Ph 205 LECTURE 2

MECHANICS OF A SYSTEM OF PARTICLES

WE NOW SHOW THAT TREATING AN EXTENDED OBJECT AS A COLLECTION OF POINT PARTICLES LEADS TO A REMARKABLE SELF-CONSISTENCY IN THE FORM OF THE LAWS OF MECHANICS FOR LARGE AND SMALL SYSTEMS.

A VERY IMPORTANT RESULT OF A SIMILAR NATURE IS THAT THE GRAVITATIONAL ATTRACTION OF A SPHERICAL SHELL OF MASS IS THE SAME AS IF ALL THE MASS WERE CONCENTRATED AT A POINT AT THE CENTER OF THE SHELL.

WE CONSIDER AN OBJECT CONTAINING n POINT PARTICLES EACH WITH MASS m_i AND POSITION $\bar{r}_i$, $i = 1, 2 \ldots n$.

AS MUCH AS POSSIBLE WE WANT TO FORMULATE A MECHANICS OF THE SYSTEM AS A WHOLE, IGNORING THE DETAILS OF THE INDIVIDUAL PARTICLE MOTIONS.

CLEARLY WE WILL DEAL WITH THE TOTAL MASS $M = \sum_i m_i$

TO DESCRIBE THE POSITION OF THE WHOLE SYSTEM BY A SINGLE VECTOR, WE USE THE CENTER OF MASS POSITION $\bar{R} = \frac{1}{M} \sum_i m_i \bar{r}_i$

WE ABBREVIATE THE CENTER OF MASS AS C.M.

1. MOMENTUM. THE VELOCITY OF THE C.M. IS

$$\bar{V} = \frac{d\bar{R}}{dt} = \frac{1}{M} \sum_i m_i \bar{v}_i$$

HENCE $\bar{P} = M\bar{V} = \sum_i m_i \bar{v}_i = \sum_i \bar{p}_i$ = TOTAL MOMENTUM

THIS SHOWS THE DESIRED TREND IN WHICH VARIABLES DESCRIBING THE WHOLE SYSTEM OBEY RELATIONS IDENTICAL TO THOSE OF A SINGLE PARTICLE.

WHAT ABOUT CHANGES IN MOMENTUM: $d\bar{P}/dt$?

EACH PARTICLE OBEYS $d\bar{p}_i/dt = \bar{F}_i$

WE BREAK UP $\bar{F}_i = \bar{F}_i^e + \sum_j \bar{f}_{ij}$

$\bar{F}_i^e$ IS THE FORCE ON PARTICLE i DUE TO CAUSES EXTERNAL TO THE SYSTEM.

$\bar{f}_{ij}$ = FORCE ON PARTICLE i DUE TO PARTICLE j. WE ASSUME $\bar{f}_{ii} = 0$ FOR A POINT PARTICLE.

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THEN $\frac{d\bar{P}}{dt} = \sum_i \frac{d\bar{p}_i}{dt} = \sum_i \bar{F}_i^e + \sum_{ij} \bar{f}_{ij}$

WE INVOKE NEWTON'S 3RD LAW TO SET $\bar{f}_{ij} + \bar{f}_{ji} = 0 \Rightarrow \sum_{ij} \bar{f}_{ij} = 0$

FOR THIS IT IS <u>NOT</u> NECESSARY THAT $\bar{f}_{ij}$ BE DIRECTED ALONG THE LINE JOINING i AND j.

WE DEFINE THE <u>TOTAL</u> <u>EXTERNAL</u> <u>FORCE</u> $\bar{F}^e = \sum_i \bar{F}_i^e$

THEN $\boxed{d\bar{P}/dt = \bar{F}^e}$

SINCE $\bar{P} = M\bar{V}$, $d\bar{P}/dt = M\,d\bar{V}/dt = M\bar{A}$ IF ALL MASSES ARE CONSTANT. $\bar{A}$ = ACCELERATION OF THE C.M.

HENCE $\boxed{\bar{F}^e = M\bar{A}}$

THIS IS NEWTON'S 2ND LAW FOR A SYSTEM OF PARTICLES. NOTE THAT ONLY EXTERNAL FORCES DETERMINE THE MOTION OF THE C.M.

IF $\bar{F}^e = 0$, THEN $\bar{P}$ = CONSTANT $\Longleftrightarrow$ MOMENTUM CONSERVATION OF A SYSTEM. OF COURSE, $\bar{f}_{ij}$ NEED **NOT** BE ZERO.

WE USED NEWTON'S 3RD LAW TO OBTAIN THE RESULT

$$\sum_{ij} \bar{f}_{ij} = 0 \qquad \text{FOR INTERNAL FORCES.}$$

WE CAN ARRIVE AT THIS CONCLUSION BY A DIFFERENT ARGUMENT. SUPPOSE ALL PARTICLES IN THE SYSTEM ARE DISPLACED BY THE SAME AMOUNT $\delta\bar{r}$. THE SEPARATIONS BETWEEN ALL PAIRS OF PARTICLES REMAIN FIXED. OUR KEY ASSUMPTION IS THAT IN THIS TRANSLATION OF THE SYSTEM THE INTERNAL FORCES DO NO NET WORK. THAT IS, WE ASSUME THE TOTAL <u>INTERNAL</u> ENERGY OF THE SYSTEM IS INDEPENDENT OF WHERE THE SYSTEM IS LOCATED, SO LONG AS THE CONFIGURATION IS THE SAME.

DURING THE DISPLACEMENT THE WORK DONE ON PARTICLE i BY INTERNAL FORCES IS $\delta W_i = \sum_j \bar{f}_{ij} \cdot \delta\bar{r}$

THE TOTAL WORK DONE IS $\delta W = \sum_i \sum_j \bar{f}_{ij} \cdot \delta\bar{r} = 0$ BY OUR ASSUMPTION. BUT δr IS ARBITRARY, SO $\sum_{ij} \bar{f}_{ij} = 0$.

THIS DOES NOT NECESSARILY REQUIRE $\bar{f}_{ij} + \bar{f}_{ji} = 0$ PAIR BY PAIR.

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2. ANGULAR MOMENTUM

THE TOTAL ANGULAR MOMENTUM OF THE SYSTEM IS

$$\bar{L} = \sum_i \bar{r}_i \times \bar{p}_i$$

IN ANALOGY TO $\bar{P} = M\bar{V}$, CAN WE WRITE $\bar{L} = \bar{R} \times \bar{P}$ WHERE $\bar{R}$ = POSITION OF THE C.M.?

NO! A WHEEL SPINNING ABOUT A FIXED AXIS HAS LOTS OF ANGULAR MOMENTUM, BUT $\bar{P} = 0$.

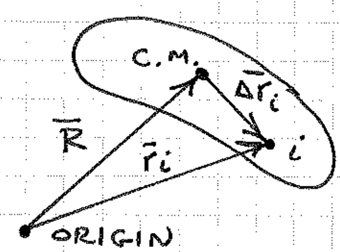
WE SHALL TRY TO SEPARATE THE MOTION INTO 2 PARTS:

1. THE MOTION OF THE C.M.
2. THE MOTION OF PARTICLES RELATIVE TO THE C.M.

FOR EACH PARTICLE WRITE $\bar{r}_i = \bar{R} + \Delta\bar{r}_i$

SO $\bar{v}_i = \bar{V} + \Delta\bar{v}_i$

USING $\Delta\bar{v}_i = \dfrac{d\,\Delta\bar{r}_i}{dt}$



$$\bar{L} = \sum_i (\bar{R} + \Delta\bar{r}_i) \times m_i(\bar{V} + \Delta\bar{v}_i)$$

$$= \sum_i m_i(\bar{R} \times \bar{V}) + \sum_i \Delta\bar{r}_i \times m_i \Delta\bar{v}_i + \sum_i \bar{R} \times m_i \Delta\bar{v}_i + \sum_i m_i \Delta\bar{r}_i \times \bar{V}$$

BUT $\sum_i m_i \Delta\bar{r}_i = 0$ SINCE THIS IS JUST $M\bar{R}_{cm}$ IN A COORDINATE SYSTEM WITH THE C.M. AS THE ORIGIN! LIKEWISE

$$\sum_i m_i \Delta\bar{v}_i = 0$$

SO $$\boxed{\bar{L} = \bar{R} \times \bar{P} + \sum_i \Delta\bar{r}_i \times m_i \Delta\bar{v}_i}$$

THAT IS, $\bar{L}$ = (ANGULAR MOMENTUM OF THE C.M. MOTION)

+ (ANGULAR MOMENTUM OF MOTION ABOUT THE C.M.)

NOTE THAT THE FIRST TERM DEPENDS ON OUR CHOICE OF ORIGIN, WHILE THE SECOND DOES NOT.

IF THE C.M. IS FIXED, THE ANGULAR MOMENTUM IS THE SAME ABOUT ANY ORIGIN (AT REST).

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YOU CAN EASILY VERIFY THAT THE ANGULAR MOMENTUM ABOUT THE C.M. CAN BE WRITTEN SEVERAL WAYS:

$$\bar{L}_{\text{ABOUT C.M.}} = \sum_i \Delta\bar{r}_i \times m_i \Delta\bar{v}_i = \sum_i \bar{r}_i \times m_i \Delta\bar{v}_i = \sum_i \Delta\bar{r}_i \times m_i \bar{v}_i$$

WE NOW CONSIDER CHANGES IN $\bar{L}$.

$$\frac{d\bar{L}}{dt} = \frac{d}{dt}\sum_i \bar{r}_i \times \bar{p}_i = \sum_i \bar{v}_i \times \bar{p}_i + \sum_i \bar{r}_i \times \frac{d\bar{p}_i}{dt} = \sum_i \bar{r}_i \times \bar{F}_i$$

AGAIN WE WRITE $\bar{F}_i = \bar{F}_i^e + \sum_j \bar{f}_{ij}$

$$\frac{d\bar{L}}{dt} = \sum_i \bar{r}_i \times \bar{F}_i^e + \sum_{ij} \bar{r}_i \times \bar{f}_{ij}$$

NEWTON'S 3RD LAW $\Rightarrow$ $\bar{f}_{ij} = -\bar{f}_{ji}$

SO $\sum_{ij} \bar{r}_i \times \bar{f}_{ij} = \frac{1}{2}\sum_{ij}\left[(\bar{r}_i \times \bar{f}_{ij}) + (\bar{r}_j \times \bar{f}_{ji})\right] = \frac{1}{2}\sum_{ij}(\bar{r}_i - \bar{r}_j) \times \bar{f}_{ij}$

IF IN ADDITION TO $\bar{f}_{ij} = -\bar{f}_{ji}$ WE ALSO HAVE THAT $\bar{f}_{ij}$ ACTS ALONG THE LINE JOINING i TO j, THEN $\bar{f}_{ij} \sim \bar{r}_i - \bar{r}_j$

AND $\sum_{i} (\bar{r}_i - \bar{r}_j) \times \bar{f}_{ij} = 0$

THEN $\underline{\dfrac{d\bar{L}}{dt} = \sum_i \bar{r}_i \times \bar{F}_i^e = \bar{N}^e}$ = <u>TOTAL EXTERNAL TORQUE</u>

NOTE HOWEVER THAT $\bar{N}^e \neq R \times \bar{F}^e$

IF THE TOTAL <u>EXTERNAL</u> TORQUE VANISHES, ANGULAR MOMENTUM IS CONSERVED.

INSTEAD OF INVOKING THE STRONG FORM OF NEWTON'S 3RD LAW, YOU CAN CONVINCE YOURSELF THAT $\sum_{ij} \bar{r}_i \times \bar{f}_{ij} = 0$

SO LONG AS THE INTERNAL FORCES DO NO NET WORK IN ANY ROTATION ABOUT AN AXIS IN WHICH THE RELATIVE POSITIONS OF ALL PARTICLES REMAIN UNCHANGED.

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3. Work and Energy

The work done on the whole system in going from configuration 1 to configuration 2 is

$$W_{12} = \sum_i \int_1^2 \bar{F}_i \cdot d\bar{s}_i$$

As before, each $\bar{F}_i \cdot d\bar{s}_i = m_i \frac{d\bar{v}_i}{dt} \cdot \frac{d\bar{s}_i}{dt}\, dt = d\left(\frac{1}{2} m_i v_i^2\right)$

$$\text{so} \quad W_{12} = T_2 - T_1$$

where the total kinetic energy $= T = \sum_i \frac{1}{2} m_i v_i^2$

Can we write $T = \frac{1}{2} M V^2$ using C.M. quantities only?

Clearly not! But as in the case of angular momentum, we can write $\bar{v}_i = \bar{V} + \Delta\bar{v}_i$

$$\text{so} \quad T = \sum_i \frac{1}{2} m_i \left(V^2 + 2\bar{V} \cdot \Delta\bar{v}_i + \Delta v_i^2\right)$$

$$\text{or} \quad T = \frac{1}{2} M V^2 + \sum_i \frac{1}{2} m_i \Delta v_i^2 \qquad \text{noting } \sum_i m_i \Delta\bar{v}_i = 0$$

T = (kinetic energy of C.M. motion)

+ (kinetic energy of motion relative to the C.M.)

We now consider the potential energy.

Again we write $\bar{F}_i = \bar{F}_i^e + \sum_j \bar{f}_{ij}$

If the external forces are conservative, then

$\bar{F}_i^e = -\bar{\nabla} V_i$ where V_i is the potential energy of the i th particle in the external force field.

$$\text{Thus} \quad W_{12} = \sum_i \int_1^2 \bar{F}_i \cdot d\bar{s}_i = \sum_i \int_1^2 \bar{F}_i^e \cdot d\bar{s}_i + \sum_{ij} \int_1^2 \bar{f}_{ij} \cdot d\bar{s}_i$$

$$= \sum_i \left[V_i(1) - V_i(2)\right] + \sum_{ij} \int_1^2 \bar{f}_{ij} \cdot d\bar{s}_i$$

What about the $\bar{f}_{ij}$? While we can presumably determine whether $\bar{F}_i^e$ is conservative, this may not be so easy for the $\bar{f}_{ij}$.

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BUT SUPPOSE THE $\bar{f}_{ij}$ ARE INDEED CONSERVATIVE FORCES.

THEN WE EXPECT THE POTENTIAL V_{ij} CORRESPONDING TO $\bar{f}_{ij}$ TO BE IDENTICAL TO POTENTIAL V_{ji} CORRESPONDING TO $\bar{f}_{ji}$.

THAT IS, IF THE V_{ij} REPRESENT A KIND OF STORED ENERGY WHICH DEPENDS ON THE POSITIONS OF PARTICLES i AND j, THEN V_{ij} AND V_{ji} BOTH REPRESENT THE SAME ENERGY.

$$V_{ij} = V_{ij}(\bar{r}_i, r_j) = V_{ij}(\bar{r}_j, \bar{r}_i) = V_{ji}$$

IF SPACE IS HOMOGENEOUS AND ISOTROPIC THEN V_{ij} CAN DEPEND ONLY ON THE DISTANCE BETWEEN i AND j:

$$V_{ij} = V_{ij}(r_{ij}) \quad , \quad r_{ij} = |\bar{r}_i - \bar{r}_j| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}$$

THIS IS ENOUGH TO INSURE THAT THE STRONG FORM OF NEWTON'S 3RD LAW HOLDS:

$$f_{ij}\big|_x = -\frac{\partial}{\partial x_i} V_{ij}(r_{ij}) = -\frac{dV_{ij}}{dr_{ij}} \frac{x_i - x_j}{r_{ij}} = +\frac{\partial}{\partial x_j} V_{ij} = -f_{ji}\big|_x$$

SO $\bar{f}_{ij} = -\bar{f}_{ji}$ AND $\bar{f}_{ij} \sim \bar{r}_i - \bar{r}_j$

ALSO $\sum_{ij} \int_1^2 \bar{f}_{ij} \cdot d\bar{s}_i = \sum_{ij} \left[V_{ij}(1) - V_{ij}(2) \right] = \frac{1}{2} \sum_{\text{PAIRS}} \left[V_{ij}(1) - V_{ij}(2) \right]$

SINCE EACH PAIR (i, j) OCCURS TWICE IN THE SUM $\sum_{ij}$

COMPARING TO p18,

$$W_{12} = \sum_i \left[V_i(1) - V_i(2) \right] + \frac{1}{2} \sum_{\text{PAIRS}} \left[V_{ij}(1) - V_{ij}(2) \right]$$

WE DEFINE THE TOTAL POTENTIAL ENERGY AS

$$V = \sum_i V_i + \frac{1}{2} \sum_{\text{PAIRS}} V_{ij}$$

THEN $E = T + V$ IS A CONSTANT.

Ph 205 LECTURE 2

4. SEPARATION INTO MOTION OF THE C.M. AND MOTION RELATIVE TO THE C.M.

IN DISCUSSING THE ANGULAR MOMENTUM AND THE KINETIC ENERGY OF A SYSTEM OF PARTICLES WE FOUND IT CONVENIENT TO ASSIGN A PIECE OF EACH OF THE PARTICLE MOTIONS TO MOTION OF THE SYSTEM AS A WHOLE (C.M. MOTION) AND ANOTHER PIECE TO MOTION ABOUT THE C.M.

THE EQUATIONS GOVERNING CHANGES OF THE MOTION CAN ALSO BE SEPARATED IN A SIMILAR MANNER.

WE HAVE ALREADY SEEN THAT IF $\bar{R}$ = C.M. POSITION, THEN

$$M \frac{d^2\bar{R}}{dt^2} = \bar{F}^e = \text{TOTAL EXTERNAL FORCE}$$

IN DETERMINING THE C.M. MOTION WE CAN COMPLETELY IGNORE THE MOTION OF PARTICLES RELATIVE TO THE C.M., AND SIMPLY PRETEND THAT ALL THE EXTERNAL FORCES ON THE SYSTEM ACT DIRECTLY ON THE C.M.

LIKEWISE, THE MOTION OF PARTICLES ABOUT THE C.M. CAN BE DETERMINED BY PRETENDING THE C.M. IS FIXED (BY A FICTITIOUS FORCE $-\bar{F}^e$) BUT ALL FORCES ON THE PARTICLES ACT AS BEFORE.

WE SHALL NOW DEMONSTRATE THIS.

AS EARLIER WE WRITE $\bar{r}_i = \bar{R} + \Delta\bar{r}_i$

SO THE ANGULAR MOMENTUM BECOMES

$$\bar{L} = \bar{R} \times \bar{P} + \sum_i \Delta\bar{r}_i \times M_i \Delta\bar{v}_i$$

AND $$\frac{d\bar{L}}{dt} = \bar{R} \times \frac{d\bar{P}}{dt} + \sum_i \Delta\bar{r}_i \times M_i \Delta\bar{a}_i = \bar{N}^e$$

HENCE $$\sum_i \Delta\bar{r}_i \times M_i \Delta\bar{a}_i = \bar{N}^e - \bar{R} \times \bar{F}^e$$

THUS THE <u>RELATIVE</u> MOTION IS DETERMINED BY THE TOTAL TORQUE PLUS A TORQUE EXACTLY EQUAL TO THAT OF A FORCE $-\bar{F}^e$ APPLIED TO THE C.M. SO AS TO FIX THE C.M.

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THE LEFT HAND SIDE OF THE ABOVE RESULT CONTAINS ONLY QUANTITIES MEASURED RELATIVE TO THE C.M.. WE CAN ARRANGE THE SAME THING ON THE RIGHT HAND SIDE BY TAKING OUR ORIGIN AT THE C.M. AND CALCULATING ALL TORQUES ABOUT THE C.M.

$$\text{THEN} \quad \sum_i \Delta\bar{r}_i \times M_i \Delta\bar{a}_i = \bar{N}^e_{\text{ABOUT THE C.M.}}$$

THE MOTION OF A SYSTEM RELATIVE TO THE C.M. CAN BE DETERMINED FROM THE TORQUE ABOUT THE C.M. EVEN WHEN THE C.M. IS ACCELERATING.

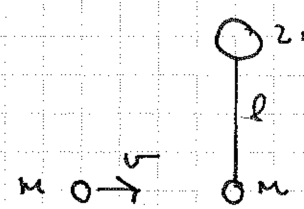
THIS USE OF AN 'ACCELERATED COORDINATE SYSTEM' IS ESPECIALLY SIMPLE. NOTE THAT THE DIRECTIONS OF THE X, y & Z AXES MUST BE KEPT FIXED WHEN USING THIS METHOD. A MORE GENERAL DISCUSSION OF ACCELERATED FRAMES WILL BE GIVEN LATER IN THE COURSE.

AN INTERESTING EXAMPLE OF THE SEPARATION OF THE MOTION IS THE EARTH-MOON-SUN SYSTEM. NEWTON SHOWED THAT THE GRAVITATIONAL ATTRACTION BETWEEN A POINT MASS AND A SPHERICAL SHELL IS EQUIVALENT TO THAT BETWEEN THE POINT AND A SECOND POINT AT THE CENTER OF THE SHELL.

IF THE EARTH ACTUALLY OBEYS THIS RESULT, THEN THE GRAVITY OF THE SUN AND THE MOON ACTS ONLY ON THE C.M. MOTION OF THE EARTH AND CANNOT AFFECT ITS ROTATION. HOWEVER THE AXIS OF ROTATION OF THE EARTH IS KNOWN TO PRECESS, AND THIS IS COMMONLY ASCRIBED TO THE EFFECT OF THE SUN AND THE MOON. WE SHALL CONSIDER LATER HOW THIS COMES ABOUT.

ANOTHER EXAMPLE IS TAKEN FROM THE Ph 103 LEARNING GUIDE. A MASS M COLLIDES WITH A DUMBBELL OF LENGTH ℓ AND MASSES M AND 2M.

IF THE TWO MASSES STICK TOGETHER, WHAT IS THE ANGULAR VELOCITY ABOUT THE C.M. AFTER COLLISION?



WE SOLVE THIS 3 WAYS.

1. MOMENTUM CONSERVATION IN THE COLLISION. (IS ENERGY CONSERVED?) JUST AFTER THE COLLISION THE LOWER MASS 2M HAS VELOCITY $v/2$, WHILE THE UPPER 2M IS STILL AT REST.

$$\therefore \; V_{cm} = v/4. \quad \text{AND} \quad \omega = \frac{v/2 - v/4}{\ell/2} = \frac{v}{2\ell}$$

2. ANGULAR MOMENTUM CONSERVATION ABOUT THE INITIAL POSITION OF THE UPPER 2M.

$$L_i = mvl = L_f = L_{OF\ CM} + L_{ABOUT\ CM} = 4mV_{cm}\frac{l}{2} + 2(2m)\left(\frac{l}{2}\right)^2\omega$$

$$= mvl/2 + ml^2\omega$$

SO $\omega = v/2l$

3. ANGULAR MOMENTUM CONSERVATION ABOUT THE C.M.

$$L_i = m(v - v_{cm})\,l/2 + 2m\,v_{cm}\,l/2 - m\,V_{cm}\,l/2 = mv\frac{l}{2} = L_f = ml^2\omega$$

AGAIN $\omega = v/2l$

Rigid Bodies

A RIGID BODY IS A SYSTEM OF PARTICLES IN WHICH THE SEPARATION BETWEEN ANY PAIR OF PARTICLES IS CONSTANT IN TIME.

IN RIGID BODIES THE INTERNAL FORCES DO NO WORK, SO WE DON'T NEED TO KNOW WHETHER THESE FORCES ARE CONSERVATIVE OR NOT. (SEE PROBLEM SET 2 FOR A PROOF OF THIS STATEMENT).

OUR RELATIONS $\frac{d\bar{P}}{dt} = \bar{F}^e$, $\frac{d\bar{L}}{dt} = \bar{N}^e$

ARE IMMEDIATELY APPLICABLE TO RIGID BODIES.

IN ENERGY CONSIDERATIONS THE QUESTION OF AN INTERNAL POTENTIAL ENERGY IS IRRELEVANT. WE JUST DEFINE IT TO BE ZERO AND IT WILL REMAIN SO. THUS WE NEVER NEED TO KNOW WHAT MAKES A RIGID BODY STICK TOGETHER WHEN STUDYING MECHANICS. BUT THEN WE WILL NEVER LEARN WHY IT DOES EITHER!

General Motion of a Rigid Body

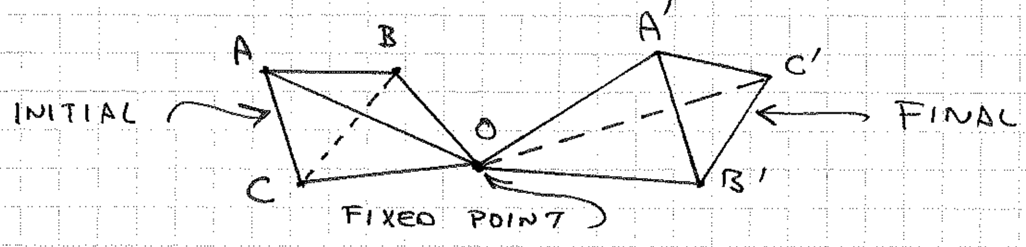
A RESULT OF SOLID GEOMETRY, CHASLES' THEOREM STATES THAT ANY MOTION OF A RIGID BODY CAN BE DUPLICATED BY A SIMPLE TRANSLATION FOLLOWED BY A ROTATION ABOUT AN AXIS. THE AXIS CAN BE CHOSEN TO PASS THRU ANY DESIRED POINT, SUCH AS THE C.M.

['CHASLES' IS PRONOUNCED MUCH LIKE 'SHAWL']

Ph 205 LECTURE 2

SKETCH OF A PROOF:

FIRST JUST TRANSLATE THE BODY UNTIL THE DESIRED POINT IS IN ITS FINAL POSITION. NOW WE MUST SHOW THAT ONLY A SINGLE ROTATION ABOUT SOME AXIS THRU THAT POINT SUFFICES TO BRING THE BODY TO ITS FINAL CONFIGURATION.



CONSIDER ANY LINE PASSING THRU O WHICH LIES IN THE PLANE BISECTING LINE AA'. A SUITABLE ROTATION ABOUT THIS LINE BRINGS A TO A'.

SIMILARLY A ROTATION ABOUT ANY LINE THRU O IN THE PLANE BISECTING BB' BRINGS B INTO B'.

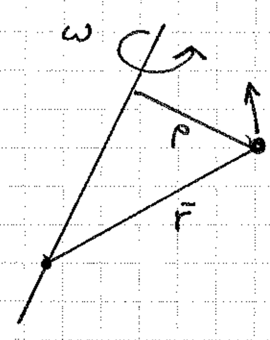
SO CONSIDER THE LINE OF INTERSECTION OF THE TWO BISECTING PLANES. THIS IS THE DESIRED AXIS, BELIEVE IT OR NOT

WE ARE MAINLY INTERESTED IN THE CASE THAT THE MOTION TAKES PLACE DURING SOME SHORT TIME dt. THEN WE CAN SAY THAT THE MOTION CONSISTS OF A TRANSLATION OF THE C.M., PLUS A ROTATION ABOUT AN "INSTANTANEOUS AXIS" THRU THE C.M.

IN GENERAL THE DIRECTION OF THE INSTANTANEOUS AXIS CHANGES WITH TIME. THIS MAKES LIFE DIFFICULT AND WILL BE CONSIDERED LATER IN THE COURSE.

Rigid Body Rotation about a Fixed Axis

EACH PARTICLE TRAVELS IN A CIRCLE ABOUT THIS AXIS, WITH VELOCITY PROPORTIONAL TO THE DISTANCE FROM THE AXIS



$v = \omega \rho$

ω = ANGULAR VELOCITY

WE CAN DESCRIBE THIS IN VECTOR LANGUAGE:

$$\bar{v} = \bar{\omega} \times \bar{r}$$

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WHERE $\bar{r}$ = POSITION VECTOR WITH RESPECT TO SOME ORIGIN ON THE AXIS,
AND $\bar{\omega}$ IS PARALLEL TO THE AXIS.

THE ANGULAR MOMENTUM ABOUT THE ORIGIN ON THE AXIS IS

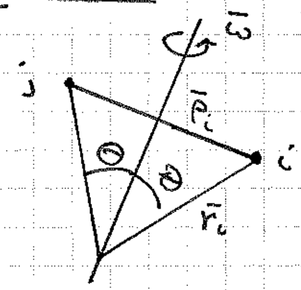
$$\bar{L} = \sum_i \bar{r}_i \times m_i \bar{v}_i = \sum_i m_i \bar{r}_i \times (\bar{\omega} \times \bar{r}_i) = \sum_i m_i \left[r_i^2 \bar{\omega} - (\bar{r}_i \cdot \bar{\omega}) \bar{r}_i \right]$$

THIS IS STILL MESSY IN GENERAL, SO CONSIDER

RIGID BODY ROTATION ABOUT A SYMMETRY AXIS

FOR EACH m_i AT $\bar{r}_i$ THERE IS AN EQUAL MASS $m_j = m_i$ AT $\bar{r}_j = \bar{r}_i - 2\bar{\rho}_i$

NOTE $\bar{r}_i = r_i \cos\theta\, \hat{\omega} + r_i \sin\theta\, \hat{\rho}_i$



THEN

$$\bar{L} = \sum_i m_i \left[r_i^2 \bar{\omega} - r_i \omega \cos\theta \left(r_i \cos\theta\, \hat{\omega} + r_i \sin\theta\, \hat{\rho}_i \right) \right]$$

$$= \sum_i m_i r_i^2 \sin^2\theta\, \bar{\omega} \qquad \text{INVOKING THE SYMMETRY}$$

$$= \omega \sum_i m_i \rho_i^2$$

WE DEFINE $\underline{I = \sum_i m_i \rho_i^2}$ = <u>MOMENT OF INERTIA ABOUT THE AXIS</u>

I DOES NOT DEPEND ON WHICH POINT ON THE AXIS IS TAKEN AS THE ORIGIN. OF COURSE, THE C.M. MUST LIE ON A SYMMETRY AXIS.

CLEARLY OUR DEFINITION OF I IS NOT RESTRICTED TO THE USE OF A SYMMETRY AXIS. BUT FOR ROTATION ABOUT SYMMETRY AXES WE CAN WRITE

$$\underline{\bar{L} = I \bar{\omega}}.$$

IN GENERAL, $\bar{L}$ AND $\bar{\omega}$ ARE NOT PARALLEL! – C.F. TOP OF THIS PAGE.

{ WE WILL RETURN TO THE GENERAL RELATION BETWEEN $\bar{L}$ AND $\bar{\omega}$ IN LECTURE 17. HERE WE REMARK THAT FOR ANY RIGID BODY THERE ARE (AT LEAST) 3 AXES ALONG WHICH $\bar{L} = I_{AXIS} \bar{\omega}$ HOLDS.... }

FOR CASES WHERE $\bar{L} = I\bar{\omega}$ HOLDS, THE EQUATION OF MOTION IS

$$\frac{d\bar{L}}{dt} = I \bar{\alpha} = \bar{N}^e \qquad \text{WHERE } \bar{\alpha} = \frac{d\bar{\omega}}{dt} = \text{ANGULAR ACCELERATION.}$$

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THE ROTATIONAL KINETIC ENERGY IS JUST

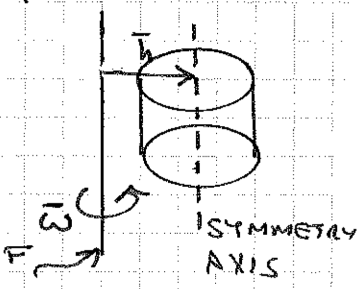
$$\underline{T_{ROT}} = \sum_i{}' \tfrac{1}{2} m_i v_i^2 = \tfrac{1}{2} \sum{}' m_i \rho_i^2 \omega^2 = \underline{\tfrac{1}{2} I \omega^2}$$

REMEMBER THAT IF THE C.M. IS MOVING, THE TOTAL KINETIC ENERGY WILL BE $\frac{1}{2} M V_{cm}^2 + T_{ROT}$.

ROTATION ABOUT AN AXIS PARALLEL TO A SYMMETRY AXIS

WE CAN EXTEND THE SIMPLE RELATION $\bar{L} = I\bar{\omega}$ TO A SOMEWHAT LARGER CLASS OF ROTATIONS.

CONSIDER ROTATION ABOUT AN AXIS PARALLEL TO A SYMMETRY AXIS BUT DISTANCE h AWAY.



$$\text{THEN} \quad \bar{L} = M \bar{h} \times \bar{V}_{cm} + I_{cm} \bar{\omega}$$
$$= (M h^2 + I_{cm}) \bar{\omega}$$
$$= I \bar{\omega}$$

WHERE WE HAVE USED A NUMBER OF INTERESTING FACTS:

- $\bar{L} = \bar{L}_{OF\ cm} + \bar{L}_{ABOUT\ THE\ C.M.}$
- WE WRITE I_{cm} FOR THE MOMENT OF INERTIA ABOUT THE SYMMETRY AXIS THRU THE C.M.
- $\bar{V}_{cm} = \bar{\omega} \times \bar{h}$
- THE <u>RATE</u> OF ROTATION ABOUT THE SYMMETRY AXIS IS THE SAME AS ABOUT THE FIXED AXIS, EVEN THO THE SYMMETRY AXIS IS MOVING. (PROVE THIS TO YOURSELF) HENCE $\bar{L}$ ABOUT THE C.M. $= I_{cm}\bar{\omega}$
- FINALLY WE USE THE

PARALLEL AXIS THEOREM: $\underline{I = I_{cm} + M h^2}$

A GEOMETRICAL RESULT YOU SHOULD ALSO PROVE TO YOURSELF.

PERPENDICULAR AXIS THEOREM.

ANOTHER GEOMETRICAL RESULT, OF MORE LIMITED APPLICATION, IS THAT FOR A <u>THIN PLATE</u> LYING IN THE X-Y PLANE,

$$I_x + I_y = I_z \qquad \text{(PROVE IT!)}$$

EXAMPLE: A THIN DISC HAS $I_z = \frac{1}{2} m r^2$, FOR $z \perp$ TO THE DISC THRU ITS C.M.

$$\therefore I_x = I_y = \tfrac{1}{4} m r^2$$

Ph 205 LECTURE 2

STATIC EQUILIBRIUM

A CLASSICAL BRANCH OF MECHANICS IS THE STUDY OF CONDITIONS SUCH THAT A RIGID BODY (OR A SYSTEM OF RIGID BODIES) REMAINS AT REST, I.E. IN STATIC EQUILIBRIUM.

CLEARLY THE C.M. MUST NOT MOVE $\Rightarrow \bar{F}^e = 0$

ALSO THERE MUST BE NO ROTATION ABOUT THE C.M. $\Rightarrow \bar{N}^e_{CM} = 0$

IN FACT THERE MUST BE NO ROTATION ABOUT ANY POINT.

HENCE $\bar{N}^e_{\text{ANY POINT}} = 0$

HOWEVER WE DO NOT HAVE AN INFINITE SET OF INDEPENDENT CONDITIONS TO BE SATISFIED. IF $\bar{r} = \bar{R} + \Delta\bar{r}$ IS THE POSITION OF AN ARBITRARY POINT, WHILE $\bar{R}$ = C.M. POSITION, THEN

$$\bar{N}^e_{\text{ABOUT } \bar{r}} = \sum_i (\bar{r}_i - \bar{r}) \times \bar{F}_i^e = \sum_i (\bar{r}_i - \bar{R}) \times \bar{F}_i^e + \sum_i (\bar{R} - \bar{r}) \times \bar{F}_i^e$$

$$= \bar{N}^e_{\text{ABOUT CM}} \qquad \text{IF } \sum_i \bar{F}_i^e = 0$$

HENCE WE HAVE ONLY 6 INDEPENDENT EQUATIONS FOR STATIC EQUILIBRIUM.

IS THIS ENOUGH?

YES - BECAUSE IT TAKES EXACTLY 6 COORDINATES TO DESCRIBE THE POSITION OF A RIGID BODY.

CERTAINLY IF WE KNOW THE POSITIONS OF ANY 3 NON-COLLINEAR POINTS IN THE BODY, WE KNOW EVERYTHING ABOUT THE POSITION OF THE WHOLE BODY. IT TAKES 3 COORDINATES TO SPECIFY THE LOCATION OF THE 1ST POINT, BUT ONLY 2 MORE TO SPECIFY THE 2ND, SINCE THE DISTANCE FROM 1 TO 2 IS FIXED. THEN THE 3RD POINT IS DESCRIBED BY ONLY 1 MORE COORDINATE AS ITS DISTANCES TO BOTH 1 AND 2 ARE FIXED.

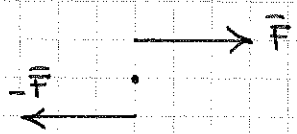
WE SAY THAT THE MOTION OF A RIGID BODY HAS 6 <u>DEGREES OF FREEDOM</u>, 3 OF TRANSLATION AND 3 OF ROTATION.

Ph 205 LECTURE 2

THERE ARE A LARGE NUMBER OF SPECIAL RULES WHICH HAVE BEEN DEVELOPED TO ASSIST IN THINKING ABOUT STATICS PROBLEMS. WE SHALL MENTION ONLY 3.

1. ANY SYSTEM OF FORCES IS EQUIVALENT TO THE RESULTANT FORCE APPLIED AT AN ARBITRARY POINT, PLUS A <u>COUPLE</u> ABOUT THAT SAME POINT.

A COUPLE IS A PAIR OF EQUAL AND OPPOSITE FORCES WHOSE LINES OF APPLICATION ARE NOT THE SAME.

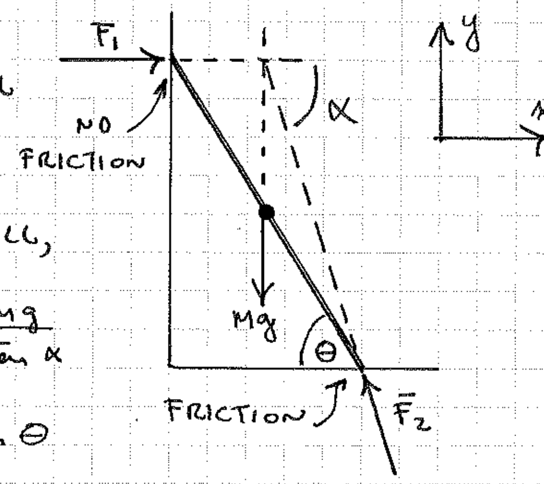


THE RESULTANT OF A COUPLE IS ZERO, BUT ITS TORQUE IS NOT.

[SEE THE FALLING CHIMNEY PROBLEM ON SET 2]

2. IF THERE ARE EXACTLY 3 EXTERNAL FORCES AND THEY ALL LIE IN A PLANE, THEN $\bar{N} = 0$ IS EQUIVALENT TO THE REQUIREMENT THAT THE LINES OF APPLICATION OF THE 3 FORCES MEET IN A POINT.

EXAMPLE: A LADDER LEANING ON A WALL



θ, M ARE GIVEN. WE WANT F_1 AND F_2

IF THERE IS NO FRICTION AT THE VERTICAL WALL,

$F_{2y} = Mg$ AND $F_{2x} = F_1 = \dfrac{F_{2y}}{\tan\alpha} = \dfrac{Mg}{\tan\alpha}$

OUR RULE TELLS US THAT $\tan\alpha = 2\tan\theta$

SO $F_1 = \dfrac{Mg}{2}\cot\theta$

HOW WOULD THE ANSWER CHANGE IF THERE IS FRICTION AT THE WALL?

[SEE ALSO THE NAPKIN RING PROBLEM ON SET 2.]

3. <u>THE PRINCIPLE OF VIRTUAL WORK</u>

IF A SYSTEM IS IN STATIC EQUILIBRIUM THEN THE FORCE ON EACH POINT MUST BE ZERO. HENCE IF EACH POINT IS DISPLACED FROM ITS EQUILIBRIUM POSITION BY A SMALL AMOUNT $\delta\bar{r}_i$, THEN THE TOTAL WORK DONE ON THE SYSTEM VANISHES:

$$\delta W = \sum_i \bar{F}_i \cdot \delta\bar{r}_i = 0$$

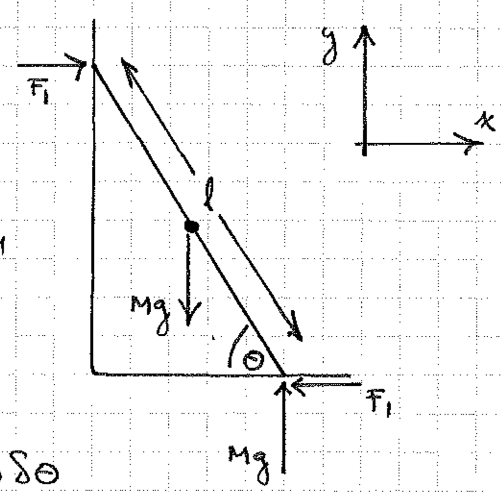
Ph 205 Lecture 2

This is true whether or not the displacements $\delta \vec{r}_i$ are physically possible. It is sufficient to imagine them.

Our principle is then that <u>the 'virtual work' vanishes in any virtual displacement of a system from equilibrium.</u>

A clever choice of the displacements will often allow us to determine some unknown forces if others are already known.

Example: The ladder problem.



Imagine the lower end of the ladder is fixed and we make a rotation by $\delta\theta$ about this point. The wall miraculously moves so as to keep contact with the ladder, always applying force F_1.

$$\delta W_{GRAVITY} = -Mg\,\delta y = -Mg\,\delta\left(\frac{l}{2}\sin\theta\right) = -\frac{Mgl}{2}\cos\theta\,\delta\theta$$

Minus since y is up and gravity is down

$$\delta W_{WALL} = F_1\,\delta x = -F_1\,\delta(l\cos\theta) = F_1\,l\sin\theta\,d\theta$$

This must be + since F_1 is parallel to δx

$$\delta W_{TOTAL} = 0 = \left(F_1\,l\sin\theta - \frac{Mgl}{2}\cos\theta\right)\delta\theta$$

so $F_1 = \frac{Mg}{2}\cot\theta$ as before.