# Ph 205 Lecture 20

## Motion of a Rigid Body which is Rolling Without Slipping

An interesting but difficult class of problems is that of a rigid body - usually a disc or a sphere — which is rolling without slipping on some surface.

The rolling constraint is non-holonomic: Lagrange's method would require the use of Lagrange multipliers!

We now give some examples of a vectorial approach, introduced in Set 10, which is very powerful and fairly intuitive. It is the subject of a book: 'Vectorial Mechanics' by C. A. Milne.

The starting point is that in rolling without slipping the velocity of contact is zero (if the support surface is at rest). Then Chasles' theorem allows us to write

$$\bar{v}_{\text{contact}} = \bar{v}_{cm} + \bar{\omega} \times \bar{a} = 0$$

where $\bar{\omega}$ = total angular velocity of the body

$\bar{a}$ = vector from the c.m. to point of contact

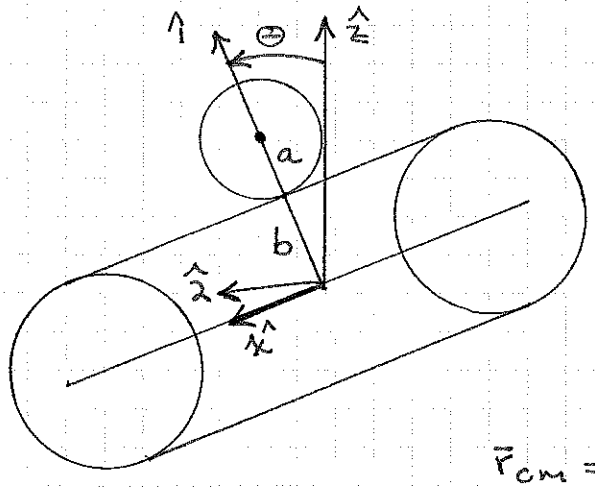
For a body acted on only by gravity, $-mg\hat{z}$, and the unknown force of contact, $\bar{F}$, the equations of motion are

$$\bar{F} - mg\hat{z} = m\frac{d^2\bar{r}_{cm}}{dt^2} \quad \text{and} \quad \bar{N}_{cm} = \bar{a} \times \bar{F} = \frac{d\bar{L}_{cm}}{dt}$$

In the homework set you have the chance to consider several interesting examples. We now illustrate the method with two somewhat academic examples.

**Example** A sphere of radius $a$ rolls without slipping on the surface of a fixed horizontal cylinder of radius $b$.

If the sphere is spinning, is there some 'stable' motion in which it doesn't fall off the cylinder?




LET $\hat{1}$ BE FROM THE POINT OF CONTACT TO THE CENTER OF THE SPHERE

$\hat{z}$ IS VERTICAL

$\hat{x}$ IS HORIZONTAL ALONG THE AXIS OF THE CYLINDER

$\hat{2} = \hat{x} \times \hat{1}$ IS IN THE $\hat{1}$-$\hat{z}$ PLANE

$\hat{1}, \hat{2}, \hat{x}$ IS OUR LOCAL COORD. SYSTEM

$$\bar{r}_{CM} = x\,\hat{x} + (a+b)\,\hat{1}$$

$\bar{a} = -a\,\hat{1}$ IS FROM C.M. TO POINT OF CONTACT

THE ROLLING CONSTRAINT IS $\dfrac{dx}{dt}\hat{x} + (a+b)\dfrac{d\hat{1}}{dt} = -\bar{\omega} \times \bar{a} = -a\,\hat{1} \times \bar{\omega}$

WE ELIMINATE THE UNKNOWN FORCE $\bar{F}$ FROM THE EQUATIONS OF MOTION:

$$\frac{d\bar{L}}{dt} = I\frac{d\bar{\omega}}{dt} = mg\,\bar{a} \times \hat{z} + m\bar{a} \times \frac{d^2\bar{r}}{dt^2} = mga\sin\theta\,\hat{x} - ma\,\hat{1} \times \left\{\frac{d^2x}{dt^2}\hat{x} + (a+b)\frac{d^2\hat{1}}{dt^2}\right\}$$

USING $\bar{L} = I\,\bar{\omega}$ FOR A SPHERE

WE WISH TO BRING ALL TERMS INTO COMPONENTS ALONG THE $\hat{1}, \hat{2}$ OR $\hat{x}$ AXES.

TO GET $\bar{\omega}$, WE PLAY A TRICK WITH THE ROWING CONSTRAINT:

MULTIPLY BY $\hat{1}$: $\dfrac{dx}{dt}\,\hat{1} \times \hat{x} + (a+b)\,\hat{1} \times \dfrac{d\hat{1}}{dt} = -a\,\hat{1} \times (\hat{1} \times \bar{\omega}) = -a\left\{(\bar{\omega}\cdot\hat{1})\hat{1} - \bar{\omega}\right\}$

$$\text{OR} \quad \bar{\omega} = \omega_1\,\hat{1} - \frac{1}{a}\frac{dx}{dt}\,\hat{2} + \frac{a+b}{a}\,\hat{1} \times \frac{d\hat{1}}{dt}$$

NOW $\hat{1}$ & $\hat{2}$ ROTATE ABOUT THE AXIS OF THE CYLINDER AT RATE $\dot{\theta}\,\hat{x}$ (WATCH SIGNS) $\Rightarrow$ $\dfrac{d\hat{1}}{dt} = \dot{\theta}\,\hat{x} \times \hat{1} = \dot{\theta}\,\hat{2} \Rightarrow \hat{1} \times \dfrac{d\hat{1}}{dt} = \dot{\theta}\,\hat{x}$

$$\text{HENCE} \quad \bar{\omega} = \omega_1\,\hat{1} - \frac{1}{a}\frac{dx}{dt}\,\hat{2} + \frac{a+b}{a}\,\dot{\theta}\,\hat{x}$$

WE ALSO NEED $\dfrac{d\hat{2}}{dt} = \dot{\theta}\,\hat{x} \times \hat{2} = -\dot{\theta}\,\hat{1}$ ; $\dfrac{d^2\hat{1}}{dt^2} = \ddot{\theta}\,\hat{2} + \dot{\theta}\dfrac{d\hat{2}}{dt} = \ddot{\theta}\,\hat{2} - \dot{\theta}^2\,\hat{1}$

WE NOW HAVE EVERYTHING NEEDED IN THE EQUATION OF MOTION.

$$\frac{d\bar{L}}{dt} = I\left\{\dot{\omega}_1\,\hat{1} + \omega_1\dot{\theta}\,\hat{2} - \frac{1}{a}\ddot{x}\,\hat{2} + \frac{1}{a}\dot{x}\dot{\theta}\,\hat{1} + \frac{a+b}{a}\ddot{\theta}\,\hat{x}\right\}$$

$$= mga\sin\theta\,\hat{x} + ma\ddot{x}\,\hat{2} - ma(a+b)\ddot{\theta}\,\hat{x}$$

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The components of the equations of motion are

$\hat{\uparrow}$: $I\left(\dot{\omega}_1 + \frac{\dot{\theta}\dot{x}}{a}\right) = 0$ or $\underline{a\dot{\omega}_1 = -\dot{\theta}\dot{x}}$

$\hat{2}$: $I\left(\omega_1\dot{\theta} - \frac{\ddot{x}}{a}\right) = ma\ddot{x}$ or $\underline{Ia\omega_1\dot{\theta} = (I+ma^2)\ddot{x}}$

$\hat{x}$: $I\frac{a+b}{a}\ddot{\theta} = mga\sin\theta - ma(a+b)\ddot{\theta}$

or $\underline{(I+ma^2)(a+b)\ddot{\theta} = mga^2\sin\theta}$

The $\hat{x}$ equation shows that no stability is possible! Once $\theta$ starts to increase, it continues to do so no matter what the spin $\omega_1$ about the point of contact.

But some further manipulations are possible.

We can get a kind of energy equation if we multiply the $\hat{x}$ equation by $\dot{\theta}$ and integrate.

$$(I+ma^2)(a+b)\dot{\theta}\ddot{\theta} = mga^2\sin\theta\,\dot{\theta}$$

$$\Rightarrow \quad (I+ma^2)(a+b)\frac{\dot{\theta}^2}{2} = mga^2(\cos\theta_0 - \cos\theta)$$

The sphere falls off the cylinder if the radial component of the contact force vanishes, i.e. when $(\bar{F}\cdot\hat{\uparrow}) = 0$

But $\bar{F} = mg\hat{z} + m\frac{d^2\bar{r}}{dt^2} = mg\hat{z} + m\left(\ddot{x}\hat{x} + (a+b)\frac{d^2\hat{\uparrow}}{dt^2}\right)$

$\bar{F}\cdot\hat{\uparrow} = 0 = mg\cos\theta - m(a+b)\dot{\theta}^2$ or $g\cos\theta = (a+b)\dot{\theta}^2$ at falloff

Using the energy relation: $g\cos\theta = \frac{2ma^2g(\cos\theta_0 - \cos\theta)}{I+ma^2}$

or $\cos\theta = \frac{2ma^2\cos\theta_0}{I+3ma^2}$ at falloff. [Compare (10) b), Set 3]

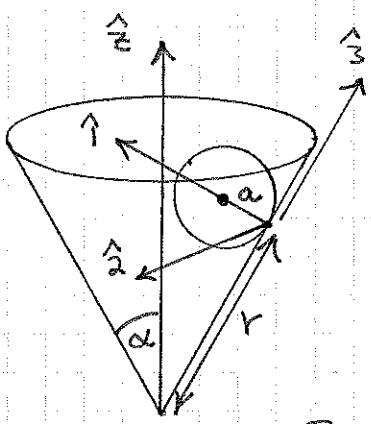
For the $x$ motion, we can combine the $\hat{\uparrow}$ and $\hat{2}$ equations:

$$(I+ma^2)\ddot{x} = -Ia^2\frac{\omega_1\dot{\omega}_1}{\dot{x}} \quad \text{or} \quad (I+ma^2)\dot{x}\ddot{x} = -Ia^2\omega_1\dot{\omega}_1$$

So $(I+ma^2)\dot{x}^2 = Ia^2(\omega_{1_0}^2 - \omega_1^2)$, another energy equation.

Finally, if $\omega_{1_0} \neq 0$, $x$ will vary as the sphere falls down. Because if $x$ = const, then $\ddot{x} = 0$ and the $\hat{2}$ eq. would $\Rightarrow \dot{\theta} = 0$ also. But this is impossible according to the $\hat{x}$ equation!

EXAMPLE




A SPHERE OF RADIUS $a$ ROLLS WITHOUT SLIPPING ON THE INSIDE OF A FIXED VERTICAL CONE OF HALF ANGLE $\alpha$. WHAT KIND OF STEADY MOTION IS POSSIBLE?

WE CHOOSE AXES

$\hat{1}$ FROM POINT OF CONTACT TO C.M. OF SPHERE

$\hat{2}$ HORIZONTAL

$\hat{3}$ ALONG THE CONE THRU POINT OF CONTACT

FOR STEADY MOTION WE EXPECT $Y$ ALONG CONE TO BE CONSTANT, AND THAT $\bar{\omega}$ WILL HAVE NO COMPONENT ALONG THE $\hat{2}$ AXIS. $\bar{\omega} = \omega_1 \hat{1} + \omega_3 \hat{3}$ WITH $\omega_1, \omega_3$ CONSTANT

ALL 3 AXES WILL PRECESS ABOUT THE VERTICAL AT RATE $\Omega \hat{z}$

WE WANT A RELATION BETWEEN $\Omega, \omega_1, \omega_3$ AND $r$.

NOW $\hat{z} = \hat{1}\sin\alpha + \hat{3}\cos\alpha$, SO

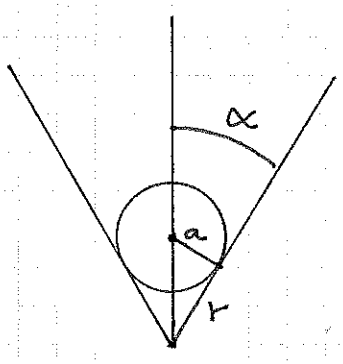
$$\frac{d\hat{1}}{dt} = \Omega\hat{z}\times\hat{1} = \Omega\cos\alpha\,\hat{2};\quad \frac{d\hat{2}}{dt} = \Omega\hat{z}\times\hat{2} = \Omega(\sin\alpha\,\hat{3} - \cos\alpha\,\hat{1});\quad \frac{d\hat{3}}{dt} = -\Omega\sin\alpha\,\hat{2}$$

THE ROLLING CONSTRAINT IS $\frac{d\bar{r}_{cm}}{dt} + \bar{\omega}\times\bar{a} = 0$

WHERE $\bar{r}_{cm} = a\hat{1} + r\hat{3}$ AND $\bar{a} = -a\hat{1}$

SO $a\frac{d\hat{1}}{dt} + r\frac{d\hat{3}}{dt} = -\bar{\omega}\times\bar{a} = (\omega_1\hat{1} + \omega_3\hat{3})\times a\hat{1} = a\omega_3\hat{2} = \frac{d\bar{r}_{cm}}{dt}$

OR $a\omega_3 = \Omega(a\cos\alpha - r\sin\alpha) \Rightarrow \underline{\omega_3 = -\Omega\cos\alpha\left(\frac{r}{a}\tan\alpha - 1\right)}$




NOTE THAT IF THE SPHERE JUST SITS IN THE BOTTOM OF THE CONE, $\tan\alpha = \frac{a}{r}$.

HENCE $\frac{r}{a}\tan\alpha > 1$ IN GENERAL.

AS BEFORE, WE COMBINE THE EQUATIONS OF MOTION

$$\bar{F} - mg\hat{z} = m\frac{d^2\bar{r}_{cm}}{dt^2} \quad \text{AND} \quad \bar{N} = \bar{a}\times\bar{F} = \frac{d\bar{L}}{dt} = I\frac{d\bar{\omega}}{dt}$$

TO GET $$I\frac{d\bar{\omega}}{dt} = \bar{a}\times mg\hat{z} + m\bar{a}\times\frac{d^2\bar{r}_{cm}}{dt^2}$$

NOW $$\frac{d^2\bar{r}_{cm}}{dt^2} = \frac{d}{dt}\frac{d\bar{r}_{cm}}{dt} = a\omega_3\frac{d\hat{2}}{dt} = a\Omega\omega\left(-\cos\alpha\,\hat{1} + \sin\alpha\,\hat{3}\right)$$

HENCE $$I\frac{d\bar{\omega}}{dt} = I\Omega(\omega_1\cos\alpha - \omega_3\sin\alpha)\hat{2} = mga\cos\alpha\,\hat{2} + ma^2\Omega\omega_3\sin\alpha\,\hat{2}$$

$$-(I+ma^2)\Omega\omega_3\sin\alpha + I\Omega\omega_1\cos\alpha - mga\cos\alpha = 0$$

$$\Omega^2 I'\left(\frac{r}{a}\tan\alpha - 1\right)\sin\alpha + I\omega_1\Omega - mga = 0 \qquad [I' \equiv I + ma^2]$$

$$\Omega = \frac{-I\omega_1 \pm \sqrt{I^2\omega_1^2 + 4mga\sin\alpha\, I'\left(\frac{r}{a}\tan\alpha - 1\right)}}{2I'\sin\alpha\left(\frac{r}{a}\tan\alpha - 1\right)}$$

FOR $\alpha < 90°$, THIS MOTION IS ALWAYS STABLE, EVEN IF $\omega_1 = 0$! (PROVE IT!)

AT $\alpha \to 90°$, $\Omega \to 0$. A SPHERE WILL JUST SIT ON A FLAT PLANE!
FOR $\alpha > 90°$, WE CAN STILL HAVE STABILITY IF $\omega_1$ IS BIG ENOUGH!

STRICTLY FOR ENTERTAINMENT WE NOW CONSIDER SOME FORMAL METHODS IN ADDITION TO LAGRANGE'S METHOD. THESE DO NOT ADD MUCH TO PROBLEM SOLVING IN CLASSICAL MECHANICS, BUT PROVIDE THE LANGUAGE USED IN QUANTUM MECHANICS.

## HAMILTON'S EQUATIONS

WE CONSIDER ONLY CONSERVATIVE FORCES, AND SUPPOSE $T$ AND $V$ ARE KNOWN FUNCTIONS OF THE GENERALISED COORDINATES:

$$T = T(q_j, \dot{q}_j, t) \quad , \quad V = V(q_j)$$

IF $T$ DOES NOT DEPEND ON TIME EXPLICITLY, $\frac{\partial L}{\partial t} = 0$

AND $H = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L$ IS A CONSTANT.

IT IS OFTEN JUST THE ENERGY $E = T + V$.

ON OCCASION WE HAVE EXTRACTED THE EQUATIONS OF MOTION DIRECTLY FROM $E = T + V$ RATHER THAN $L = T - V$. HAMILTON HAS SUGGESTED DOING THIS IN GENERAL.

RECALL THAT $p_j \equiv \frac{\partial L}{\partial \dot{q}_j}$ = GENERALISED MOMENTUM.

HAMILTON'S IDEA IS TO REPLACE $\dot{q}_j$ BY $p_j$ AS A VARIABLE,

YIELDING $H(q_j, p_j, t) = \sum_j \dot{q}_j p_j - L$

WHETHER OR NOT $L$ DEPENDS EXPLICITLY ON TIME.

$H$ IS OF COURSE THE <u>HAMILTONIAN</u>.

TO EXTRACT THE EQUATIONS OF MOTION WE TAKE DERIVATIVES:

$$dL = \frac{\partial L}{\partial t} dt + \sum_j \left( \frac{\partial L}{\partial q_j} dq_j + \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j \right)$$

$$= \frac{\partial L}{\partial t} dt + \sum_j \left( \dot{p}_j \, dq_j + p_j \, d\dot{q}_j \right)$$

USING $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j}$

THEN $dH = \sum_j (\dot{q}_j \, dp_j + p_j \, d\dot{q}_j) - dL$

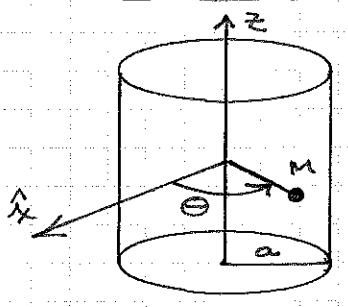
$$= \sum_j (\dot{q}_j \, dp_j - \dot{p}_j \, dq_j) - \frac{\partial L}{\partial t} dt$$

HENCE $\boxed{\dot{q}_j = \frac{\partial H}{\partial p_j} \qquad \dot{p}_j = -\frac{\partial H}{\partial q_j} \qquad \text{AND} \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}}$

THESE ARE THE SO-CALLED <u>HAMILTON'S EQUATIONS</u>

THEY CONTAIN NOTHING NEW, BUT THEY ARE VERY ELEGANT.

EXAMPLE




A PARTICLE SLIDES WITHOUT FRICTION ON THE SURFACE OF A CYLINDER OF RADIUS $a$, SUBJECT TO A FORCE $\bar{F} = -k\bar{r}$ TOWARDS A POINT ON THE AXIS. (NO GRAVITY)

ENERGY IS CONSERVED, SO $H = T + V$

USE $\theta$ AND $z$ AS COORDINATES.

$$T = \tfrac{1}{2} M (a^2 \dot{\theta}^2 + \dot{z}^2) \qquad V = \tfrac{1}{2} k r^2 = \tfrac{1}{2} k (a^2 + z^2)$$

THE GENERALISED MOMENTA ARE $p_\theta = M a^2 \dot{\theta}$ AND $p_z = M \dot{z}$

SO $$H = \frac{p_\theta^2}{2Ma^2} + \frac{p_z^2}{2M} + \tfrac{1}{2} k (a^2 + z^2)$$

HAMILTON'S EQUATIONS ARE $\dot{\theta} = \dfrac{\partial H}{\partial p_\theta} = \dfrac{p_\theta}{Ma^2}$ ; $\dot{p}_\theta = -\dfrac{\partial H}{\partial \theta} = 0$

HENCE $\dot{\theta} = $ CONST.

ALSO $\dot{z} = \dfrac{\partial H}{\partial p_z} = \dfrac{p_z}{M}$ ; $\dot{p}_z = -\dfrac{\partial H}{\partial z} = -kz$

SO $\ddot{z} = -\dfrac{k}{M} z \Rightarrow z$ OSCILLATES!

GREAT – BUT IT'S NOT THE QUICKEST METHOD

IF $H$ DOES NOT DEPEND EXPLICITLY ON TIME, IT IS CONSTANT, AS WE KNOW FROM $\partial H/\partial t = -\partial L/\partial t$, ETC. BUT THIS RESULT IS SELF CONTAINED WITHIN HAMILTON'S FORMALISM:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_j \left( \frac{\partial H}{\partial q_j} \dot{q}_j + \frac{\partial H}{\partial p_j} \dot{p}_j \right) = \frac{\partial H}{\partial t} + \sum_j \frac{\partial H}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial p_j} \frac{\partial H}{\partial q_j}$$

$$= \frac{\partial H}{\partial t}. \quad \text{SO IF } \frac{\partial H}{\partial t} = 0, \ \frac{dH}{dt} = 0 \text{ ALSO.}$$

WHEN DETERMINING THE RELATIONS $p_j = p_j(q_j, \dot{q}_j, t)$, WE NOTE THAT $p_j = \dfrac{\partial L}{\partial \dot{q}_j} = \dfrac{\partial T}{\partial \dot{q}_j}$ SINCE $V$ DOES NOT DEPEND ON $\dot{q}_j$.

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The quest for formal elegance in mechanics has been advanced by Poisson. He noted that if $f(q_j, p_j, t)$ is some function of a system with Hamiltonian $H$, then

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_j \left( \frac{\partial f}{\partial q_j}\dot{q}_j + \frac{\partial f}{\partial p_j}\dot{p}_j \right) = \frac{\partial f}{\partial t} + \sum_j \left( \frac{\partial f}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial H}{\partial q_j} \right)$$

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial t} + [f, H]}$$

where $[f, g] = \sum_j \left( \frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j} \right)$

$[f, g]$ is the <u>Poisson Bracket</u> of functions $f$ & $g$.

If $f$ does not depend explicitly on time, then the condition $[f, H] = 0 \Rightarrow \frac{df}{dt} = 0$

$\Rightarrow f$ is a <u>constant of the motion</u>

We note that not all constants appearing in a problem are "constants of the motion" in this sense.

<u>Example</u> A mass falls vertically, $x$ & $y$ remaining constant. Is $x = x_0$ a 'constant of the motion'?

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + mgz$$

$$[x, H] = (1)\left(\frac{p_x}{m}\right) - (0)(0) + (0)\left(\frac{p_y}{m}\right) - (0)(0) + (0)\left(\frac{p_z}{m}\right) - (0)(mg) = \frac{p_x}{m}$$

So $[x, H]$ is not identically zero, but only vanishes by virtue of the initial conditions. $\therefore x_0$ is not a constant of the motion.

Poisson Brackets obey marvelous mathematical properties:

$$[f, f] = 0$$

$$[f, g] = -[g, f]$$

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$$[f_1 + f_2, g] = [f_1, g] + [f_2, g]$$

$$[f_1 f_2, g] = f_1 [f_2, g] + f_2 [f_1, g]$$

SPECIAL RESULTS OBTAIN WHEN $f$ OR $g$ IS A $p$ OR $q$

$$[f, q_j] = -\frac{\partial f}{\partial p_j} \qquad [f, p_j] = \frac{\partial f}{\partial q_j}$$

HENCE $[q_i, q_j] = 0$, $[p_i, p_j] = 0$, BUT $[q_j, p_j] = \delta_{ij}$

VERY STRANGE!

A RESULT WHICH IS NOT IMMEDIATELY EVIDENT (L & L p.136) IS JACOBI'S IDENTITY:

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$$

A FINAL RESULT OF POISSON IS THAT IF $f$ & $g$ ARE BOTH CONSTANTS OF THE MOTION, THEN $[f, g]$ IS ALSO.

IF $f$ & $g$ DON'T DEPEND EXPLICITLY ON TIME, THIS FOLLOWS FROM JACOBI'S IDENTITY. NOTE THAT

$$[f, H] = 0 = [g, H] \Rightarrow [H, [f, g]] = 0 \Rightarrow \frac{d}{dt}[f, g] = 0$$

<u>EXAMPLE</u>: ANGULAR MOMENTUM COMPONENTS.

$$L_x = y p_z - z p_y \qquad L_y = z p_x - x p_z \qquad L_z = x p_y - y p_x$$

SO $[L_x, L_y] = (0)(0) - (0)(0) + p_z(0) - (-z)(0) + (-p_y)(-x) - y p_x = L_z$

$$[L_y, L_z] = L_x \qquad [L_z, L_x] = L_y \qquad \text{AMAZING!}$$

HENCE IF $L_x$ & $L_y$ ARE CONSTANTS OF THE MOTION, SO IS $L_z$!

THIS IS <u>NOT</u> THE SAME THING AS $L_x, L_y = \text{CONST} \Rightarrow L_z = \text{CONST}$.

A BODY MOVING IN THE PLANE $z = 0$ HAS $L_x = L_y = 0$, BUT $L_z \neq$ CONST IN GENERAL. HOWEVER $L_x$ AND $L_y$ ARE NOT CONSTANTS OF THE MOTION IN THE SENSE THAT $[L_x, H] = 0$ IDENTICALLY.

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## LIOUVILLE'S THEOREM

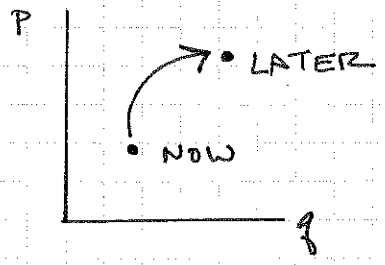
THIS FINAL RESULT DOES CONTAIN AN IMPORTANT PHYSICS MESSAGE, AND IS NOT JUST FORMAL TRICKERY. LIKE THE VIRIAL THEOREM, IT IS STATISTICAL IN CHARACTER.

TO COMPLETELY SPECIFY THE MOTION OF A PARTICLE OR SYSTEM OF PARTICLES, WE NEED TO GIVE BOTH THE POSITION AND VELOCITY. NEWTON'S LAW $\bar{F} = m\bar{a}$ GUARANTEES THAT THIS IS SUFFICIENT. HAMILTON SUGGESTS WE USE THE GENERALISED COORDINATES AND GENERALISED MOMENTA TO SPECIFY THE MOTION.

AT ANY MOMENT OUR SYSTEM IS LOCATED AT SOME POINT IN THE SPACE WHOSE COORDINATES ARE

$$(q_j, p_j) \quad j = 1 \ldots N$$

THIS $2N$ DIMENSIONAL SPACE IS CALLED PHASE SPACE.



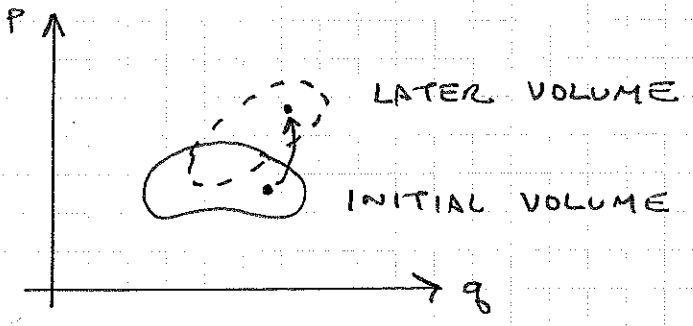

IF WE HAVE MANY IDENTICAL PARTICLES (OR SYSTEMS) WHICH ARE INDEPENDENT OF ONE ANOTHER, WE CAN DESCRIBE THEM AS A SWARM OF POINTS IN PHASE SPACE.

LIOUVILLE ASKS US TO CONSIDER THE DENSITY OF SUCH A SWARM OF POINTS IN PHASE SPACE: $\rho = \frac{dN}{d\nu}$

WHERE $d\nu = dq_1\, dq_2 \ldots dq_N\, dp_1\, dp_2 \ldots dp_N$

THE THEOREM IS THAT $\rho$ IS CONSTANT IN TIME - IF WE ADOPT A POINT OF VIEW MOVING WITH THE SWARM.

IN PICTURES:




SUPPOSE ALL POINTS ARE INITIALLY CONTAINED WITHIN SOME (MINIMAL) VOLUME IN PHASE SPACE. AFTER A WHILE THEY WILL BE CONTAINED IN A NEW VOLUME. LIOUVILLE CLAIMS THE SIZE OF THESE TWO VOLUMES IS THE SAME.

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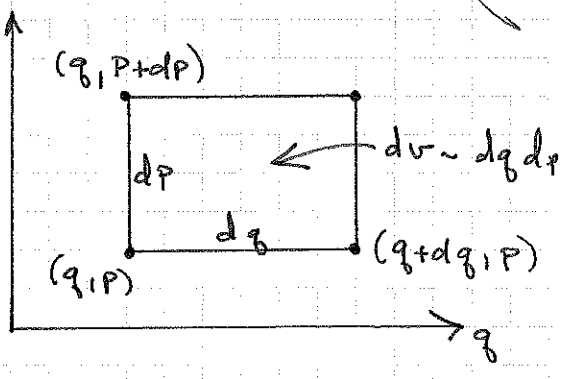
CONSIDER A SMALL RECTANGULAR VOLUME IN PHASE SPACE

$$dv = dq\,dp$$




AFTER TIME $dt$, THE POINT WITH COORDINATE $q$ HAS MOVED TO

$$q \to q + \dot{q}\,dt = q + \frac{\partial H(q,p)}{\partial p}\,dt$$

ACCORDING TO HAMILTON.

$$q + dq \to q + dq + \frac{\partial H(q+dq, p)}{\partial p}\,dt = q + dq + \left(\frac{\partial H(q,p)}{\partial p} + \frac{\partial^2 H(q,p)}{\partial p \partial q}\,dq\right)dt$$

HENCE $$dq \to dq\left(1 + \frac{\partial^2 H}{\partial p \partial q}\,dt\right)$$

ALSO $$p \to p + \dot{p}\,dt = p - \frac{\partial H(q,p)}{\partial q}\,dt$$

AND $$dp \to dp\left(1 - \frac{\partial^2 H(q,p)}{\partial q \partial p}\,dt\right)$$

THUS $$dv = dq\,dp \to dq\,dp\left(1 + \frac{\partial^2 H}{\partial p \partial q}\,dt\right)\left(1 - \frac{\partial^2 H}{\partial q \partial p}\,dt\right)$$

$$= dq\,dp \text{ TO 1ST ORDER IN } dt$$

AND PHASE VOLUME IS CONSERVED !

<u>EXAMPLE</u> FALLING RAIN DROPS. AT $t = 0$ A LAYER OF HEIGHT $h_0$ IS UNIFORMLY FILLED WITH RAIN DROPS WITH DOWNWARD VELOCITIES BETWEEN 0 AND $v_0$.

AT TIME $t$, $\quad v = v_i + gt$

$$h = h_i + v_i t + \tfrac{1}{2} g t^2$$

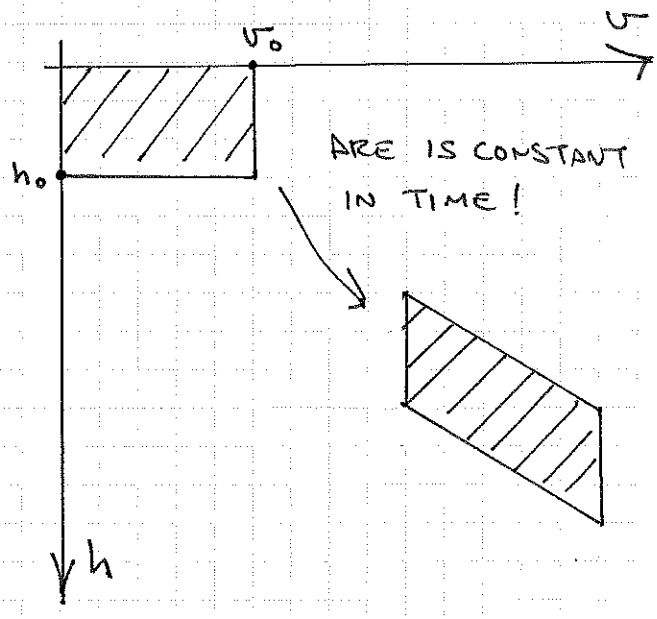
$$(h, v) = (0, 0) \to \left(\tfrac{1}{2} g t^2,\ gt\right)$$

$$(0, v_0) \to \left(v_0 t + \tfrac{1}{2} g t^2,\ v_0 + gt\right)$$

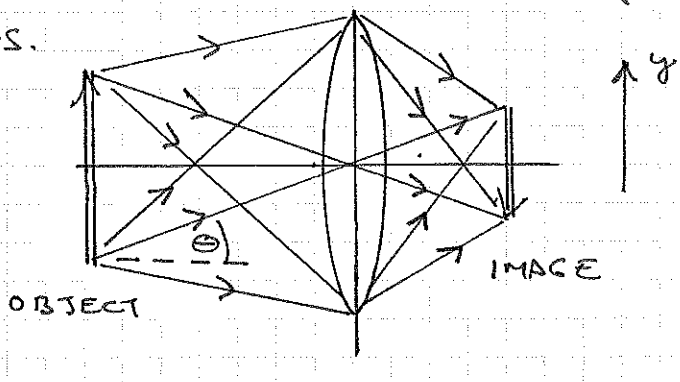
$$(h_0, 0) \to \left(h_0 + \tfrac{1}{2} g t^2,\ gt\right)$$

$$(h_0, v_0) \to \left(h_0 + v_0 t + \tfrac{1}{2} g t^2,\ v_0 + gt\right)$$

THE AREA IS $(h, v)$ PHASE SPACE IS CLEARLY CONSTANT IN TIME.



EXAMPLE LIGHT RAYS THRU A LENS.

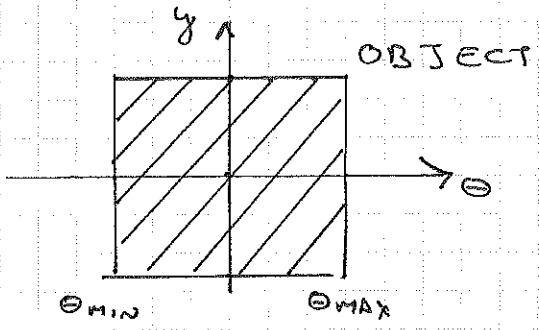


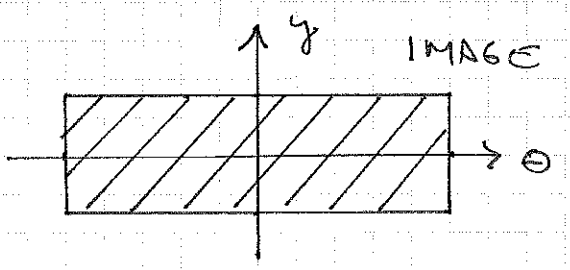

AS COORDINATES WE USE

$$q = y$$

THEN 'MOMENTUM' $p = p_y = (\hbar/\lambda)\tan\theta$

OR $p \propto \theta$



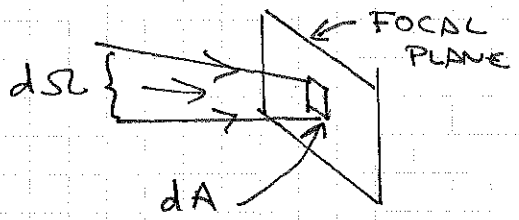




THE PRICE OF SQUEEZING THE LIGHT IN $y$ IS SPREADING IT OUT IN $\theta$ – THE PHASE-SPACE VOLUME MUST REMAIN CONSTANT.

A COROLLARY IS THE SO-CALLED BRIGHTNESS THEOREM

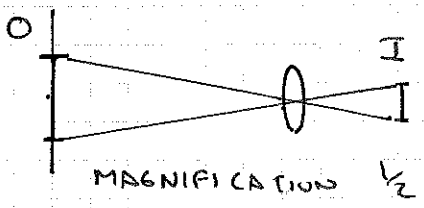
THE "BRIGHTNESS" OF AN OBJECT AS OBSERVED THRU AN OPTICAL SYSTEM IS DEFINED AS

$$\frac{dI}{dA\,d\Omega}$$

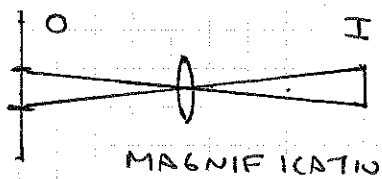



$I$ = INTENSITY
$dA$ = REGION OF OBSERVATION IN FOCAL PLANE
$d\Omega$ = SOLID ANGLE SUBTENDED BY LIGHT FROM THE OBJECT.

LIOUVILLE CLAIMS THIS QUANTITY IS AN INVARIANT. BY ADDING LENSES WE CAN CHANGE $dA$, OR $d\Omega$, BUT NOT $dA\,d\Omega$.




MAGNIFICATION ½




MAGNIFICATION 1

THE 2ND SETUP GIVES A BIGGER IMAGE (RELATIVE TO OBJECT SIZE) BUT IT GATHERS LESS LIGHT. THE 'BRIGHTNESS' IS THE SAME.

A TELESCOPE CAN MAKE THE SUN APPEAR BIGGER BUT NOT BRIGHTER!