

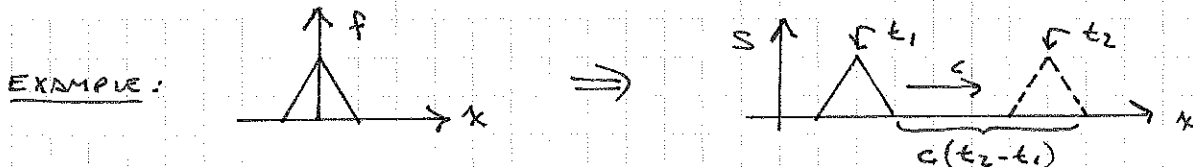
(THE LAST LECTURE WILL MEET FRIDAY, DEC. 15, 1 P.M.)

TRAVELLING WAVES

THE WAVE EQUATION FOR A STRETCHED STRING IS

$$\ddot{s} = c^2 s'' \quad \text{WHERE } c^2 = T/\rho$$

WE NOTED EARLIER THAT $s = f(x-ct) + g(x+ct)$ IS A SOLUTION FOR ANY FUNCTIONS $f(x)$ & $g(x)$.



FOR $s = f(x-ct)$ OR $g(x+ct)$ THE WAVE DOESN'T CHANGE SHAPE WITH TIME - BUT IT IS DISPLACED BY A DISTANCE $\pm ct$. THESE ARE THE TRAVELLING WAVES

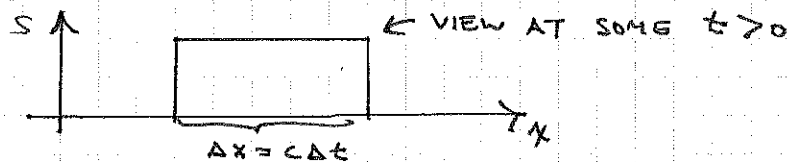
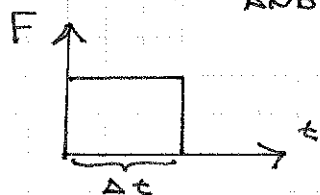
EXAMPLE SUPPOSE YOU SHAKE ONE END OF A VERY LONG STRING (WHICH IS UNDER TENSION) ACCORDING TO $s(0,t) = F(t)$



PHYSICALLY WE EXPECT ONLY THE OUTWARD GOING WAVE $f(x-ct)$

THEN $s(0,t) = f(-ct) = F(t)$ OR $f(z) = F(-\frac{z}{c})$

AND $s(x,t) = F(t - x/c)$



NOTE THAT FOR $t < 0$ OUR SOLUTION CONSISTS OF A 'FICTITIOUS' PULSE AT $x < 0$ TRAVELLING IN THE $+x$ DIRECTION, JUST WAITING TO EMERGE INTO REALITY AT $t = 0$.

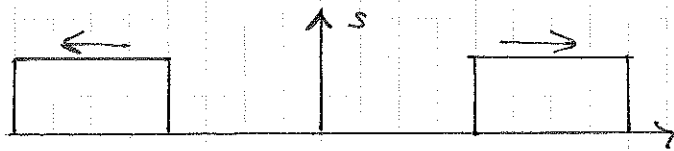
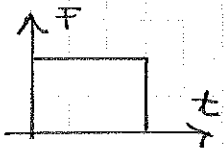
EXAMPLE SUPPOSE WE SHAKE THE MIDDLE ($x=0$) OF A LONG STRING ACCORDING TO $s(0,t) = F(t)$.

NOW WE EXPECT WAVES IN BOTH THE $+x$ AND $-x$ DIRECTIONS

$$s(x,t) = f(x-ct) + g(x+ct)$$

BUT THE SITUATION IS SYMMETRIC ABOUT $x=0 \Rightarrow f=g$

HENCE $s(x,t) = \frac{1}{2} F(t - \frac{x}{c}) + \frac{1}{2} F(t + \frac{x}{c})$



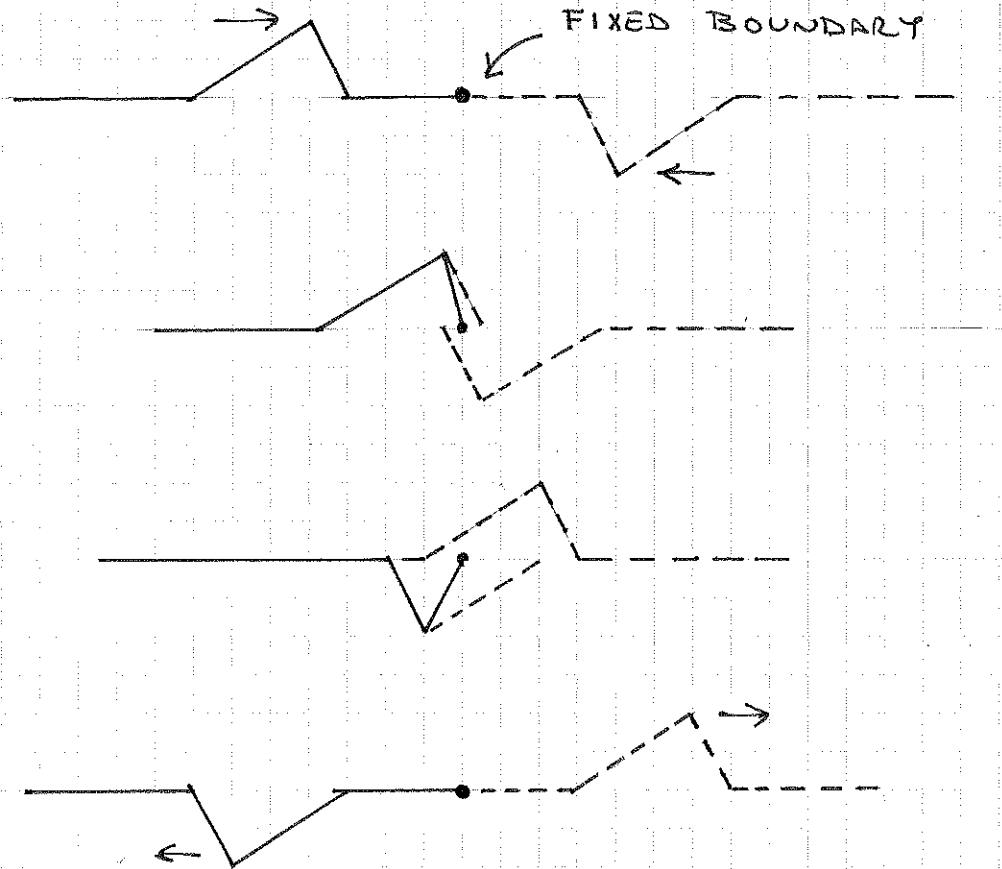
REFLECTION AT A BOUNDARY

WHAT HAPPENS TO A TRAVELLING WAVE AT THE END OF ITS ROPE?
 AS BEFORE THERE ARE 2 CASES.

a) FIXED END. $s(l, t) = 0$

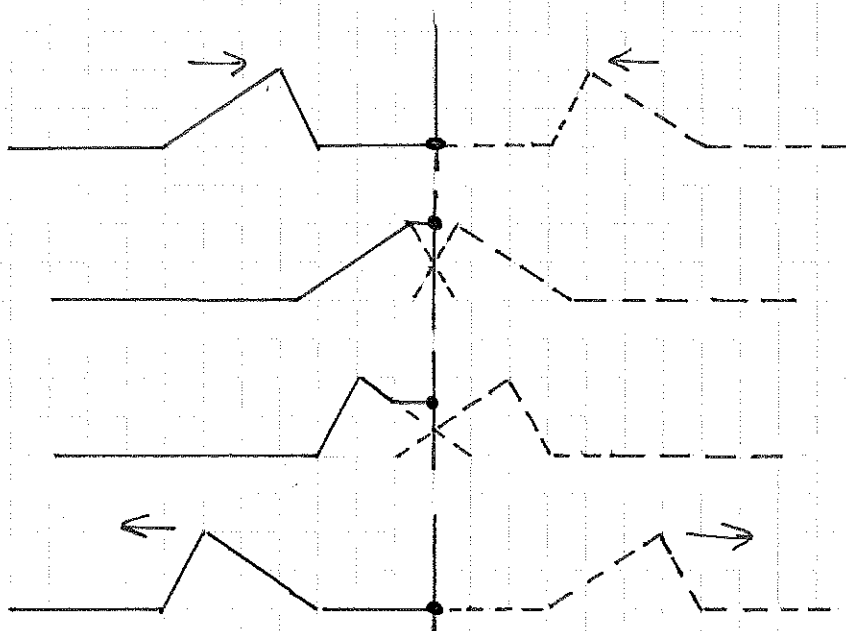
OUR PULSE $f(x-ct)$ NEVER VARIES IN SHAPE. BUT WE
 CAN IMAGINE A 'FICTITIOUS' PULSE $-f(-x-ct+2l)$

HEADING TOWARDS THE BOUNDARY FROM THE 'UNPHYSICAL' REGION $x > l$.
 THE TWO PULSES ADD (THE WAVE EQUATION IS LINEAR!) TO GIVE
 ZERO AT $x=l$, SATISFYING THE BOUNDARY CONDITION.
 FOR LARGE t , THE ORIGINAL PULSE BECOMES 'FICTITIOUS',
 AND WE SEE AN INVERTED WAVE MOVING IN THE $-x$ DIRECTION.



b) FREE END $s'(l, t) = 0$

THE REFLECTED WAVE HAS THE SAME SIGN,



STANDING WAVES AS A SPECIAL CASE OF TRAVELLING WAVES

IF THE STRING IS FIXED AT BOTH ENDS, THE WAVE IS 'TRAPPED', AND BOUNCES BACK AND FORTH. BUT ANY WAVE ON A FINITE STRING CAN BE EXPRESSED AS A SUM OF STANDING WAVE MODES BY MEANS OF A FOURIER SERIES.

LIKEWISE, ANY WAVE CAN BE EXPRESSED AS A SUM OF TRAVELLING WAVES.

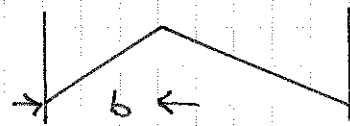
FOR EXAMPLE: $\sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$ IS A STANDING WAVE.

BY A TRIG. IDENTITY THIS IS ALSO

$$\frac{1}{2} \left[\sin \frac{n\pi}{l} (x-ct) + \sin \frac{n\pi}{l} (x+ct) \right]$$

WHICH IS THE SUM OF TWO TRAVELLING SINE WAVES OF THE SAME SHAPE - BUT MOVING IN OPPOSITE DIRECTIONS.

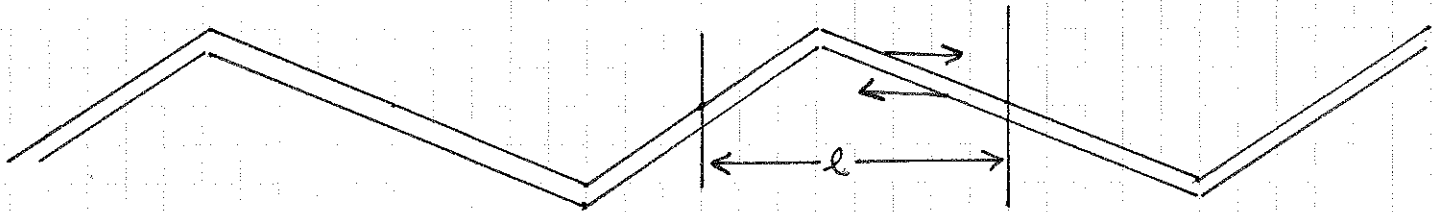
EXAMPLE THE PLUCKED STRING



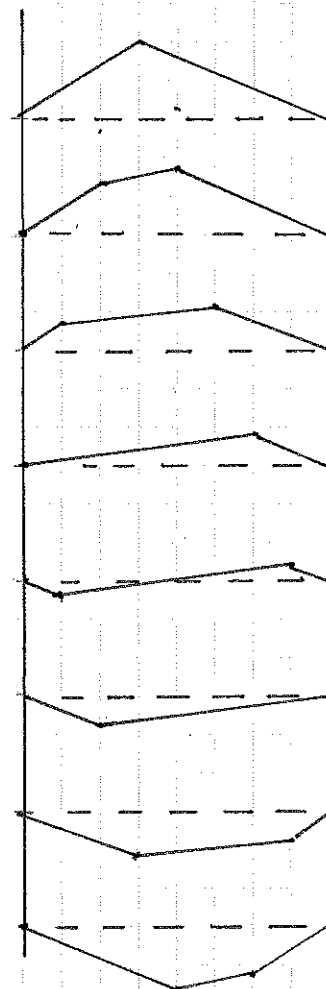
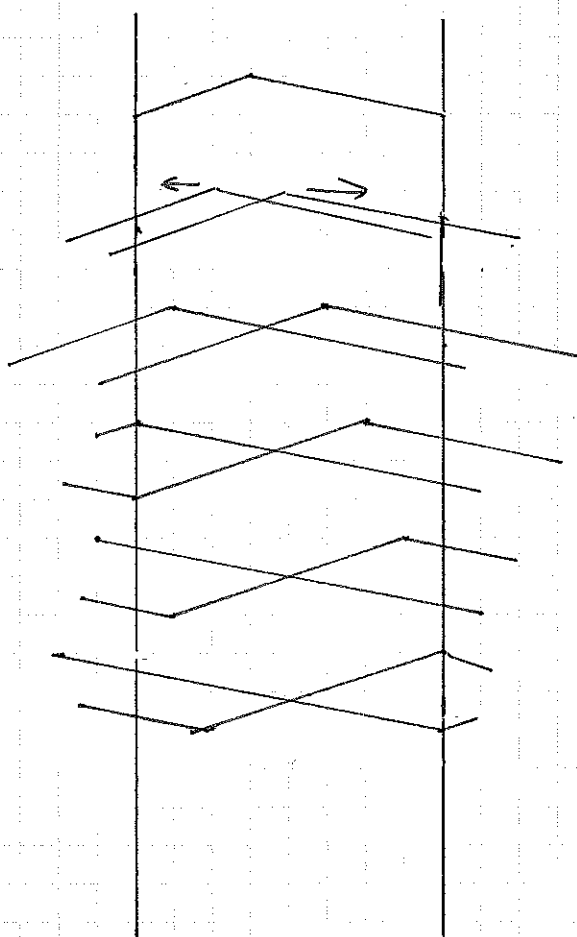
WE FOUND
$$S(x,t) = \frac{2s_0 b^2}{\pi^2 b(l-b)} \sum_n \frac{1}{n^2} \frac{\sin \frac{n\pi b}{l}}{l} \frac{\sin \frac{n\pi x}{l}}{l} \cos \frac{n\pi ct}{l}$$

IN TERMS OF STANDING WAVE MODES. BUT WE DID NOT GET A CLEAR PICTURE OF WHAT $S(x,t)$ 'LOOKS' LIKE. (AS OPPOSED TO WHAT IT 'SOUNDS' LIKE, WHICH WE LEARN FROM FOURIER).

NOW WE SEE THAT THE PLUCKING ALSO GENERATES 2 TRAVELLING WAVES WHICH PROPAGATE IN OPPOSITE DIRECTIONS, AND ARE REFLECTED, WITH INVERSION, AT THE FIXED ENDS. WE CAN IMAGINE ADDITIONAL PIECES OF THE TRAVELLING WAVES TO PROVIDE THE PAPER REFLECTIONS



WE SEE ONLY THE SUM OF THESE TWO BETWEEN 0 AND l

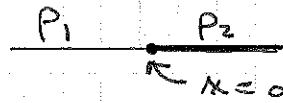


$\frac{1}{16}$ CYCLE

$\frac{1}{4}$ CYCLE

REFLECTION AND TRANSMISSION

SUPPOSE THE STRING IS MADE OF 2 PIECES OF DIFFERENT DENSITIES



WHAT HAPPENS TO A TRAVELLING WAVE AT THE JUNCTION?

ON EACH PIECE THE WAVES HAVE VELOCITIES

$$c_1 = \sqrt{T/\rho_1} \quad \text{AND} \quad c_2 = \sqrt{T/\rho_2} \quad (T_1 = T_2 \text{ OF COURSE})$$

AT THE JUNCTION WE MUST HAVE $S_1(0,t) = S_2(0,t)$

AND ALSO $S_1'(0,t) = S_2'(0,t)$ (NO MASS AT THE JUNCTION)

SUPPOSE WE HAVE AN INCIDENT WAVE ON PIECE 1, $F(t - \frac{x}{c_1})$

AFTER IT HITS THE JUNCTION WE MIGHT EXPECT A REFLECTED WAVE

ALSO, SO $S_1(x,t) = F(t - \frac{x}{c_1}) + G(t + \frac{x}{c_1})$ [$x < 0$]

ON THE 2ND PIECE WE WILL HAVE ONLY A TRANSMITTED WAVE:

$$S_2 = H(t - \frac{x}{c_2}) \quad [x > 0]$$

THEN AT $x=0$, $F(t) + G(t) = H(t)$

$$\text{AND} \quad -\frac{1}{c_1} \dot{F}(t) + \frac{1}{c_1} \dot{G}(t) = -\frac{1}{c_2} \dot{H}(t) \quad \left[\frac{d}{dx} f(t - \frac{x}{c}) \Big|_{x=0} = -\frac{1}{c} \frac{df}{dt} \right]$$

THE 1ST EQUATION SUGGESTS (AIDED BY A BIT OF 'INTUITION')

$$G = A_R F \quad \text{AND} \quad H = A_T F$$

$$\text{AND HENCE} \quad 1 + A_R = A_T$$

$$\text{THE 2ND EQUATION} \Rightarrow \quad -\frac{1}{c_1} + \frac{A_R}{c_1} = -\frac{A_T}{c_2}$$

$$\text{AFTER SOME ALGEBRA:} \quad A_R = \frac{c_2 - c_1}{c_2 + c_1} \quad A_T = \frac{2c_2}{c_2 + c_1}$$

SOME SPECIAL CASES ARE:

a) $c_1 = c_2 \Rightarrow$ REALLY JUST ONE STRING

THEN $A_R = 0$, AND $A_T = 1$ AS EXPECTED.

b) $\rho_2 \rightarrow \infty \Rightarrow c_2 \rightarrow 0$ AND THE STRING IS FIXED AT $x=0$.

$\Rightarrow A_R = -1$, $A_T = 0$ AS EXPECTED FROM P 248

c) $\rho_2 \rightarrow 0 \Rightarrow c_2 \rightarrow \infty$ AND THE STRING IS 'FREE' AT $x=0$.

$\Rightarrow A_R = +1$ BUT $A_T = 2$. $A_R = +1$ IS EXPECTED FROM P 249.

ENERGY CONSIDERATIONS

THE INCIDENT WAVE CARRIES ENERGY WITH IT!

$$\text{K.E.} = \frac{\rho_1}{2} \int \dot{F}^2 dx = \frac{T}{2c_1^2} \int \dot{F}^2 dx$$

$$\text{P.E.} = \frac{T}{2} \int F'^2 dx = \frac{T}{2c_1^2} \int \dot{F}^2 dx \quad \text{SINCE } F' = -\frac{1}{c_1} \dot{F}$$

$$E = \frac{T}{c_1^2} \int \dot{F}^2 dx \sim \frac{1}{c_1^2}, \text{ AND } E \text{ MOVES WITH THE WAVE.}$$

FOR AN OBSERVER AT FIXED x , THE MORE RELEVANT QUANTITY IS THE ENERGY FLOW PAST x . SINCE THE WAVE MOVES WITH VELOCITY c_1 , THE FLOW IS $Ec_1 \sim \frac{1}{c_1}$ (INCIDENT WAVE).

AT THE JUNCTION WE EXPECT THE SAME AMOUNT OF ENERGY TO FLOW OUT AS FLOWS IN.

FOR THE REFLECTED WAVE, THE FLOW IS $\sim \frac{A_R^2}{c_1}$

FOR THE TRANSMITTED WAVE, THE FLOW IS $\sim \frac{A_T^2}{c_2}$

THE TOTAL FLOW OUT IS THEN \sim

$$\frac{A_R^2}{c_1} + \frac{A_T^2}{c_2} = \frac{1}{c_1} \left\{ \frac{(c_2 - c_1)^2}{(c_2 + c_1)^2} + \frac{c_1}{c_2} \frac{4c_2^2}{(c_2 + c_1)^2} \right\} = \frac{1}{c_1}$$

SO ENERGY IS INDEED CONSERVED AT THE JUNCTION.

NOTE THAT IN THE LIMITING CASES b) AND c) ABOVE, NO ENERGY IS TRANSMITTED ACROSS THE JUNCTION.

TRAVELLING WAVES ON A STIFF STRING

WE WISH TO CONSIDER THE EFFECT OF 'STIFFNESS' OF OUR STRETCHED STRING — ITS RESISTANCE TO BENDING. THE STIFF STRING IS ESSENTIALLY AN ELASTIC BAR UNDER TENSION. WE STILL ASSUME THE MIDPLANE OF THE BAR OR STRING HAS CONSTANT LENGTH. WE CONSIDER ONLY TRANSVERSE WAVES HERE.

WE MUST ADD THE EFFECT OF TENSION TO OUR DERIVATION OF THE EQUATION OF MOTION GIVEN ON P 238. THE KINETIC ENERGY REMAINS

AS BEFORE: $KE = \frac{\rho A}{2} \int \dot{s}^2 dx$ (NEGLECTING ROTARY K.E.)

BUT THE P.E. HAS TERMS IN ADDITION TO $\frac{YI}{2\rho} \int s''^2 dx$.

THERE IS A NEW PIECE $\frac{I}{2} \int s'^2 dx$ AS FOR A STRETCHED STRING (T=TENSION)

ALSO, THE INTEGRAND $\frac{YI}{2\rho} s''^2$ REQUIRES MODIFICATION. IT WAS DERIVED BY CONSIDERING THAT THE FORCE ACROSS AN ELEMENT OF THE BAR IS $YA \Delta l / l_0$. BUT DUE TO THE TENSION, THE EQUILIBRIUM LENGTH OF THE BAR PRIOR TO BENDING IS l RATHER THAN l_0 , AS RELATED BY $\frac{l}{l_0} = Y \frac{l - l_0}{l_0}$ OR $\frac{l}{l_0} = \frac{T + YA}{YA}$. HENCE

FORCE DUE TO STRETCH $\Delta l = YA \frac{\Delta l}{l_0} = (T + YA) \Delta l / l \Rightarrow \frac{YI}{2\rho} s''^2 \rightarrow \left(Y + \frac{T}{A}\right) \frac{I}{2\rho} s''^2$

AND $PE = \frac{1}{2} \left(Y + \frac{T}{A}\right) \frac{I}{\rho} \int s''^2 dx + \frac{I}{2} \int s'^2 dx$

THE EQUATION OF MOTION IS THEN AT ONCE

$$\rho A \ddot{s} - T s'' + \left(Y + \frac{T}{A}\right) \frac{I}{\rho} s'''' = 0$$

WE DEFINE $a^2 = T/\rho$, $b^2 = (YA + T)/\rho$ AND $d^2 = I/A$

THEN $\boxed{\ddot{s} = a^2 s'' - b^2 d^2 s''''}$

IF THE 'STIFFNESS' IS IGNORED, $\ddot{s} = a^2 s''$, AND a IS THE WAVE VELOCITY.

WE LEAVE IT TO YOU TO SOLVE THIS EQUATION FOR STANDING WAVES

WHAT ABOUT TRAVELLING WAVES?

THIS WAVE EQUATION DOES NOT HAVE SOLUTIONS LIKE

$s(x,t) = f(x-ct) + g(x+ct)$ FOR ARBITRARY f & g .

BUT WE CAN LOOK FOR OSCILLATORY SOLUTIONS, AND TRY TO BUILD THE GENERAL SOLUTION OUT OF THESE, A LA FOURIER.
(THE DIFFERENTIAL EQUATION IS LINEAR \Rightarrow SUPERPOSITION WORKS!)

WE TRY $S(x,t) = \cos(kx \pm \omega t)$

[THIS MAY WORK SINCE $\cos'''(x) = -\cos''(x) = \cos(x)$]

PLUGGING IN: $\omega^2 = a^2 k^2 + b^2 d^2 k^4$

THE VELOCITY OF THIS WAVE IS $C = \frac{\omega}{k} = \sqrt{a^2 + b^2 d^2 k^2}$

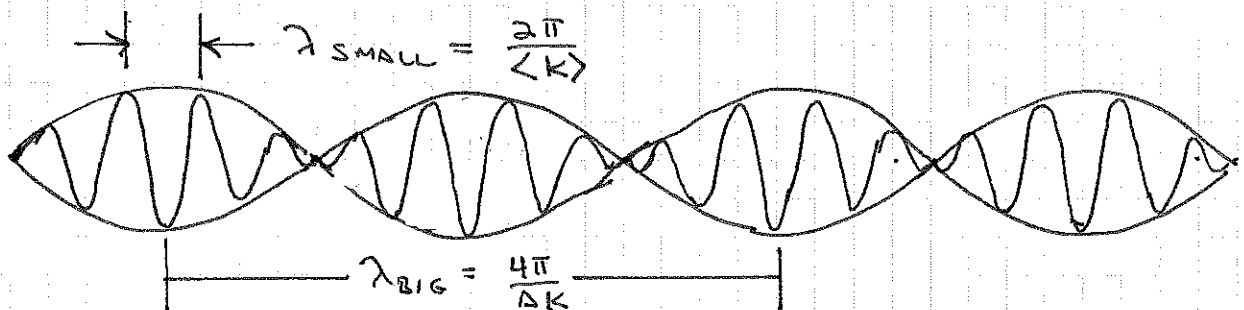
WHICH DEPENDS ON THE WAVELENGTH ($\lambda = 2\pi/k$), OR EQUIVALENTLY, ON THE FREQUENCY.

THIS PHENOMENON IS CALLED DISPERSION. IF A WAVE IS THE SUPERPOSITION OF SEVERAL SINE WAVES OF DIFFERENT FREQUENCY, IT DOES NOT HAVE A UNIQUE VELOCITY - AND THE SHAPE OF THE WAVE CHANGES WITH TIME AS THE VARIOUS COMPONENT SINE WAVES 'DISPERSE'!

PHASE VELOCITY AND GROUP VELOCITY

WHEN WE COMBINE WAVES OF VERY SIMILAR FREQUENCY, AN IMPORTANT RESULT EMERGES.

EXAMPLE $f(x,t) = \cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)$
 $= 2 \cos\left(\frac{k_1+k_2}{2} x - \frac{\omega_1+\omega_2}{2} t\right) \cos\left(\frac{k_1-k_2}{2} x - \frac{\omega_1-\omega_2}{2} t\right)$



THE SHORT WAVELENGTH RIPPLES MOVE WITH VELOCITY

$V_p = \frac{\langle \omega \rangle}{\langle k \rangle} = \text{PHASE VELOCITY}$

THE LONG WAVELENGTH ENVELOPE IS A KIND OF 'BEAT' PHENOMENON, AND MOVES WITH VELOCITY

$V_g = \frac{\Delta \omega}{\Delta k} = \text{GROUP VELOCITY}$ (NAMES DUE TO RAYLEIGH)

EXAMPLE IN A BAR WITHOUT TENSION, $a=0$, AND TRANSVERSE WAVES

$$\text{OBEY } \omega = bdk^2$$

$$\Rightarrow v_p = bdk \quad \text{WHILE } v_g = \frac{d\omega}{dk} = 2bdk = 2v_p$$

THE ENVELOPE MOVES AHEAD WITH TWICE THE VELOCITY OF THE WAVES WHICH IT IS COMPOSED OF!

FOURIER ANALYSIS

WE WISH TO CONSIDER HOW WE CAN BUILD UP AN ARBITRARY WAVE SOLUTION $f(x,t)$, SUPPOSING WE KNOW THE INITIAL PULSE SHAPE $f(x,0) = F(x)$.

IF THE STRING WERE OF FINITE LENGTH l , WE KNOW WE COULD WRITE

$$F(x) = \sum_n A_n \cos \frac{n\pi x}{l} + B_n \sin \frac{n\pi x}{l}$$

AS $l \rightarrow \infty$, $k = \frac{n\pi}{l}$ TAKES ON ESSENTIALLY A CONTINUUM OF VALUES, AND WE MIGHT EXPECT

$$F(x) = \int_0^{\infty} (A(k) \cos kx + B(k) \sin kx) dk$$

WITH A AND B REAL FUNCTIONS OF THE WAVE NUMBER k .

$$\text{RECALL } \cos kx = \frac{e^{ikx} + e^{-ikx}}{2} \quad \sin kx = \frac{e^{ikx} - e^{-ikx}}{2i}$$

$$\text{SO } F(x) = \int_0^{\infty} \left(\frac{A(k) - iB(k)}{2} \right) e^{ikx} dk + \int_0^{\infty} \left(\frac{A(k) + iB(k)}{2} \right) e^{-ikx} dx$$

$$\text{LET } C(k) = \frac{A - iB}{2} \quad \text{AND SO } C^* = \frac{A + iB}{2} \quad (k > 0)$$

IF WE DEFINE $C(-k) = C^*(k)$, THEN

$$\int_0^{\infty} C^*(k) e^{-ikx} dk = \int_{-\infty}^0 C(k) e^{ikx} dk$$

$$\text{AND } \boxed{F(x) = \int_{-\infty}^{\infty} C(k) e^{ikx} dk}$$

$$\text{THE INVERSE IS } \boxed{C(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(x) e^{-ikx} dx}$$

THE INVERSE RELATION CAN BE VERIFIED BY NOTING

$$\int_{-\infty}^{\infty} e^{i(k'-k)x} dx = 2\pi \delta(k'-k) \left[= \lim_{a \rightarrow \infty} \int_{-a}^a e^{i(k'-k)x} dx = \lim_{a \rightarrow \infty} \frac{2 \sin(k'-k)a}{k'-k} \right]$$

FINALLY THE PROPAGATING WAVE IS

$$f(x,t) = \int_{-\infty}^{\infty} C(k) e^{i(kx - \omega t)} dk$$

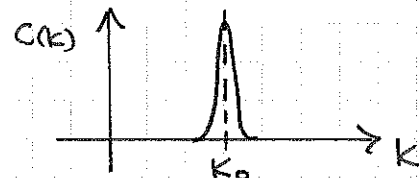
WHERE $\omega = \omega(k)$ IS KNOWN FROM THE DISPERSION RELATION DERIVED FROM THE WAVE EQUATION AS ON P 254.

[OF COURSE, WE TAKE THE REAL PART OF $f(x,t)$ AT THE END OF THE CALCULATION.]

GROUP VELOCITY REVISITED

CONSIDER A WAVE THAT IS ALMOST A PURE FREQUENCY.

⇒ ITS FOURIER COMPONENT SPECTRUM IS VERY NARROW



$$\text{THEN } \omega(k) \approx \omega(k_0) + \left. \frac{d\omega}{dk} \right|_{k_0} (k - k_0)$$

$$\begin{aligned} \text{AND } f(x,t) &= \int_{-\infty}^{\infty} C(k) e^{i \left[kx - \omega(k)t - \left. \frac{d\omega}{dk} \right|_{k_0} (k - k_0)t \right]} dk \\ &= e^{-i \left[\omega(k_0) - \left. \frac{d\omega}{dk} \right|_{k_0} \right] t} \int_{-\infty}^{\infty} C(k) e^{i k \left(x - \left. \frac{d\omega}{dk} \right|_{k_0} t \right)} dk \end{aligned}$$

$$\approx F \left(x - \left. \frac{d\omega}{dk} \right|_{k_0} t \right) \quad \text{WHERE } F(x) = \int_{-\infty}^{\infty} C(k) e^{ikx} dk$$

I.E. $F(x)$ IS THE SHAPE AT $t=0$

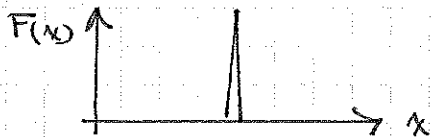
THUS THE WAVE KEEPS ITS SHAPE AND MOVES WITH VELOCITY

$$v_g = \frac{d\omega}{dk} = \underline{\text{GROUP VELOCITY}}$$

UNCERTAINTY RELATIONS

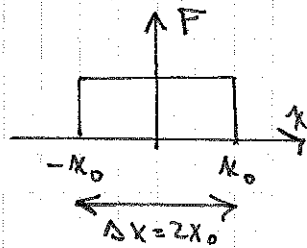
A VERY SHARP PULSE MUST BE COMPOSED OUT OF A LARGE RANGE OF FREQUENCIES, ACCORDING TO OUR FOURIER ANALYSIS.

$$F(x) = \delta(x) \implies C(k) = \frac{1}{2\pi} \int e^{ikx} \delta(x) dx = \frac{1}{2\pi}$$

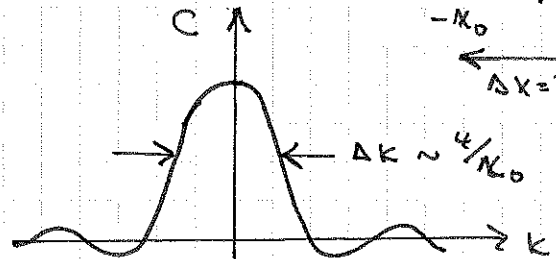


LIKEWISE A WAVE OF A PURE FREQUENCY MUST EXTEND OVER ALL SPACE!

AN INTERMEDIATE CASE MIGHT BE $F(x) = \begin{cases} 1 & -x_0 < x < x_0 \\ 0 & \text{OTHERWISE} \end{cases}$



THEN $C(k) = \frac{1}{2\pi} \int_{-x_0}^{x_0} e^{ikx} dx = \frac{\sin kx_0}{\pi k}$



THE WIDTHS OF THE FUNCTIONS $F(x)$ AND $C(k)$ OBEY $\Delta x \Delta k \sim 8$

A GENERAL RESULT OF FOURIER ANALYSIS IS THAT $\Delta x \Delta k \geq 8 \ln 2$

THIS IS A KIND OF UNCERTAINTY RELATION IN THAT ONE CANNOT HAVE SIMULTANEOUSLY A NARROW PULSE IN SPACE AND IN FREQUENCY.

THE MINIMUM PRODUCT $\Delta x \Delta k$ IS ACHIEVED BY GAUSSIAN PULSES:

$$F(x) = e^{-\frac{x^2}{2\sigma^2}} \implies C(k) = \frac{1}{2\pi} \int e^{ikx} e^{-\frac{x^2}{2\sigma^2}} dx = \frac{1}{2\pi} e^{-\frac{\sigma^2 k^2}{2}} \int e^{-\frac{(x - i\sigma^2 k)^2}{2\sigma^2}} dx$$

SO $C(k) \sim e^{-\frac{\sigma^2 k^2}{2}}$ = GAUSSIAN ALSO!

CONSTANT NOT A FUNCTION OF k

SO THE FWHM OF F IS $2\sqrt{2 \ln 2} \sigma = 2.35 \sigma$

WHILE THE FWHM OF C IS $\frac{2.35}{\sigma}$

$$\implies \Delta x \Delta k = (2.35)^2 = 5.54$$

[IT IS PERHAPS MORE MEMORABLE TO USE THE R.M.S. WIDTHS, FOR WHICH $\sigma_x \sigma_k = 1$]

Group Velocity

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(December 4, 1996)

It seems that the concept of group velocity was first enunciated by Hamilton in 1841 in published abstracts of works that never appeared.¹ Hamilton considered a wave $\cos(kx - \omega t)$ defined only for negative x at $t = 0$ and incident on a dispersive medium that occupies the region $x > 0$. He concluded “that the velocity with which such vibration spreads into those portions of the vibratory medium which were previously undisturbed is in general different for the velocity of passage of a given phase from one particle to another within that portion of the medium which is already fully agitated; since we have velocity of transmission of phase $= \omega/k$ but velocity of propagation of vibrating motion $= d\omega/dk$.” However, these results were largely ignored.

The group-velocity concept became widely known after being (re)introduced by Stokes in 1876 in a hydrodynamic context,² and the greater generality of the concept emphasized by Rayleigh in 1877 in sec. 191 of his book *The Theory of Sound*.³ The early history of the group-velocity concept is well summarized in the book *The Propagation of Disturbances in Dispersive Media* by T.H. Havelock (Cambridge U. Press, 1914).

I give two answers to the question of how one knows that wave energy propagates with the group velocity, both of which are “standard”.⁴ The discussion will be restricted to wave motion along the x axis for brevity.

1. The total energy E associated with a wave of amplitude $f(x, t)$ at a time t can in general be written

$$E(t) = \int (Af^2 + B\dot{f}^2) dx, \quad (1)$$

where $\dot{f} = \partial f/\partial t$ and either of A or B might be zero depending on the physical system. Typically, the term Af^2 is associated with energy stored in the wave medium due to the strain of the wave while $B\dot{f}^2$ is the kinetic energy of the medium due to the wave motion. For example, $B = 0$ for an electromagnetic wave while $A = 0$ for the granular systems recently studied by Swinney *et al.*, *Nature* **382**, 793 (1996).⁵

A question arises when one wishes to interpret the quantity $Af^2 + B\dot{f}^2$ as an energy density. As the wave changes with time it is possible that the energy moves in space. If the wave amplitude has the form of a travelling wave, $f(x - vt)$, then both f^2 and \dot{f}^2 are functions of a single variable, $x - vt$, and the energy can be said to be propagating with velocity v .

¹http://puhep1.princeton.edu/~mcdonald/examples/optics/hamilton_pria_1_267_41.pdf

http://puhep1.princeton.edu/~mcdonald/examples/optics/hamilton_pria_1_341_41.pdf

²Problem 11 of the Smith’s Prize examination papers (Feb. 2, 1876), in *Mathematical and Physical Papers*, Vol. 5 (Johnson Reprint Co., New York, 1966), p. 362.

³See also, http://puhep1.princeton.edu/~mcdonald/examples/optics/rayleigh_nature_25_52_81.pdf

⁴A third generic argument has been given by G.B. Whitham, *Comm. Pure Appl. Math.* **14**, 675 (1961),

http://puhep1.princeton.edu/~mcdonald/examples/fluids/whitham_cpam_14_675_61.pdf

See also sec. 2.1 of <http://puhep1.princeton.edu/~mcdonald/examples/biaxial.pdf>

⁵http://puhep1.princeton.edu/~mcdonald/examples/fluids/umbanhowar_nature_382_793_96.pdf

The concept of group velocity arises when a waveform is Fourier analyzed into a set of harmonic waves

$$f(x, t) = \int F(k) e^{i(kx - \omega t)} dk, \quad (2)$$

characterized by wave number k and frequency $\omega(k)$ where the latter relation can be non-trivial due to dispersion in the wave medium. The harmonic wave of frequency ω has phase velocity $v_p = \omega/k$ which is not necessarily equal to the velocity v of the localized waveform. (In this discussion only the real part of f has physical significance.)

The spectral function $F(k)$ can be determined by the Fourier inverse relation for the wave at a fixed time, say $t = 0$:

$$F(k) = \frac{1}{2\pi} \int f(x, 0) e^{-ikx} dx. \quad (3)$$

However, we don't need to use this result in the present case.

The usual argument asks us to restrict our attention to waveforms whose spectral function $F(k)$ is narrow enough that the dispersion relation can be approximated as

$$\omega = \omega(k_0) + \frac{d\omega(k_0)}{dk} (k - k_0), \quad (4)$$

i.e., the leading terms in a Taylor expansion about some central wave number k_0 . (The sign of k_0 determines whether the pulse moves in the $+x$ or $-x$ direction.) Certainly this approximation breaks down for very short pulses in highly dispersive media.

In the approximation (4) we have

$$\begin{aligned} f(x, t) &= e^{i[k_0(d\omega(k_0)/dk) - \omega_0]t} \int F(k) e^{ik[x - (d\omega(k_0)/dk)t]} dk \\ &= e^{i[k_0(d\omega(k_0)/dk) - \omega_0]t} f(x - (d\omega(k_0)/dk)t, 0). \end{aligned} \quad (5)$$

That is, to within a phase factor of unit modulus the waveform $f(x, t)$ is a function of a single variable, $x - (d\omega(k_0)/dk)t$, and so can be said to propagate with the group velocity

$$v_{\text{group}} = \frac{d\omega(k_0)}{dk}. \quad (6)$$

As argued above, the wave energy propagates with this velocity as well.

If the waveform is highly localized in space it will have a broad spectral content and the linear approximation to the dispersion relation may not suffice. If so, the waveform will change shape (disperse) as it propagates and the group velocity is not well defined.

This well-known argument appears to be due to Lord Kelvin, Proc. Roy. Soc. London **42**, 80 (1887),⁶ and is reproduced in much the above form in sec. 3 of the book by Havelock.

2. In 1877 both Reynolds and Rayleigh published articles relating energy flow to group velocity. Reynolds' discussion⁷ is based on water waves and can be found in sec. 273 of the book *Hydrodynamics* by H. Lamb, as well as in sec. 26 of *Mechanics of Deformable Bodies* by A. Sommerfeld (1946).

⁶http://puhepl.princeton.edu/~mcdonald/examples/fluids/thomson_prsl_42_80_87.pdf

⁷http://puhepl.princeton.edu/~mcdonald/examples/fluids/reynolds_nature_16_343_77.pdf

Rayleigh's argument⁸ has been reprinted in the Appendix to Vol. 1 of his book *The Theory of Sound* and is based on the general observation that dispersion in a physical system is always accompanied by absorption. While the latter can often be ignored as a first approximation it should not be left out of discussions of energy flow.

Here I repeat Rayleigh's argument, which seems to be less well-known than Kelvin's.

For a steady wave, energy is being transported into the medium at the same rate at which it is being absorbed, when averages are taken over a whole cycle of the wave. The power P absorbed by a mass m in the medium is $P = Fv$ where v is the velocity of the mass and $F = \gamma mv$ the dissipative force. Thus $P = \gamma mv^2$, and summing over all masses in some region, $P = 2\gamma T$, where T is the kinetic energy. Taking the time average, $\langle P \rangle = 2\gamma \langle T \rangle = \gamma E$, where E is the total energy. In writing $E = 2\langle T \rangle$ we suppose that the wave motion is a small departure from equilibrium so the restoring forces can be described by a quadratic potential, for which $E = 2\langle T \rangle$ according to the virial theorem of classical mechanics.

Consider a pure harmonic wave, $f = f_0 e^{i(kx - \omega t)}$, incident on a dispersive medium that occupies the half space $x > 0$. Because of absorption in the medium this wave dies out over some characteristic distance d . That is, the amplitude of the wave can be written $f_0 e^{-x/d}$ in the medium. Then the time average energy density is $\langle Af^2 + B\dot{f}^2 \rangle \equiv C f_0^2 e^{-2x/d}$.

The time-averaged power absorbed for $x > x_0$ in the medium is then

$$\langle P(x_0) \rangle = \gamma \int_{x_0}^{\infty} \langle Af^2 + B\dot{f}^2 \rangle dx = C\gamma f_0^2 \int_{x_0}^{\infty} e^{-2x/d} dx = \frac{C\gamma d f_0^2}{2} e^{-2x_0/d}. \quad (7)$$

The (time-averaged) rate of energy flow per unit area across the plane $x = x_0$ is the (time-averaged) energy density there times the desired velocity of energy flow, v_E . The rate of energy flow is thus $Cv_E f_0^2 e^{-2x_0/d}$. Comparing with eq. (7), we see that the energy flow velocity is given by

$$v_E = \frac{\gamma d}{2}. \quad (8)$$

To find distance d , we suppose that in the absence of absorption the harmonic solutions obey a known dispersion relation, $k = k(\omega)$. Then the equation of motion including absorption, taken to be velocity dependent, differs from that without absorption only by replacing the second time derivative $\partial^2/\partial t^2$ with the form

$$\frac{\partial^2}{\partial t^2} + \gamma \frac{\partial}{\partial t}, \quad (9)$$

where γ , whose dimensions are 1/time, characterizes the absorption process. The new dispersion relation that results on inserting our trial harmonic solution into the wave equation differs only in the term ω^2 being replaced by $\omega^2 + i\gamma\omega$. For weak absorption this is equivalent to replacing ω by $\omega + i\gamma/2$. The corresponding wave number is therefore $k(\omega + i\gamma/2) \approx k(\omega) + i(\gamma/2)(dk/d\omega)$, again ignoring terms of order γ^2 . The wave solution in the presence of absorption is therefore approximately

$$f(x > 0, t) = f_0 e^{-(\gamma/2)(dk/d\omega)x} e^{i(kx - \omega t)}. \quad (10)$$

⁸http://puhep1.princeton.edu/~mcdonald/examples/fluids/rayleigh_plms_9_21_77.pdf

Thus, the characteristic attenuation length is

$$d = \frac{2}{\gamma} \frac{d\omega}{dk}. \quad (11)$$

From eq. (8) the velocity of energy flow is

$$v_E = \frac{d\omega}{dk} = v_{\text{group}}. \quad (12)$$

An objection to this argument would be that it doesn't apply if the absorption is too strong. It may be that the heroic efforts of Sommerfeld and Brillouin (1914)⁹ to clarify signal propagation in the case of highly absorptive anomalous dispersion in optical media (where v_{group} exceeds the speed of light) have left the impression that the more ordinary case is similarly intricate.

⁹L. Brillouin, *Wave Propagation and Group Velocity* (Academic Press, New York, 1960).