Ph 205 LECTURE 13

MORE ABOUT OSCILLATIONS

ALONG WITH THE $1/r^2$ FORCE LAW, THE SPRING FORCE, $-kr$, IS HIGHLY ANALYZABLE. ITS IMPORTANCE ARISES AS IT IS THE FIRST APPROXIMATION TO THE RESTORING FORCE IN ANY DEPARTURE FROM EQUILIBRIUM.

WE CONSIDER SEVERAL TOPICS:

1) THE DAMPED HARMONIC OSCILLATOR
2) SINUSDIODAL DRIVING FORCES
3) PERIODIC DRIVING FORCES
4) ARBITRARY DRIVING FORCES

ALTHOUGH THE ARITHMETIC IS SOMEWHAT LENGTHY, THE BASIC IDEAS ARE RELATIVELY SIMPLE. IT IS IMPORTANT TO FORM AN INTUITIVE PICTURE OF THE NATURE AND METHODS OF SOLUTION, WITHOUT UNDO WORRY ABOUT EVERY LAST CONSTANT IN THE EQUATIONS.

REFERENCES: B & O SECS 1-7, 8, 9 ; L & L SECS 21, 22, 25, 26

1) THE DAMPED HARMONIC OSCILLATOR

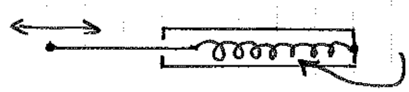
IN REALITY MOST OSCILLATORY SYSTEMS CONTAIN SOME FRICTION. THE CASE OF FRICTION PROPORTIONAL TO VELOCITY IS EASILY SOLVED, SO WE EMPHASIZE IT.

$$F = ma \Rightarrow m\ddot{x} = -kx - b\dot{x}$$

k = SPRING CONSTANT

b = DRAG COEFFICIENT [MECHANICAL ENERGY IS NOT CONSERVED IF $b \neq 0$]

THIS EQUATION HAS A WELL-KNOWN APPLICATION IN L-R-C ELECTRICAL CIRCUITS. AS A TYPICAL MECHANICAL EXAMPLE, CONSIDER THE SHOCK ABSORBER OF A CAR.



GOO, TO DAMP THE SPRING

WE REWRITE OUR EQUATION $\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$

AS $\underline{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0}$ $\qquad \omega_0^2 = k/m \qquad 2\beta = b/m$

THERE IS A VERY USEFUL TRICK IN SOLVING THIS EQUATION:

TRY A SOLUTION $x = Ae^{\alpha t}$ WITH A, α COMPLEX

WHAT WE REALLY MEAN IS $x = \mathrm{Re}\left[Ae^{\alpha t}\right]$, BUT WE USUALLY LEAVE OUT THIS NOTATION, AND SIMPLY TAKE THE REAL PART OF THE SOLUTION AT THE LAST MOMENT.

RECALL THAT ANY COMPLEX NUMBER CAN BE WRITTEN

$$A = \mathrm{Re}\,A + i\,\mathrm{Im}\,A = Be^{i\theta} \quad \text{WITH } B, \theta \text{ REAL}$$

$$e^{i\theta} = \cos\theta + i\sin\theta \;;\; B = \sqrt{AA^*} \;;\; A^* = \text{COMPLEX CONJUGATE} = Be^{-i\theta}$$

$$\tan\theta = \mathrm{Im}A / \mathrm{Re}A \quad \text{ETC.}$$

WE NOW SUBSTITUTE OUR TRIAL SOLUTION $x = Ae^{\alpha t}$ INTO

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

$$\Longrightarrow \quad Ae^{\alpha t}\left[\alpha^2 + 2\beta\alpha + \omega_0^2\right] = 0$$

THE NON-TRIVIAL SOLUTION IS $\alpha = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$.

WE HAVE REDUCED THE DIFFERENTIAL EQUATION TO A MERE QUADRATIC EQUATION! THERE ARE TWO ROOTS — SINCE A SECOND ORDER DIFFERENTIAL SHOULD HAVE 2 SOLUTIONS:

$$x = e^{-\beta t}\left(A_1 e^{\sqrt{\beta^2 - \omega_0^2}\,t} + A_2 e^{-\sqrt{\beta^2 - \omega_0^2}\,t}\right)$$

THERE A 3 CATEGORIES OF SOLUTION, DEPENDING ON $\beta^2 - \omega_0^2$

a) <u>WEAK DAMPING</u> $\iff \beta < \omega_0$ [NO DAMPING $\iff \beta = 0$]

$$\alpha = -\beta \pm i\sqrt{\omega_0^2 - \beta^2}$$

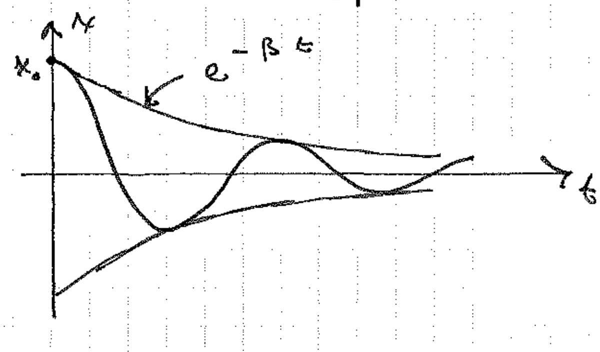
WE DEFINE $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$ SO ω_1 IS REAL

$$x = e^{-\beta t}\left(A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t}\right) \qquad A_1, A_2 \text{ COMPLEX}$$

TAKING THE REAL PART, $x = e^{-\beta t}\left(C\cos\omega_1 t + D\sin\omega_1 t\right)$, C, D REAL

EXAMPLE: $x(0) = x_0 \quad \dot{x}(0) = 0 \Rightarrow C = x_0\,,\; D = \frac{\beta}{\omega_1}x_0$

$$x = x_0 e^{-\beta t}\left(\cos\omega_1 t + \frac{\beta}{\omega_1}\sin\omega_1 t\right)$$



b) STRONG DAMPING $\Longleftrightarrow$ $\beta > \omega_0$ (OVER-DAMPING)

α IS REAL $\Longrightarrow$ NO OSCILLATORY MOTION

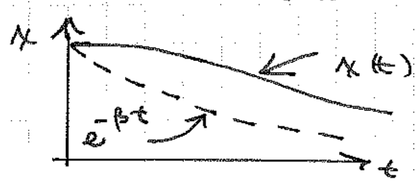
$\alpha = -\beta \pm \sqrt{\beta^2 - \omega_0^2} < 0$ FOR EITHER SIGN

$$x = e^{-\beta t}\left(A_1 e^{\sqrt{\ }\,t} + A_2 e^{-\sqrt{\ }\,t}\right) = e^{-\beta t}\left(C \cosh\sqrt{\ }\,t + D \sinh\sqrt{\ }\,t\right)$$

WITH A_1, A_2, C, D REAL

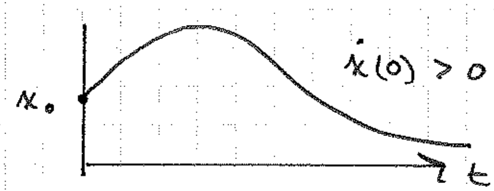
EXAMPLE: $x(0) = x_0$, $\dot{x}(0) = 0$ $\Rightarrow$ $C = x_0$, $D = \dfrac{\beta x_0}{\sqrt{\ }}$

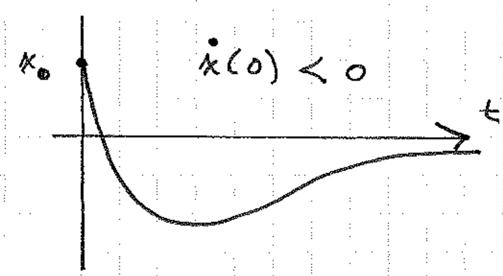
$$x = x_0 e^{-\beta t}\left(\cosh\sqrt{\ }\,t + \frac{\beta}{\sqrt{\ }}\sinh\sqrt{\ }\,t\right)$$



AS $t \to \infty$, $x \to x_0 e^{-(\beta - \sqrt{\ })t}\left[\dfrac{1 + \frac{\beta}{\sqrt{\ }}}{2}\right]$

IF $\dot{x}(0) \neq 0$, THE MOTION CAN HAVE ONE 'OSCILLATION'





c) CRITICAL DAMPING $\Longleftrightarrow$ $\beta = \omega_0$

$\alpha = -\beta$ $\Rightarrow$ ONLY ONE ROOT

BUT WE EXPECT 2 SOLUTIONS $\Rightarrow$ 1 IS MISSING ??

CONSIDER THE LIMIT $\beta \to \omega_0$ WHILE $\beta > \omega_0$, WITH $x(0) = A$ & $\dot{x}(0) = 0$:

$$x = e^{-\beta t}\left(A \cosh\sqrt{\beta^2 - \omega_0^2}\,t + \frac{A\beta}{\sqrt{\beta^2-\omega_0^2}}\sinh\sqrt{\beta^2 - \omega_0^2}\,t\right)$$

$$\sim e^{-\beta t}\left(A + \beta A t\right)$$

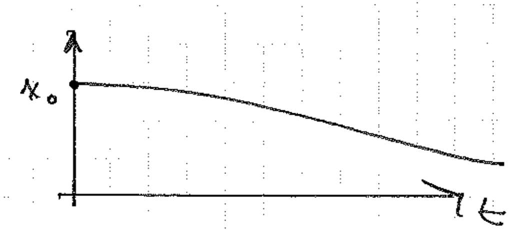
THIS SUGGESTS OUR DESIRED SOLUTION IS

$$x = e^{-\beta t}\left(A_1 + A_2 t\right)$$

WHICH CAN BE VERIFIED BY DIRECT SUBSTITUTION.

EXAMPLE: $x(0) = x_0$, $\dot{x}(0) = 0$

$\Rightarrow x = x_0 e^{-\beta t}(1 + \beta t)$



SIMILAR TO STRONG DAMPING

2) Sinusoidally Driven Oscillations

An important oscillation problem is the case of an external force applied to the system. The simplest case is a sinusoidal force, $F(t) = F_0 \cos \omega t$. [ω not necessarily $= \omega_0$]

In our example of an automobile shock absorber, the external force might be due to driving over a rutted, or 'washboard' road. We know that for certain speeds ($\Rightarrow$ certain frequencies) large oscillations occur.

Our differential equation is

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F_0}{M}\cos\omega t = \mathrm{Re}\left[\frac{F_0}{M} e^{i\omega t}\right]$$

We solve this by first noting an important simplification. If the force is suddenly turned on at some moment, the oscillator begins to respond, but the damping causes the 'turn-on transient' effect to die out — leaving only a steady motion. We shall solve only for the steady motion, and leave the question of the transient response to section 4 below.

If the motion is to be steady, it must follow the driving force. If the driving force is periodic, so must be the steady motion. We make the enlightened guess that

$x = A e^{i\omega t}$ (A complex) is the steady-state solution.

The frequency is the same as that of the driving force! Plugging in:

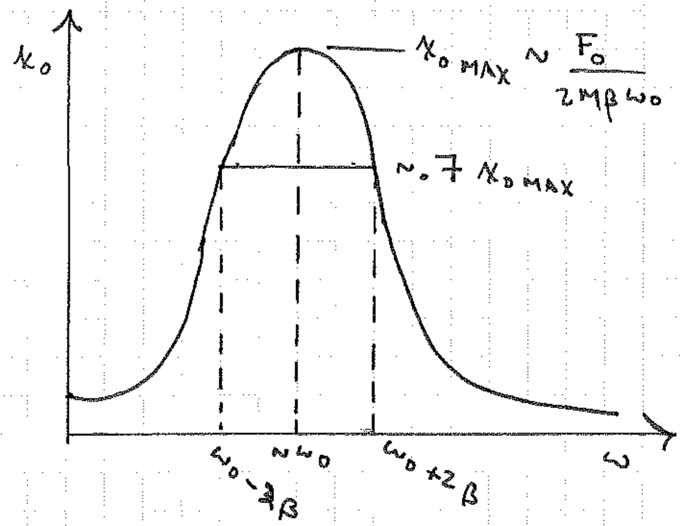
$$A e^{i\omega t}\left(-\omega^2 + 2i\beta\omega + \omega_0^2\right) = \frac{F_0}{M} e^{i\omega t}$$

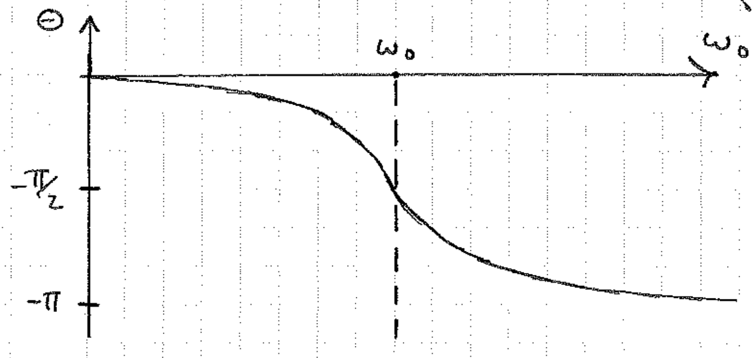
Clearly we have a solution if $A = \dfrac{F_0}{M}\dfrac{1}{\omega_0^2 - \omega^2 + 2i\beta\omega}$

It is useful to write $A = x_0 e^{i\Theta} \Rightarrow x = \mathrm{Re}\left(x_0 e^{i(\omega t + \Theta)}\right) = x_0 \cos(\omega t + \Theta)$

$$A = \frac{F_0}{M}\frac{\omega_0^2 - \omega^2 - 2i\beta\omega}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \qquad \tan\Theta = \frac{-2\beta\omega}{\omega_0^2 - \omega^2}$$

and $x_0 = \sqrt{AA^*} = \dfrac{F_0}{M}\dfrac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$



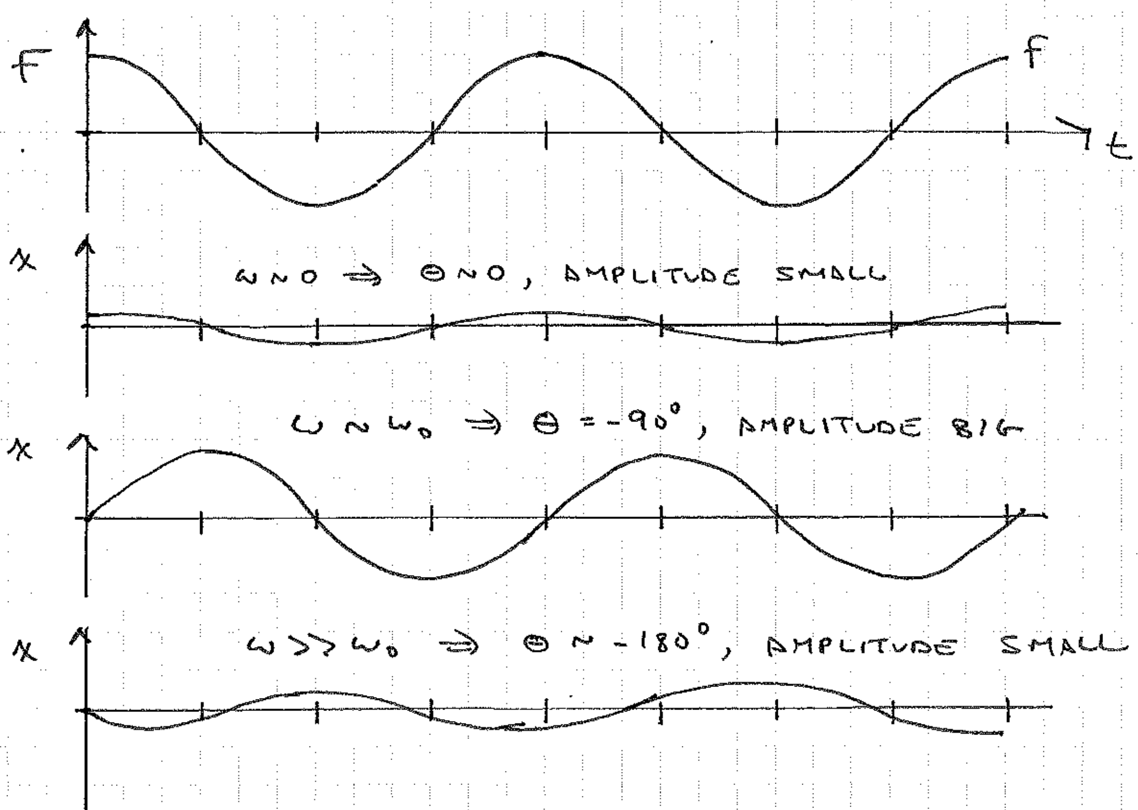


THE AMPLITUDE x_0 IS SMALL UNLESS $\omega \sim \omega_0$ AT WHICH WE SAY THAT RESONANCE OCCURS. IF THE DRIVING FREQUENCY IS THE NATURAL FREQUENCY ω_0, BIG OSCILLATIONS RESULT.

THE MAXIMUM x_0 OCCURS WHEN $\omega = \sqrt{\omega_0^2 - 2\beta^2} \sim \omega_0 - \frac{\beta^2}{\omega_0} \sim \omega_0$ IF β IS SMALL. THEN $x_{0\,MAX} \sim \frac{F_0}{2M\beta\omega_0}$.

IF $\beta \to 0$ WE GET THE UNPHYSICAL RESULT THAT $x_{0\,MAX} \to \infty$. HENCE SOME DAMPING MUST OCCUR IN ANY OSCILLATOR.

THE PHASE θ IS ALWAYS NEGATIVE $\Rightarrow$ MOTION LAGS BEHIND THE FORCE. AT RESONANCE, $\theta = -\pi/2 \Rightarrow 90°$ OUT OF PHASE.



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IF $\beta \neq 0$, MECHANICAL ENERGY IS NOT CONSERVED — THE DAMPING MECHANISM CONVERTS ENERGY INTO HEAT. A MEASURE OF THIS IS THE "Q" OF THE RESONANCE.

$$Q \equiv \left. \frac{\text{AVE. ENERGY OF THE OSCILLATOR}}{\text{ENERGY LOSS PER CYCLE}} \right|_{\text{AT RESONANCE}}$$

$$\langle E \rangle = \frac{1}{2} k x_0^2$$

AT RESONANCE $\begin{cases} F = F_0 \cos \omega_0 t \\ x = x_0 \sin \omega_0 t = \dfrac{F_0 \sin \omega_0 t}{2m\beta\omega_0} \\ v = x_0 \omega_0 \cos \omega_0 t \end{cases}$

$$\left. \frac{dE}{dt} \right|_{\text{LOST}} = F \cdot v$$

$$= x_0 \omega_0 F_0 \cos^2 \omega_0 t$$

$\Rightarrow$ LOSS PER CYCLE $= \left(\frac{1}{2} x_0 \omega_0 F_0\right) T = \pi x_0 F_0$

AND $Q = \dfrac{k x_0}{2\pi F_0} = \dfrac{k}{2\pi F_0} \cdot \dfrac{F_0}{2m\beta\omega_0} = \underline{\underline{\dfrac{1}{2\pi} \dfrac{\omega_0}{2\beta}}}$

CLEARLY SMALL DAMPING $\Rightarrow$ SMALL $\beta \Rightarrow$ LARGE Q

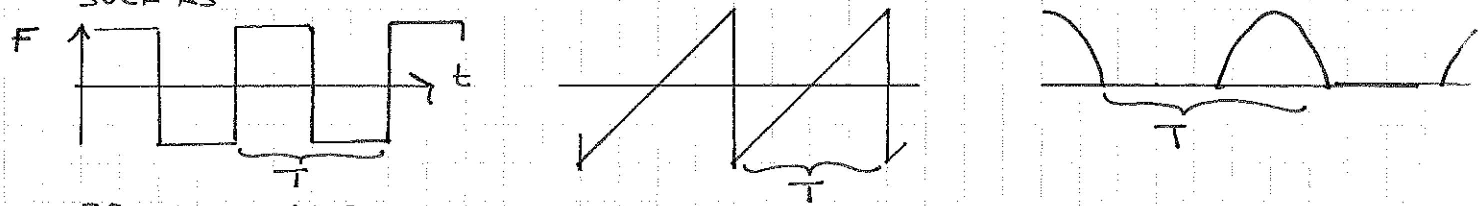
RECALL THAT $2\beta \sim$ HALF WIDTH OF RESONANCE, FROM THE PICTURE ON p 140. HENCE $Q \sim \dfrac{1}{2\pi} \dfrac{\omega_0}{\Delta\omega}$

IF THE DRIVING FORCE WAS $F(t) = F_0 \sin \omega t$
YOU COULD USE $\sin \omega t = \mathrm{Re}\left(-i e^{i\omega t}\right)$
TO VERIFY THAT $x = x_0 \sin(\omega t + \theta)$
WITH x_0 AND θ EXACTLY AS GIVEN ON p. 139.

THE NATURAL PERIOD OF OSCILLATION OF THE EARTH FOR MOTIONS LIKE  IS ABOUT 2 HOURS. IF THE EARTH AND THE MOON WERE COMBINED INTO A SINGLE BODY, CONSERVING ANGULAR MOMENTUM, THE PERIOD OF ROTATION WOULD BE ABOUT 4 HOURS. HENCE THE SOLAR TIDE EFFECT MIGHT CAUSE RESONANCE OSCILLATIONS, CAUSING THE MOON TO BE EJECTED FROM THE PRE-EARTH. (G. DARWIN, 1880'S). THIS IDEA APPARENTLY DOES NOT HOLD UP WHEN EXAMINED MORE CLOSELY.

3) Periodic Driving Force

We now consider driving forces periodic with period T, such as



SQUARE WAVE SAW TOOTH HALF-WAVE

We again seek the steady-state response, which we expect will also be periodic with period T.

The idea of Fourier is to decompose the driving force into a sum of sines and cosines.

We know the solution to a single sine or cosine force from sec. 2). Since the differential equation is linear, we can add solutions to get another solution. Thus the solution to the general problem of a periodic force can be composed out of the sine and cosine solutions.

Fourier also notes that all sines and cosines which are periodic with period T have frequencies which are integral multiples of $\omega = \frac{2\pi}{T}$ = principal frequency

That is, we look for a decomposition

$$F(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos n\omega t + \sum_{n=1}^{\infty} B_n \sin n\omega t \qquad [A_n, B_n \text{ REAL}]$$

If this can be done, then from sec. 2) the motion is just

$$x(t) = \frac{A_0}{2M\omega_0^2} + \sum_{n=1}^{\infty} \frac{A_n \cos(n\omega t + \theta_n) + B_n \sin(n\omega t + \theta_n)}{m\sqrt{(\omega_0^2 - n^2\omega^2)^2 + 4\beta^2 n^2 \omega^2}}$$

where $\tan \theta_n = \dfrac{-2n\beta\omega}{\omega_0^2 - n^2\omega^2}$

Due to the presence of the harmonics $2\omega, 3\omega \cdots$ it is now possible to have resonant motion whenever

$$\omega = \frac{\omega_0}{n} \qquad n = 1, 2, 3 \ldots$$

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HOW ARE THE FOURIER COEFFICIENTS A_n AND B_n DETERMINED?

WE NEED SOME TRIGONOMETRIC RELATIONS:

$$\int_{-T/2}^{T/2} \cos n\omega t \cos m\omega t = \frac{1}{2}\int \cos(m+n)\omega t + \frac{1}{2}\int \cos(n-m)\omega t = \begin{cases} 0 & \text{IF } n \neq m \\ T/2 & \text{IF } n=m>0 \\ T & \text{IF } n=m=0 \end{cases}$$

$$\int_{-T/2}^{T/2} \cos n\omega t \sin m\omega t = \frac{1}{2}\int \sin(n+m)\omega t + \frac{1}{2}\int \sin(n-m)\omega t = 0 \quad \text{ALWAYS}$$

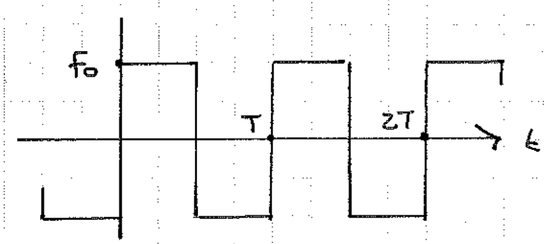
$$\int_{-T/2}^{T/2} \sin n\omega t \sin m\omega t = \frac{1}{2}\int \cos(n-m)\omega t - \frac{1}{2}\int \cos(n+m)\omega t = \begin{cases} 0 & \text{IF } n \neq m \\ T/2 & \text{IF } n=m>0 \\ 0 & \text{IF } n=m=0 \end{cases}$$

HENCE $\displaystyle\int_{-T/2}^{T/2} F(t) \cos n\omega t = \frac{T}{2} A_n \qquad \int_{-T/2}^{T/2} F(t) \sin n\omega t = \frac{T}{2} B_n$

TO SUMMARIZE:

$$\boxed{\begin{aligned} F(t) &= \frac{A_0}{2} + \sum_n{}' A_n \cos n\omega t + \sum_n{}' B_n \sin n\omega t \\ A_n &= \frac{2}{T}\int_{-T/2}^{T/2} F(t) \cos n\omega t \qquad B_n = \frac{2}{T}\int_{-T/2}^{T/2} F(t) \sin n\omega t \end{aligned}}$$

WHICH LEADS TO THE SERIES SOLUTION GIVEN ON p. 142



EXAMPLE: SQUARE WAVE

SINCE $F(-t) = -F(t)$

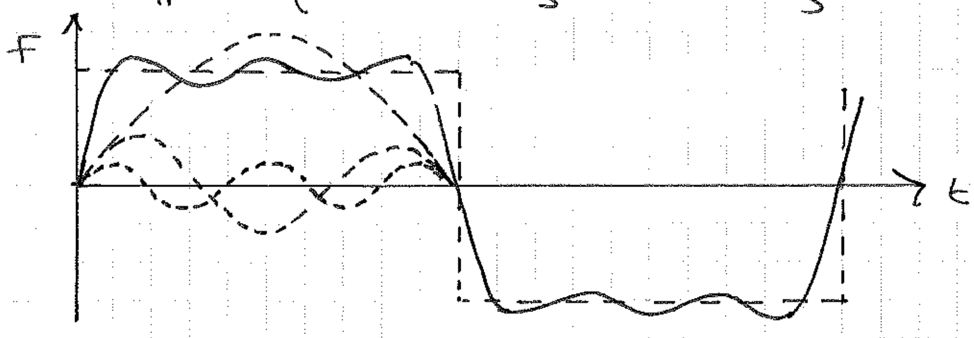
$$\int_{-T/2}^{T/2} F(t) \cos n\omega t = 0 \Rightarrow \text{ALL } A_n = 0$$

AND $$B_n = \frac{4}{T}\int_0^{T/2} F_0 \sin n\omega t = \frac{4F_0}{n\omega T}(-\cos n\omega t)\Big|_0^{T/2} = \frac{2F_0}{n\pi}(1-\cos n\pi)$$

$$\left(\text{NOTING } \omega = \frac{2\pi}{T}\right)$$

OR $$B_n = \begin{cases} \dfrac{4F_0}{n\pi} & n \text{ ODD} \\ 0 & n \text{ EVEN} \end{cases}$$

$$F(t) = \frac{4}{\pi} F_0 \left(\sin \omega t + \frac{1}{3} \sin 3\omega t + \frac{1}{5} \sin 5\omega t + \frac{1}{7} \sin 7\omega t \ldots \right)$$



AFTER THE FIRST 3 TERMS THE SUM LOOKS SOMETHING LIKE

THERE IS ALWAYS SOME OVER-SHOOT IN THE FOURIER APPROXIMATION, WHICH IS WORST FOR THE 1ST RIPPLE. EVEN IN THE LIMIT $n \to \infty$, THERE IS AN 18% EXCESS JUST BEYOND THE STEP — THE SO-CALLED GIBB'S PHENOMENON

HOWEVER AT $t = T/4$, $F \to \frac{4F_0}{\pi} \{ \underbrace{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots}_{\pi/4} \} = F_0$

THE FOURIER SERIES FOR THE SQUARE WAVE CONVERGES RATHER SLOWLY — MANY HARMONICS ARE REQUIRED. HIGH FIDELITY REPRODUCTION OF A SQUARE WAVE ON YOUR STEREO IS QUITE HARD — BUT THEN A SQUARE WAVE IS NOT VERY 'MUSICAL'.

4) ARBITRARY DRIVING FORCE

A COMPLETE, FORMAL SOLUTION CAN BE GIVEN TO THE DRIVEN OSCILLATOR PROBLEM FOR ANY FORCE.

L & L DISCUSS THIS SOLUTION IN SEC. 22 USING A DEVIOUS TRICK. YOU MAY WISH TO GENERALIZE THEIR ARGUMENT TO THE CASE OF A DAMPED OSCILLATOR, TO DERIVE THE SOLUTION GIVEN BELOW.

INSTEAD, WE INTRODUCE A POWERFUL METHOD, DUE TO GREEN, IN WHICH AN ARBITRARY FORCE IS CONSIDERED TO BE THE RESULT OF A SERIES OF IMPULSES.

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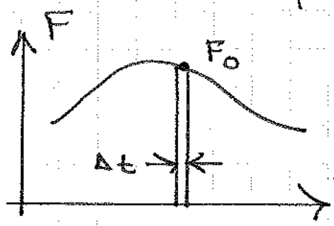


WE MUST FIRST FIND THE RESPONSE OF THE OSCILLATOR TO AN ARBITRARY IMPULSE $I = F_0 \Delta t = F(t_0)\Delta t$, AND THEN ADD TOGETHER ALL SUCH RESPONSES TO GET THE MOTION UNDER THE CONTINUOUS FORCE $F(t)$. AGAIN NOTE THAT OUR TECHNIQUE DEPENDS ON THE LINEARITY OF THE DIFFERENTIAL EQUATION.

FOR AN OSCILLATOR, WE KNOW THE MOTION AFTER THE IMPULSE IS OVER HAS THE FORM (p 137)

$$x(t) = A e^{-\beta t} \cos(\omega_1 t + \Theta) \qquad t > t_0 \qquad \omega_1 = \sqrt{\omega_0^2 - \beta^2}$$

WE SUPPOSE THE OSCILLATOR WAS AT REST BEFORE THE IMPULSE, AND SO DETERMINE THE CONSTANTS A AND Θ BY THE RELATIONS

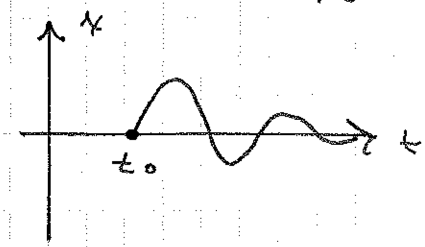
$$x(t_0) = 0 \quad \text{AND} \quad m\dot{x}(t_0) = \Delta p = \text{IMPULSE} = F_0 \Delta t$$

$$x(t_0) = 0 \Rightarrow \omega_1 t_0 + \Theta = \pi/2 \quad \text{OR} \quad \Theta = \pi/2 - \omega_1 t_0$$

$$\dot{x}(t_0) = A e^{-\beta t_0}\left\{-\beta\cos(\omega_1 t_0 + \Theta) - \omega_1 \sin(\omega_1 t_0 + \Theta)\right\} = -\omega_1 A e^{-\beta t_0} = \frac{F_0 \Delta t}{m}$$

$$\text{SO} \quad x(t) = -\frac{F_0 \Delta t}{m\omega_1} e^{-\beta(t-t_0)} \cos\left[\pi/2 - \omega_1(t-t_0)\right]$$

$$= \frac{F_0 \Delta t}{m\omega_1} e^{-\beta(t-t_0)} \sin\left[\omega_1(t-t_0)\right]$$



OF COURSE $x(t) = 0$ FOR $t < t_0$.

TO GET THE GENERAL SOLUTION FOR $x(t)$ WE ADD UP THE EFFECT OF ALL IMPULSES WHICH OCCURED BEFORE TIME t:

$$\boxed{x(t) = \int_{-\infty}^{t} dt' \, \frac{F(t')}{m\omega_1} e^{-\beta(t-t')} \sin \omega_1 (t-t')}$$

THIS IS THE SO-CALLED GREEN'S METHOD. THE FACTOR MULTIPLYING $F(t')$ IN THE INTEGRAND IS THE SO-CALLED <u>GREEN'S FUNCTION</u> OF THE DRIVEN-OSCILLATOR PROBLEM

IN PRACTICE THE INTEGRAL CAN BE DONE ANALYTICALLY ONLY FOR A FEW CASES.