Ph 205 LECTURE 10

CENTRAL FORCES, CONTINUED

SOLUTION 3 - THE ORBIT EQUATION

WE GIVE A MORE SOPHISTICATED VERSION OF OUR ELEMENTARY SOLUTION — IN WHICH WE FOUND THE ORBIT $r(\theta)$ BUT NOT $r(t)$ OR $\theta(t)$.

FROM THE EXPRESSION FOR TOTAL ENERGY $E = \frac{1}{2}\mu\dot{r}^2 + V_{eff}(r)$

WE CAN IMMEDIATELY WRITE DOWN THE EQUATION OF MOTION FOR r

$$\mu\ddot{r} = -\frac{dV_{eff}}{dr}$$

THE MAIN TRICK OF THIS METHOD IS TO CHANGE VARIABLES:

$$\underline{u = 1/r}$$

$$\dot{r} = -\frac{1}{u^2}\dot{u} = -\frac{1}{u^2}\frac{du}{d\theta}\dot{\theta} = -\frac{1}{u^2}\frac{L}{\mu r^2}\frac{du}{d\theta} = -\frac{L}{\mu}\frac{du}{d\theta}$$

USING $L = \mu r^2\dot{\theta}$

$$\ddot{r} = -\frac{L}{\mu}\frac{d}{dt}\frac{du}{d\theta} = -\frac{L}{\mu}\frac{d^2u}{d\theta^2}\dot{\theta} = -\frac{Lu^2}{\mu^2}\frac{d^2u}{d\theta^2}$$

(TO AVOID CONFUSING μ AND u, I WRITE $\mu = M =$ REDUCED MASS FROM NOW ON)

$$\text{SO } \ddot{r} = -\frac{L^2u^2}{M^2}\, M\frac{d^2u}{d\theta^2} = -\frac{dV_{eff}}{dr} = -\frac{d}{dr}\left(V(r) + \frac{L^2}{2Mr^2}\right) = F(r) + \frac{L^2}{Mr^3}$$

$$\text{OR} \quad \boxed{\frac{d^2u}{d\theta^2} + u = -\frac{M}{L^2u^2}F\left(\frac{1}{u}\right)}$$

THIS IS THE SO-CALLED ORBIT EQUATION. IT HAS THE MATHEMATICAL FORM OF THE DRIVEN HARMONIC OSCILLATOR PROBLEM (LECTURE 14).

FOR GRAVITY $\quad F = -\frac{GM_1M_2}{r^2} = -\alpha u^2$

WHERE WE INTRODUCE THE ABBREVIATION $\underline{\alpha \equiv GM_1M_2}$

$$\text{SO} \quad \frac{d^2u}{d\theta^2} + u = \frac{\alpha M}{L^2}$$

THE STANDARD MATHEMATICAL TRICK TO INTEGRATE THIS IS TO WRITE DOWN A PARTICULAR SOLUTION: $u = \frac{\alpha M}{L^2}$ $\left(\frac{d^2u}{d\theta^2} = 0\right)$

AND A GENERAL SOLUTION TO THE HOMOGENEOUS EQUATION

$$\frac{d^2u}{d\theta^2} + u = 0 \implies u = A\cos(\theta - \theta_0)$$

SINCE THE DIFFERENTIAL EQUATION IS LINEAR WE MAY ADD THESE TWO SOLUTIONS:

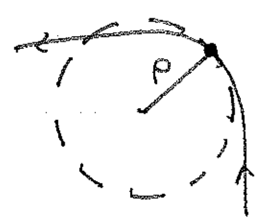
$$u = \frac{1}{r} = \frac{\alpha M}{L^2} + A\cos(\theta - \theta_0)$$

THIS IS OF THE FORM $\frac{1}{r} = \frac{1 + \epsilon\cos\theta}{a(1-\epsilon^2)}$ IF $\theta_0 = 0$

AS IN OUR PREVIOUS ELEMENTARY SOLUTION WE DO NOT YET OBTAIN SEPARATE RELATIONS FOR THE ORBIT PARAMETERS a AND ϵ.

EXERCISE: A FAMOUS HISTORICAL PROBLEM IS: GIVEN THE SHAPE OF THE ORBIT, FIND THE CENTRAL FORCE WHICH PRODUCED IT. FOR EXAMPLE, SHOW THAT IF THE ORBIT IS AN ELLIPSE WITH THE FORCE CENTER AT THE GEOMETRICAL CENTER, THEN $F(r) = -Kr$.

DIGRESSION: THE RADIUS OF CURVATURE, ρ, AT SOME POINT ON AN ORBIT IS THE RADIUS OF THE CIRCLE THAT IS TANGENT TO THE ORBIT AT THAT POINT, AND ALSO HAS THE SAME SECOND DERIVATIVE AS THE ORBIT.



FOR ORBITS UNDER AN ATTRACTIVE CENTRAL FORCE, THE RADIUS OF CURVATURE IS ALWAYS POSITIVE WITH RESPECT TO THE FORCE CENTER — I.E., THE ORBIT IS ALWAYS CONCAVE INWARDS.

TO SEE THIS, NOTE THAT FOR AN ORBIT $r(\theta)$ WE HAVE

$$\frac{1}{\rho} = \frac{r^2 + 2r'^2 - rr''}{(r^2 + r'^2)^{3/2}}$$ (PROVE IT!)

WHERE $r' \equiv \frac{dr}{d\theta}$.

IF WE WRITE $r = \frac{1}{u}$ THEN $r' = -\frac{u'}{u^2}$, AND $r'' = -\frac{u''}{u^2} + \frac{2u'^2}{u^3}$, AND $$\frac{1}{\rho} = \frac{u^3(u+u'')}{(u^2+u'^2)^{3/2}}$$

FROM THE ORBIT EQUATION ON p. 103, $\quad u''+u = -\frac{M}{Lu^2}F\left(\frac{1}{u}\right)$

FOR AN <u>ATTRACTIVE</u> CENTRAL FORCE, $F<0 \implies \rho>0$

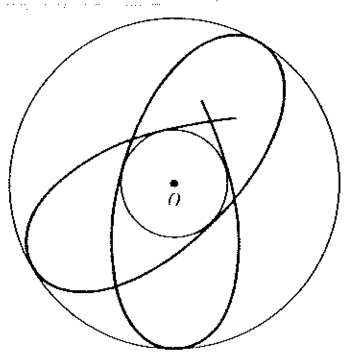


Fig. 3.34 An aperiodic bounded orbit.

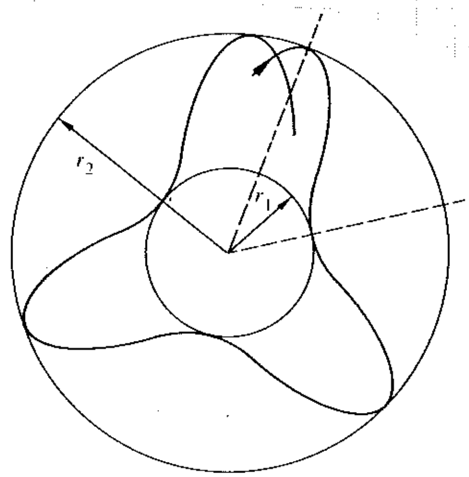


FIGURE 3–7
Schematic illustration of the nature of the orbits for bounded motion.

POSSIBLE ↑

IMPOSSIBLE ↑ (FROM GOLDSTEIN, 2ND ED. p 80)

<u>SOLUTION 4</u> : <u>TO FIND $\theta(r)$</u> (B&O SEC 4-2, L&L SECS 14, 15)

THE ENERGY RELATION $\quad E = \frac{1}{2}M\dot{r}^2 + V_{eff}(r) \quad$ (M = REDUCED MASS)

CAN BE REWRITTEN $\quad \dot{r} = \sqrt{\frac{2}{M}\left(E - V_{eff}(r)\right)}$

AGAIN $L = Mr^2\dot{\theta}$ SO $\dot{r} = \frac{dr}{d\theta}\dot{\theta} = \frac{L}{Mr^2}\frac{dr}{d\theta}$

HENCE $$d\theta = \frac{L}{\sqrt{2M}}\frac{dr}{r^2\sqrt{E-V_{eff}(r)}}$$

THIS CAN BE INTEGRATED IN PRINCIPLE – AND DIRECTLY YIELDS THE <u>PRECESSION</u> OF THE ORBIT (IF ANY) (PROB 3 p 40, L&L). ON THE PROBLEM SET WE SUGGEST YOU USE OTHER METHODS TO STUDY THE PRECESSION, HOWEVER.

FOR GRAVITY, $\quad V_{eff} = -\frac{\alpha}{r} + \frac{L^2}{2Mr^2} \quad (\alpha = GM_1M_2)$

SO $$d\theta = \frac{L}{\sqrt{2M}}\frac{dr}{r^2\sqrt{E+\frac{\alpha}{r}-\frac{L^2}{2Mr^2}}} = \frac{L\,dr}{r\sqrt{2MEr^2+2M\alpha r-L^2}}$$

THIS INTEGRAL IS 'ELEMENTARY' (DWIGHT 380.111)

$$\theta = \theta_0 + \sin^{-1}\left\{\frac{2M\alpha r - 2L^2}{r\sqrt{4M^2\alpha^2 + 8MEL^2}}\right\} = \theta_0 + \sin^{-1}\left\{\frac{1 - \frac{L^2}{M\alpha r}}{\sqrt{1 + \frac{2EL^2}{M\alpha^2}}}\right\}$$

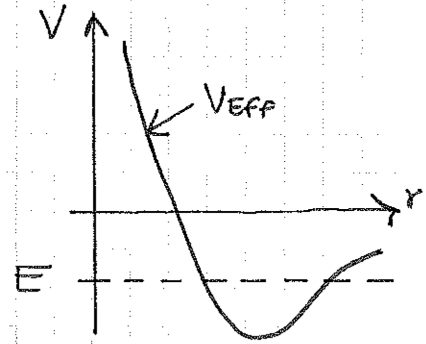
$$1 - \frac{L^2}{M\alpha r} = \sqrt{1 + \frac{2EL^2}{M\alpha^2}}\ \sin(\theta - \theta_0)$$

or $$\frac{1}{r} = \frac{M\alpha}{L^2}\left\{1 - \sqrt{1 + \frac{2EL^2}{M\alpha^2}}\ \sin(\theta - \theta_0)\right\}$$

TO PUT THIS IN OUR STANDARD FORM, CHOOSE $\theta_0 = \pi/2$

$$\frac{1}{r} = \frac{M\alpha}{L^2}\left(1 + \sqrt{1 + \frac{2EL^2}{M\alpha^2}}\cos\theta\right) = \frac{1 + \epsilon\cos\theta}{a(1-\epsilon^2)}$$

THUS $$\boxed{\epsilon = \sqrt{1 + \frac{2EL^2}{M\alpha^2}} = \text{ECCENTRICITY}}$$

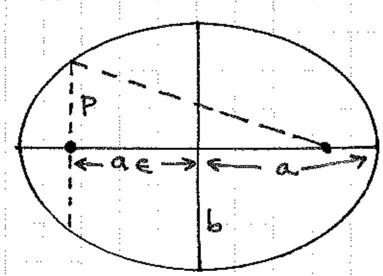


$$a(1-\epsilon^2) = \frac{L^2}{M\alpha} = a\left(\frac{-2EL^2}{M\alpha^2}\right)$$

SO $$\boxed{a = -\frac{\alpha}{2E} = \frac{\alpha}{2|E|} = \text{SEMI-MAJOR AXIS}}$$

REMEMBER, FOR A <u>BOUND</u> ORBIT, $E < 0$.

ALSO, $$b = a\sqrt{1-\epsilon^2} = \frac{\alpha}{2|E|}\sqrt{\frac{2|E|L^2}{M\alpha^2}} = \frac{L}{\sqrt{2M|E|}}$$



PEOPLE SOMETIMES DEFINE $P \equiv$ <u>LATUS RECTUM</u>

NOW $d = 2a = P + \sqrt{P^2 + 4a^2\epsilon^2}$

SO $4a^2 - 4aP + P^2 = P^2 + 4a^2\epsilon^2$

THUS $$\underline{P} = a(1-\epsilon^2) = \underline{\frac{L^2}{M\alpha}}$$

THIS COMPLETES THE RELATION OF THE KINEMATIC CONSTANTS E & L TO THE GEOMETRIC PARAMETERS OF THE ORBIT a, b, P & ϵ.

SOLUTION 5 - TO FIND THE MOTION IN TIME (L&L SEC 15)

AS IN SOLUTION 4, WE MANIPULATE THE ENERGY EQUATION TO GIVE
$$\dot{r} = \sqrt{\frac{2}{m}\left(E - V_{eff}(r)\right)}$$

SO
$$t = \sqrt{\frac{m}{2}}\int \frac{dr}{\sqrt{E - V_{eff}(r)}}$$

THIS IS A FORMAL SOLUTION TO THE CENTRAL FORCE PROBLEM. BUT CAN WE DO THE INTEGRAL?

WE CONSIDER POWER LAW POTENTIALS, $V(r) = \frac{-C}{r^{\lambda}} \Leftrightarrow F(r) = \frac{-\lambda C}{r^{\lambda+1}}$

THESE INCLUDE GRAVITY $(\lambda = 1)$ AND SPRINGS $(\lambda = -2)$

$$t = \sqrt{\frac{m}{2}}\int \frac{dr}{\sqrt{E + \frac{C}{r^{\lambda}} - \frac{L^2}{2mr^2}}} = \sqrt{\frac{m}{2}}\int \frac{dr\ r^{1+\lambda/2}}{\sqrt{E r^{\lambda+2} + C r^2 - \frac{L^2 r^{\lambda}}{2m}}}$$

IF WE LOOK IN INTEGRAL TABLES, THE ONLY SIMPLE CASES OF THIS TYPE ARE $\int \frac{dr}{\sqrt{ar^2 + br + c}}$

$\lambda = 0 \Rightarrow F = 0$ WE IGNORE THIS TRIVIAL CASE

$\lambda = -1 \Rightarrow F =$ CONSTANT (BUT RADIAL) <u>NOT</u> INTEGRABLE!

$\lambda = 1 \Rightarrow$ GRAVITY
$$t = \sqrt{\frac{m}{2}}\int \frac{dr\ r^{3/2}}{\sqrt{Er^3 + Cr^2 - \frac{L^2 r}{2m}}} = \sqrt{\frac{m}{2}}\int \frac{r\,dr}{\sqrt{Er^2 + Cr - L^2/2m}}$$

THIS IS SURELY INTEGRABLE

$\lambda = -2 \Rightarrow$ SPRINGS
$$t = \sqrt{\frac{m}{2}}\int \frac{dr}{\sqrt{E + Cr^2 - \frac{L^2}{2mr^2}}} = \sqrt{\frac{m}{2}}\int \frac{r\,dr}{\sqrt{Er^2 + Cr^4 - \frac{L^2}{2m}}}$$

LET $s = r^2$

$$t = \frac{\sqrt{2m}}{4}\int \frac{ds}{\sqrt{Es + Cs^2 - L^2/2m}}$$

THERE IS ONLY ONE MORE SIMPLE CASE!

$$\lambda = 2 \Rightarrow F = -\frac{2C}{r^3}$$

$$t = \sqrt{\frac{m}{2}}\int \frac{dr\, r^2}{\sqrt{Er^4 + Cr^2 - \frac{L^2 r^2}{2m}}} = \sqrt{\frac{m}{2}}\int \frac{r\,dr}{\sqrt{Er^2 + C - L^2/2m}}$$

WE REFER YOU TO PROBLEM (8) TO INVESTIGATE THIS CASE BY A DIFFERENT METHOD. SEE ALSO PROB 2, p 40 L & L.

WE NOW CONCENTRATE ON GRAVITY, $\lambda = 1$

AGAIN WE WRITE $\alpha = GM_1M_2$

$$t = \sqrt{\frac{m}{2}}\int \frac{r\,dr}{\sqrt{Er^2 + \alpha r - L^2/2m}}$$

RECALL THAT $E < 0$ FOR A BOUND ORBIT

NOW $E = -\frac{\alpha}{2a}$ AND $\frac{L^2}{2m} = \frac{\alpha P}{2} = \frac{\alpha a(1-\epsilon^2)}{2}$

$$t = \sqrt{\frac{m}{2\alpha}}\int \frac{r\,dr}{\sqrt{-\frac{r^2}{2\alpha} + r - \frac{a(1-\epsilon^2)}{2}}} = \sqrt{\frac{ma}{\alpha}}\int \frac{r\,dr}{\sqrt{-r^2 + 2ar - a^2 + a^2\epsilon^2}}$$

$$= \sqrt{\frac{ma}{\alpha}}\int \frac{r\,dr}{\sqrt{a^2\epsilon^2 - (r-a)^2}}$$

TO INTEGRATE THIS WE SUBSTITUTE

$$r - a = -a\epsilon\cos\tau \qquad dr = a\epsilon\sin\tau\, d\tau$$

$$t = \sqrt{\frac{ma}{\alpha}}\int \frac{a(1-\epsilon\cos\tau)\,a\epsilon\sin\tau\, d\tau}{a\epsilon\sqrt{1-\cos^2\tau}} = \sqrt{\frac{ma^3}{\alpha}}\int (1-\epsilon\cos\tau)\,d\tau$$

HENCE
$$\boxed{\begin{aligned} t &= \sqrt{\frac{ma^3}{\alpha}}\,(\tau - \epsilon\sin\tau) \\ r &= a(1-\epsilon\cos\tau) \end{aligned}}$$
IS OUR SOLUTION.

THIS IS SOMETIMES CALLED KEPLER'S EQUATION OF TIME.

FOR SMALL ϵ, THE PARAMETER τ IS ESSENTIALLY THE SAME AS Θ (SEE P 98)

WE CAN DERIVE OTHER RELATIONS INVOLVING TIME.

NOW $\frac{1}{r} = \frac{1+\epsilon\cos\theta}{a(1-\epsilon^2)} = \frac{M\alpha}{L^2}(1+\epsilon\cos\theta)$

SO $\dot{r} = \frac{\alpha\epsilon}{L}\sin\theta$ USING $L = Mr^2\dot{\theta}$

THE VELOCITY IS THEN $v^2 = \dot{r}^2 + r^2\dot{\theta}^2 = \dot{r}^2 + \frac{L^2}{M^2r^2}$

OR $v = \frac{\alpha}{L}\sqrt{1+2\epsilon\cos\theta+\epsilon^2}$

$v_{MAX} = \frac{\alpha}{L}(1+\epsilon)$ AT $\theta = 0$

$v_{MIN} = \frac{\alpha}{L}(1-\epsilon)$ AT $\theta = \pi$

FINALLY $\dot{\theta} = \frac{L}{Mr^2} = \frac{M\alpha^2}{L^3}(1+\epsilon\cos\theta)^2$

THIS CAN BE INTEGRATED (DWIGHT 446.03) TO GIVE

$$t = \frac{La}{\alpha}\left\{\frac{2}{\sqrt{1-\epsilon^2}}\tan^{-1}\left[\frac{(1+\epsilon)\tan\theta/2}{\sqrt{1-\epsilon^2}}\right] + \frac{\epsilon\sin\theta}{1+\epsilon\cos\theta}\right\}$$

WHICH I'M SURE IS MORE THAN YOU EVER WANTED TO KNOW ABOUT ORBITS. (EXERCISE: DERIVE THIS (OR ITS EQUIVALENT) FROM KEPLER'S EQUATION OF TIME.)

NONETHELESS, WE CAN STILL GIVE ANOTHER SOLUTION....

SOLUTION 6 - ECCENTRICITY VECTOR (B&O SEC 4-3, L&L p39)

THE SOLUTION TO THE CENTRAL FORCE PROBLEM INVOLVES THE INTEGRATION OF THE THREE-DIMENSIONAL, SECOND-ORDER DIFFERENTIAL EQUATION $\bar{F} = M\bar{a}$. HENCE WE SHOULD EXPECT THE SOLUTION TO INVOLVE 6 CONSTANTS OF INTEGRATION. BUT SO FAR WE HAVE ONLY USED 5: THE DEFINITION OF $\theta = 0$ AT PERIHELION, TOTAL ENERGY E, AND VECTOR ANGULAR MOMENTUM $\bar{L}$ = CONSTANT. ANOTHER CONSTANT MUST EXIST. IT IS AMUSING THAT FOR THE CASE OF GRAVITY, WE CAN ACTUALLY FIND ANOTHER CONSTANT VECTOR PROPERTY OF THE MOTION. BUT WE CAN ALSO FIND 2 RELATIONS BETWEEN THIS NEW VECTOR AND OUR PREVIOUS

CONSTANTS – SO WE HAVE EXACTLY 6 INDEPENDENT CONSTANTS OF THE MOTION AS DESIRED.

THE NEW VECTOR IS

$$\bar{\epsilon} = \hat{r} + \frac{\bar{L} \times \bar{v}}{\alpha} = \hat{r} + \frac{\bar{L} \times \bar{p}}{M\alpha} \qquad \left(\bar{L} = \text{ANGULAR MOMENTUM}\right)$$

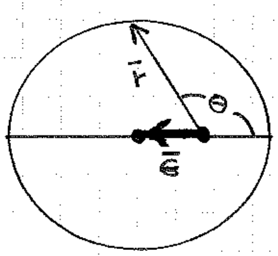
AT ONCE $\bar{\epsilon} \cdot \bar{L} = 0 \Rightarrow \bar{\epsilon}$ LIES IN THE PLANE OF THE ORBIT.

AFTER SOME ALGEBRA WE MAY SHOW THAT $\epsilon^2 = 1 + \dfrac{2EL^2}{M\alpha^2} = (\text{ECCENTRICITY})^2$

HENCE THE NAME $\bar{\epsilon}$ = <u>ECCENTRICITY VECTOR</u>.

TO PROVE THAT IT IS A CONSTANT, JUST DIFFERENTIATE – IT IS AN INTERESTING EXERCISE TO SHOW THAT $\dfrac{d\bar{\epsilon}}{dt} = 0$.
(HINT: FIRST SHOW THAT $d\hat{r}/dt = \bar{L} \times \hat{r}/Mr^2$)

KNOWING THAT $\bar{\epsilon}$ IS A CONSTANT, WE CAN SOLVE FOR THE ORBIT:



$$\bar{\epsilon} \cdot \bar{r} = -\epsilon r \cos\theta$$
$$= r + \frac{(\bar{L} \times \bar{p}) \cdot \bar{r}}{M\alpha}$$
$$= r + \frac{\bar{L} \cdot (\bar{p} \times \bar{r})}{M\alpha}$$

OR $$-\epsilon r \cos\theta = r - \frac{L^2}{M\alpha}$$

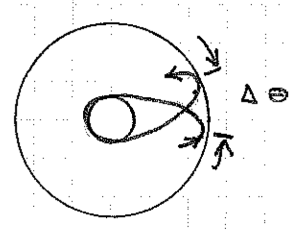
AND $$\frac{1}{r} = \frac{M\alpha}{L^2}(1 + \epsilon \cos\theta) \qquad \text{AS USUAL!}$$

A 6TH CONSTANT OF THE MOTION EXISTS FOR ANY CENTRAL FORCE PROBLEM, BUT APPARENTLY ONLY FOR GRAVITY CAN IT BE GIVEN SUCH AN INTERESTING PHYSICAL INTERPRETATION.

<u>EXAMPLE</u>: <u>PRECESSION OF ORBITS DUE TO A DUST CLOUD</u>

A FAMOUS PROBLEM IN ASTRONOMY IS THE PRECESSION OF THE PERIHELION OF MERCURY'S ORBIT BY 40'' PER CENTURY

WE HAVE SEEN THAT THE ORBITS OF A $1/r^2$ FORCE DO NOT PRECESS. BUT IN GENERAL THE ORBITS OF A CENTRAL FORCE ARE NOT SIMPLE CLOSED CURVES, AND CAN BE SAID TO PRECESS. WE MEASURE THE PRECESSION BY THE CHANGE OF ANGULAR POSITION OF THE PERIHELION OR APHELION. $\Delta\theta$ = AMOUNT OF PRECESSION IN 1 REVOLUTION.



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ESSENTIALLY ANY DEPARTURE OF GRAVITY FROM $1/r^2$ WILL CAUSE PRECESSION. HISTORICALLY MANY SUGGESTIONS WERE MADE TO EXPLAIN THE PRECESSION OF MERCURY'S ORBIT — SEE ALSO THE HOMEWORK SET.

HERE WE EXPLORE THE EFFECT OF A DUST CLOUD WHICH FILLS THE SOLAR SYSTEMS WITH MASS OF UNIFORM DENSITY ρ. WE DO NOT CONSIDER ANY FRICTIONAL EFFECT, BUT RATHER THE GRAVITATIONAL EFFECT OF THE DUST.

WE NEED TO KNOW THAT THE EFFECT ON A PLANET AT RADIUS r FROM THE SUN IS THE SAME AS IF ALL THE DUST INSIDE RADIUS r WERE AT THE ORIGIN. THE DUST OUTSIDE RADIUS r HAS NO NET EFFECT. (LIKE GAUSS'S LAW IN ELECTRICITY).

$$F_{DUST} = -\frac{G M_{EFF} m_2}{r^2} \quad \text{WHERE} \quad M_{EFF} = \frac{4\pi}{3} r^3 \rho$$

$$\text{SO} \quad F_{TOTAL} = -\frac{G m_1 m_2}{r^2} - \frac{4\pi}{3} G \rho m_2 r$$

WE WILL USE THE EFFECTIVE POTENTIAL METHOD TO ANALYZE THE MOTION.

$$V = -\int F\, dr = -\frac{G m_1 m_2}{r} + \frac{2\pi}{3} G \rho m_2 r^2$$

$$\text{SO} \quad V_{EFF} = \frac{L^2}{2 m_2 r^2} - \frac{G m_1 m_2}{r} + \frac{2\pi}{3} G \rho m_2 r^2$$

HERE WE SUPPOSE $m_2 << m_1$ SO $\mu \approx m_2$ = REDUCED MASS.

$$\frac{dV_{EFF}}{dr} = \frac{-L^2}{m_2 r^3} + \frac{G m_1 m_2}{r^2} + \frac{4\pi}{3} G \rho m_2 r$$

$$\frac{d^2 V_{EFF}}{dr^2} = \frac{3L^2}{m_2 r^4} - \frac{2 G m_1 m_2}{r^3} + \frac{4\pi}{3} G \rho m_2$$

AT EQUILIBRIUM, $\left.\frac{dV_{EFF}}{dr}\right|_{r_0} = 0$ AND $L \approx m_2 r_0^2 \Omega$

WHERE Ω = EQUILIBRIUM ANGULAR VELOCITY.

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THUS $\quad -M_2 r_0 \Omega^2 + \dfrac{G M_1 M_2}{r_0^2} + \dfrac{4\pi}{3} G \rho M_2 r_0 = 0$

$$\text{OR} \quad \frac{1}{r_0^3} = \frac{\Omega^2}{G M_1} - \frac{4\pi\rho}{3 M_1}$$

AND $\quad K_{eff} = \left.\dfrac{d^2 V_{eff}}{d r^2}\right|_{r_0} = 3 M_2 \Omega^2 - 2 G M_1 M_2 \left(\dfrac{\Omega^2}{G M_1} - \dfrac{4\pi\rho}{3 M_1}\right) + \dfrac{4\pi}{3} G \rho M_2$

$$= M_2 \Omega^2 + 4\pi G \rho M_2$$

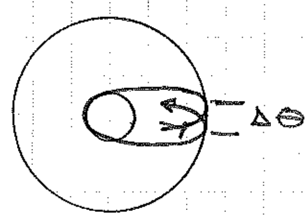
THE OSCILLATION FREQUENCY IS

$$\omega = \sqrt{\frac{K_{eff}}{M_{eff}}} \approx \sqrt{\frac{K_{eff}}{M_2}} = \sqrt{\Omega^2 + 4\pi G\rho} \approx \Omega\left(1 + 2\pi \frac{G\rho}{\Omega^2}\right) > \Omega$$

THE <u>PERIOD</u> OF OSCILLATION $\left(\frac{2\pi}{\omega}\right)$ IS GREATER THAN THE PERIOD OF ROTATION $\left(\frac{2\pi}{\Omega}\right)$.

IN ONE PERIOD OF OSCILLATION THE POLAR ANGLE ADVANCES BY $\quad \Theta = \Omega\left(\frac{2\pi}{\omega}\right) \approx 2\pi\left(1 - \frac{2\pi G\rho}{\Omega^2}\right)$

HENCE THE AMOUNT OF PRECESSION PER REVOLUTION IS $\quad \Delta\Theta \approx \dfrac{-4\pi^2 G\rho}{\Omega^2}$ RADIANS



$\Delta\Theta < 0 \Rightarrow$ THE PERIHELION (AND APHELION) <u>RETREATS</u> RATHER THAN ADVANCES.

THIS ALONE TELLS US THAT THE DUST CLOUD CANNOT EXPLAIN THE EFFECT ON MERCURY, WHOSE APHELION IS OBSERVED TO <u>ADVANCE</u> WITH TIME.

THE RATE OF CHANGE OF THE ANGLE OF APHELION (OR PERIHELION) IS $\quad \dfrac{\Delta\Theta}{\Delta t} \approx \dfrac{-4\pi^2 G\rho}{\Omega^2}\dfrac{1}{T} = -G\rho T \quad$ USING $\quad T = \dfrac{2\pi}{\Omega}$

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FOR AMUSEMENT, WE CALCULATE THE DENSITY ρ NEEDED TO PRODUCE THE MAGNITUDE OF THE EFFECT ON MERCURY, EVEN THOUGH THE SIGN IS WRONG.

$$\frac{\Delta\theta}{\Delta t} = \frac{40''}{\text{CENTURY}} \sim 7\times10^{-14} \text{ RADIANS / SEC}$$

$$T = .24 \text{ YEAR} \sim 7\times10^{6} \text{ SEC}$$

$$\rho = \frac{1}{GT}\frac{\Delta\theta}{\Delta t} \sim 10^{-10}\ \text{kg/m}^3 \sim 10^{-13}\ \text{gm/cm}^3$$

FOR A CLOUD OF HYDROGEN GAS, THIS WOULD AMOUNT TO ABOUT 10^{-9} OF ATMOSPHERIC PRESSURE — ABOUT THE ATMOSPHERIC PRESSURE ON THE MOON. THE PRESSURE BETWEEN THE PLANETS IS PRESUMABLY LESS.

<u>EXERCISE</u>: IF THE FORCE LAW IS NOT EXACTLY $\bar{F} = -\frac{k}{r^2}\hat{r}$ THEN THE LENZ VECTOR $\bar{\epsilon}$ (p. 109) IS NOT CONSTANT. SHOW THAT ITS <u>AVERAGE</u> RATE OF CHANGE CAN BE WRITTEN

$$\left\langle \frac{d\bar{\epsilon}}{dt} \right\rangle = \bar{\omega} \times \bar{\epsilon}$$

WHERE $\omega = \frac{d\theta}{dt}$ = PRECESSION RATE OF THE ORBIT.

THAT IS, $\bar{\epsilon}$ POINTS ALONG THE AVERAGE DIRECTION OF THE MAJOR AXIS, AND ROTATES WITH THE PRECESSING ORBIT. $|\bar{\epsilon}|$ IS CONSTANT.

BE VERY CAREFUL IN TAKING THE TIME AVERAGE OF $\frac{d\bar{\epsilon}}{dt}$!

[SEE AM. J. PHYS. <u>58</u>, 540 (1990).]

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<u>THREE-BODY PROBLEM</u>. THERE IS NO GENERAL SOLUTION FOR THE MOTION OF THREE BODIES THAT INTERACT ONLY VIA GRAVITY. HOWEVER, THERE IS ONE SPECIAL CASE OF INTEREST: 3 BODIES IN STEADY ROTATION ABOUT THEIR CENTER OF MASS.

A FAMOUS RESULT, DUE TO LAGRANGE (1772) IS THAT THE 3 MASSES FORM AN EQUILATERAL TRIANGLE.

TO SEE THIS, CHOOSE THE CM (ASSUMED AT REST) AS THE ORIGIN:

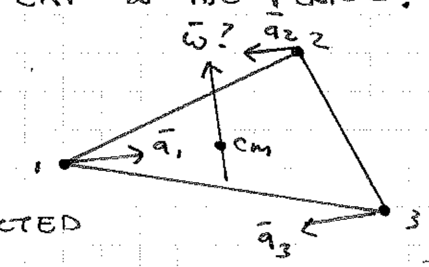
$$m_1 \bar{r}_1 + m_2 \bar{r}_2 + m_3 \bar{r}_3 = 0$$

EACH MASS MOVES WITH THE SAME ANGULAR VELOCITY, $\bar{\omega}$, WHICH IS CONSTANT BY HYPOTHESIS. THEN $\bar{v}_i = \bar{\omega} \times \bar{r}_i \quad (i = 1,2,3)$

AND THE ACCELERATION IS $\bar{a}_i = \dot{\bar{v}}_i = \dot{\bar{\omega}} \times \bar{r}_i + \omega \times \dot{\bar{r}}_i$

$$= \bar{\omega} \times (\bar{\omega} \times \bar{r}_i) = (\bar{\omega} \cdot \bar{r}_i)\, \bar{\omega} - \omega^2 \bar{r}_i$$

SINCE THE $\bar{a}_i$ ARISE FROM GRAVITY, THEY MUST LIE IN THE PLANE OF THE 3 MASSES. WE INFER THAT EITHER $\bar{\omega}$ IS IN THIS PLANE ALSO, OR THAT $\bar{\omega} \cdot \bar{r}_i = 0 \iff \bar{\omega}$ IS PERP TO THE PLANE. WE CLAIM THE LATTER.



IF $\bar{\omega}$ WERE IN THE PLANE, THEN EACH $\bar{a}_i$ IS PERP TO $\bar{\omega}$, AND AT LEAST ONE OF THESE IS NOT DIRECTED TO THE INTERIOR OF THE TRIANGLE. HOWEVER, THIS CANNOT BE, SINCE $\bar{a}_i = \frac{\bar{F}_i}{m_i}$

$\therefore$ $\bar{\omega}$ IS PERP TO THE TRIANGLE, AND $\bar{a}_i = -\omega^2 \bar{r}_i$

CONSIDER MASS 1: $\quad m_1 \bar{a}_1 = F_1 \Rightarrow -m_1 \omega^2 \bar{r}_1 = G m_1 \left[\frac{m_2(\bar{r}_2 - \bar{r}_1)}{r_{12}^3} + \frac{m_3(\bar{r}_3 - \bar{r}_1)}{r_{13}^2} \right]$

BUT SINCE $m_3 \bar{r}_3 = -m_1 \bar{r}_1 - m_2 \bar{r}_2$, WE FIND

$$\left(\frac{m_2}{r_{12}^3} - \frac{m_2}{r_{13}^2} \right) \bar{r}_2 = \left(\frac{m_1}{r_{13}^3} + \frac{m_2}{r_{12}^3} + \frac{m_3}{r_{13}^3} - \frac{\omega^2}{G} \right) \bar{r}_1$$

SINCE $\bar{r}_1$ IS NOT $\parallel$ TO $\bar{r}_2$, BOTH SIDES MUST ACTUALLY BE ZERO!

$\Rightarrow r_{12} = r_{13} \equiv R$, AND $\omega^2 = \frac{G(m_1 + m_2 + m_3)}{R^3}$. START WITH $m_2 \bar{a}_2$ TO SHOW $r_{23} = R$...