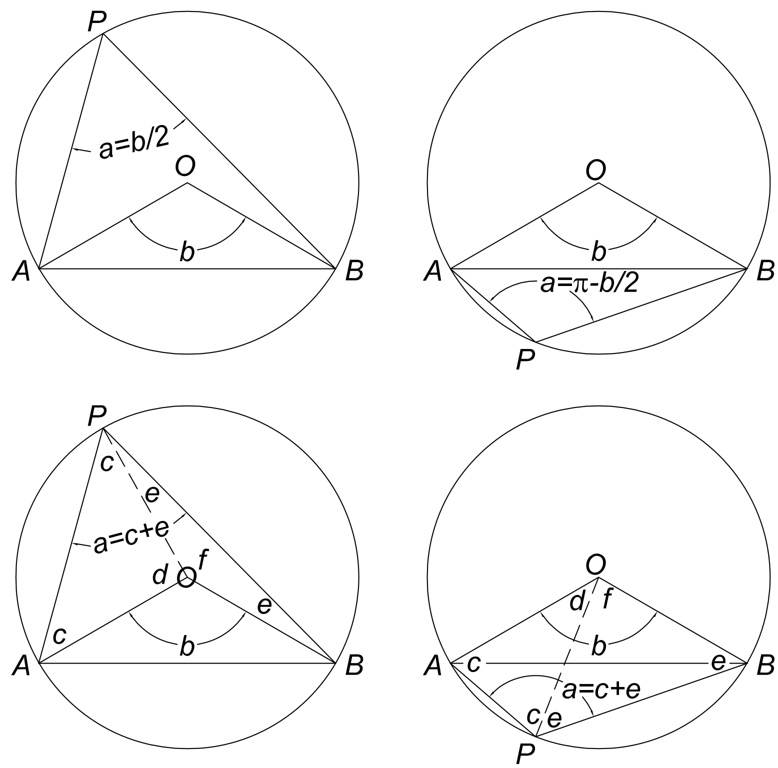

Figure 2: To show that angle $a = b/2$ (left) or $\pi - b/2$ (right).

Proof: Draw line OP from the center of the circle to point P, as shown in the lower portions of Fig. 2. Then, ΔAOP and ΔPOB (and also ΔAOB) are isosceles triangles, such that angle $c = (\pi - d)/2$, angle $e = (\pi - f)/2$, and $a = c + e = \pi - (d + f)/2$. If points O and P are on the same side of the chord AB (left figures above), then $d + f = 2\pi - b$ and $a = b/2$; while if points O and P are on opposite sides of the chord AB (right figures above), then $d + f = b$ and $a = \pi - b/2$. *QED.*

2.2 Theorem 1

The diameter of a circle subtends a 90° angle from any point P on the circumference.

In Fig. 3, the diameter AB subtends angle $b = \pi$ with respect to the center of the circle, so angle a is $\pi/2 = 90°$ according to Theorem 0.[1]

[1]Thanks to Christopher Crawford for noting that Theorems 1 and 2 follow from Theorem 0.

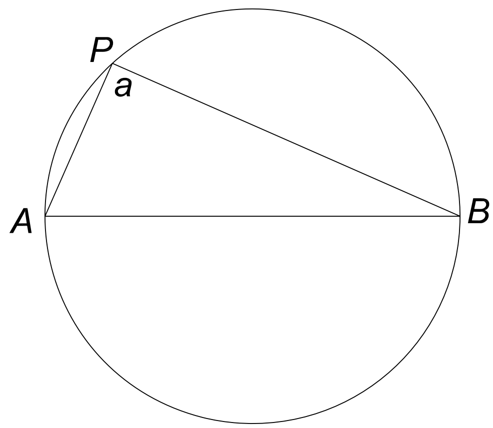

Figure 3: Angle $a = 90°$ when chord AB is a diameter.

2.3 Theorem 2

The angle a subtended from any point P on the circumference of a circle to the endpoints of two orthogonal diameters is $45°$, if the point does not lie along the $90°$ arc between the two endpoints of the diameters. If the point does lie along that arc, the angle subtended is $135°$.

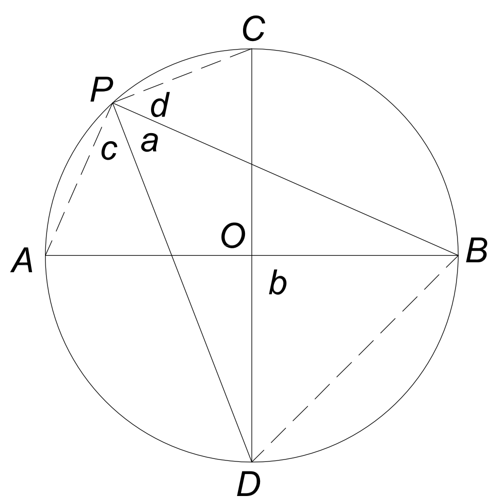

Figure 4: Angles $a = c = d = 45°$ when chords AB and CD are two orthogonal diameters of a circle.

In Fig. 4, the point is P, and the orthogonal diameters are AB and CD. Then, chord BD subtends angle $b = 90°$ with respect to the center O of the circle, and angle $a = b/2 = 45°$ if O and P are on the same side of the chord BD, according to Theorem 0.

Similarly, chords AD and BD subtends angle $90°$ with respect to the center O of the circle, and angle $c = d = 45°$ if O and P are on the same side of the chord BD,

2.4 The Four Point Construction

2.4.1 Rectangle Through Four Points Using Theorem 1

An infinite set of rectangles can be fit through four points using the result of Theorem 1, as shown in Fig. 5.

Construct four circles which have diameters as the four sides of the quadrilateral $ABCD$. Pick an arbitrary point I on the arc AB, and extend a line from point I through point B until it intersects arc BC at point J. Extend a line from point J through point C until it intersects arc CD at point K. Then, $\angle BJC = 90°$ according to Theorem 1.

Extend a line from point K through point D until it intersects arc DA at point L, forming right angle CKD. Finally, extend a line from point L through point A until it intersects arc AB at point I, forming right angles LDA and AIB. Then, quadrilateral $IJKL$ is a rectangle with points A, B, C, and D on its four sides.

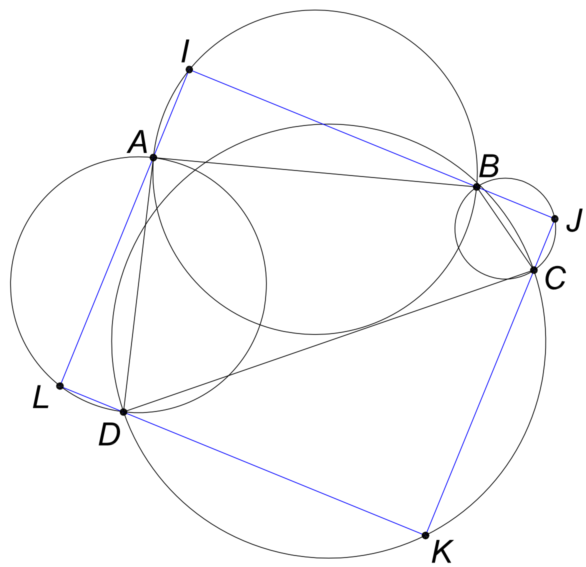

Figure 5: Construction of a rectangle $IJKL$ whose sides contain the four points A, B, C, and D, using Theorem 1.

However, in general the rectangles $IJKL$ are not squares.

2.4.2 Square Through Four Points Using Theorem 2

We seek a method to choose among the infinite set of rectangles $IJKL$ that contain points A, B, C, and D those that are squares.

In general, the diagonal IK, shown in Fig. 6, of rectangle $IJKL$ does not bisect the right angles AIB and CDK, but if it did the rectangle would be a square. That is, we desire that $\angle AIK$, $\angle BIK$, $\angle CKI$, and $\angle DKI$ all be 45°. This suggests use of Theorem 2.

For example, consider the two circles whose diameters are the lines AB and CD, as shown in Fig. 7. Construct diameters EF perpendicular to AB and GH perpendicular to

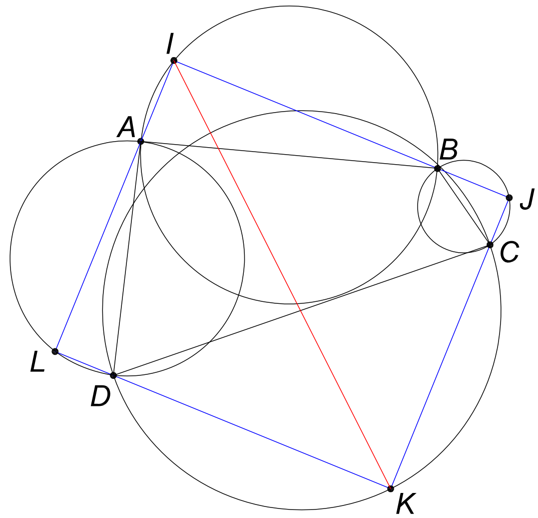

Figure 6: The rectangle $IJKL$ would be a square if the diagonal IK bisected the right angles AIB and CKD.

CD. Then, extend line FH until it intersects arc AB at point I and arc CD at point K. Theorem 2 tells us that $\angle AIK$, $\angle BIK$, $\angle CKI$, and $\angle DKI$ are all 45° as desired.

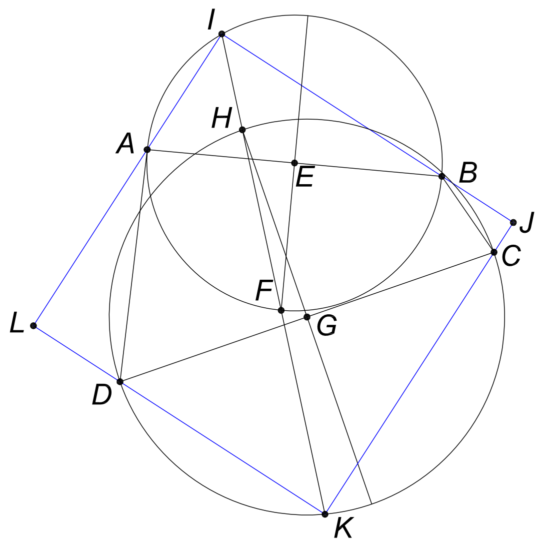

Figure 7: Construction of a square $IJKL$ whose sides contain the four points A, B, C, and D, using Theorem 2.

We complete the construction of a square by extending lines IB and CD to intersect at point J, and lines AI and DK to intersect at point L. Since $\angle JIK$ and $\angle JKI$ of ΔIJK are both 45°, $\angle IJL = 90°$. Similarly, $\angle KLI = 90°$, and so the quadrilateral $IJKL$ is a square which contains points A, B, C, and D on its sides. *QED*

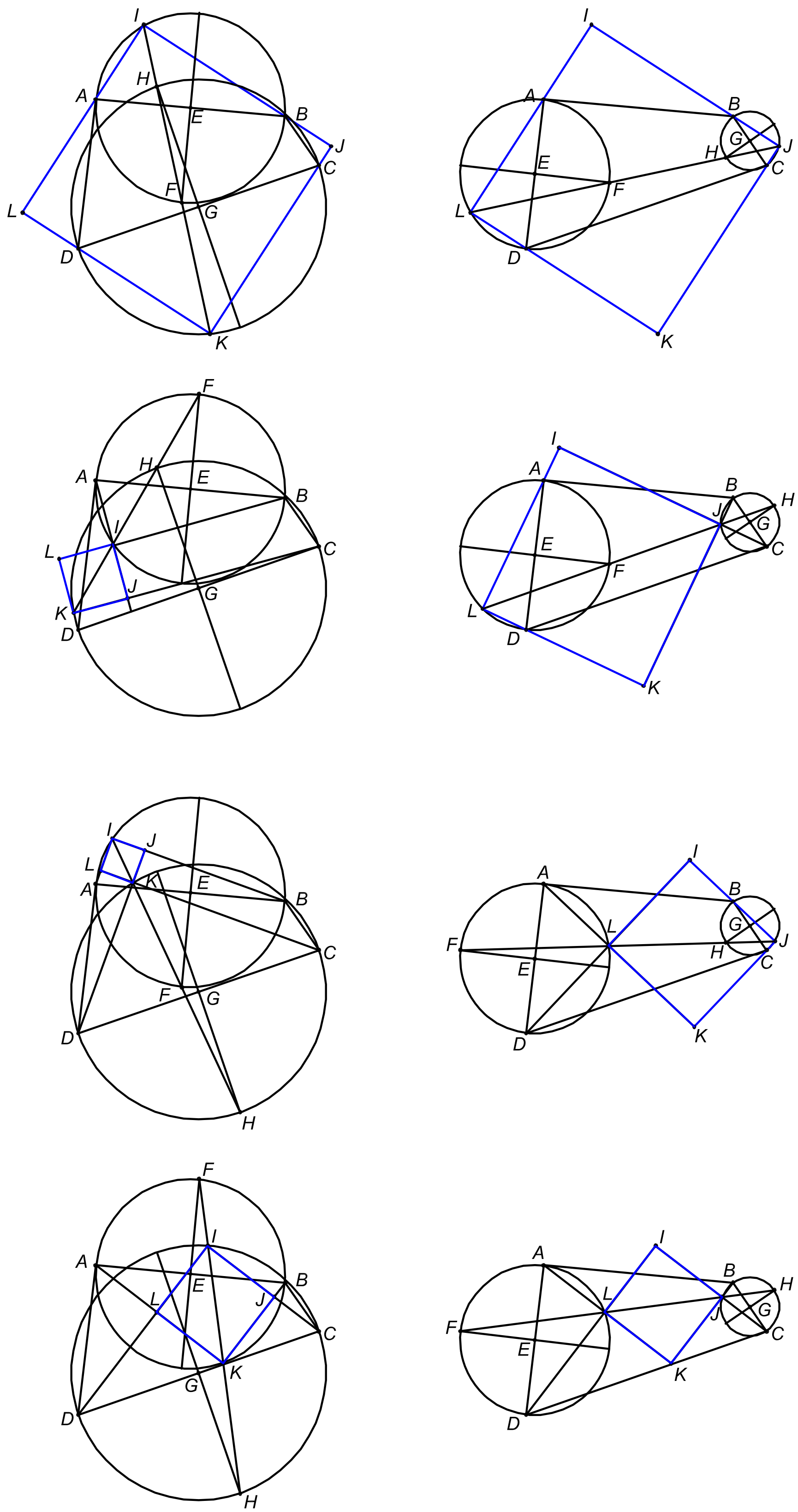

Figure 8: The 8 variants of the construction of a square $IJKL$ whose (extended) sides contain the four points A, B, C, and D.

2.4.3 The Eightfold Way

We note that the prescription given in sec. 2.4.2 can be implemented eight ways: one may chose to start with circles on sides AB and CD or on sides BC and AD (circles on adjacent sides do not work); then for each of the pair of circles, either end of the orthogonal diameter may be used to define the diagonal of the square.

The eight constructions corresponding to the four points of Fig. 1 are shown in Fig. 8. We see that for most of the constructed squares, one or more of the four original points A, B, C, and D lie on an extension of one of the sides of the square. Indeed, at most 2 of the 8 constructions can lead to squares the actually contain the four points on the sides of the square, and those 2 constructions yield the same square.

If we accept the generalization that an extended side of a square can contain one of the original four points, then we can construct squares that contain any four points. For example, Fig. 9 illustrates the construction of one of the eight squares associated with four points on a common line.

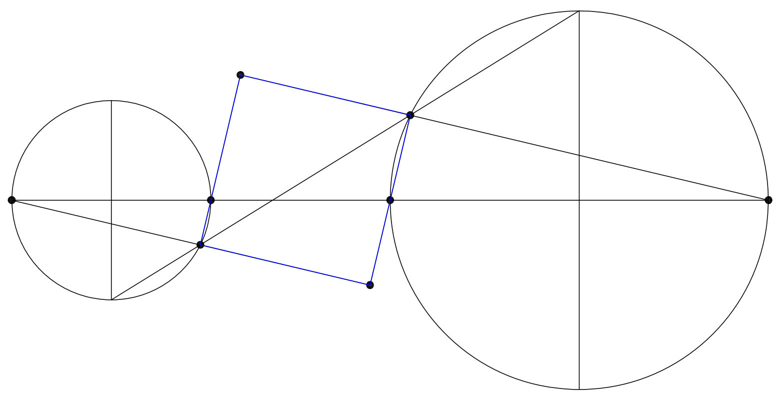

Figure 9: One of the 8 squares whose extended sides contain four points on a common line.