

WATER WAVES

FINAL EXAM: FRI JAN 25

1:30 PM., PALMER 306A.

AS A FINAL EXAMPLE OF WAVE MOTION, WE CONSIDER WAVES IN WATER. THIS IS A RATHER LARGE TOPIC. WE DERIVE ONLY A FEW RESULTS, AND REFER YOU TO THE LITERATURE THEREAFTER.

THESE NOTES ARE LARGELY TAKEN FROM 'WAVES' BY CRAWFORD (BERKELEY PHYSICS, VOL. 3), AND FROM THE CLASSIC TOME 'HYDRODYNAMICS' BY LAMB (DOVER PAPERBACK). A BOOK WITH INTERESTING PHOTOS OF WAVE PHENOMENA, AS WELL AS ADVANCED DERIVATIONS IS 'WATER WAVES' BY STOKER.

THE SCOPE OF OUR APPROACH

WE MAKE SEVERAL RESTRICTIONS TO BE ABLE TO DERIVE SOME RESULTS FAIRLY QUICKLY.

- WE CONSIDER ONLY GRAVITY AS THE RESTORING FORCE. WE IGNORE SURFACE TENSION, AND THE EFFECT OF THE AIR ABOVE THE WATER. THE WIND IS RESPONSIBLE FOR THE GENERATION OF MANY WAVES IN THE FIRST PLACE - AND THERE ARE INTERESTING COUPLINGS BETWEEN AIR WAVES AND WATER WAVES.
- WE SUPPOSE THE DENSITY OF WATER IS CONSTANT. I.E. THE WATER IS INCOMPRESSIBLE. (THIS IS A GOOD APPROXIMATION.)
- WE IGNORE THE FRICTION OR VISCOSITY OF WATER. FEYNMAN CALLS THIS 'DRY WATER' - VOL II LECTURES 40, 41)
- WE CONSIDER ONLY PLANE WAVES, NOT CIRCULAR WAVES
- IN MOST FLUID PROBLEMS ONE IS INTERESTED IN THE FLOW OF THE FLUID. BUT IN WATER WAVES A MOLECULE OF WATER DOESN'T HAVE A NET FLOW, BUT OSCILLATES ABOUT AN EQUILIBRIUM POSITION. WE WILL USE A DESCRIPTION

$$\vec{r} = (x, y, z) = \text{EQUILIBRIUM POSITION}$$

$$\vec{s} = (s_x, s_y, s_z) = \vec{s}(x, y, z, t) = \text{DISPLACEMENT}$$

$$\vec{r} = \vec{r} + \vec{s} = \text{ACTUAL POSITION}$$

THEN $\vec{v} = \dot{\vec{s}}$ = VELOCITY OF THE WATER WHICH IS AT (x, y, z) WHEN AT EQUILIBRIUM, (NOT THE VELOCITY OF THE WATER PRESENTLY AT (x, y, z)).

BEFORE CONSIDERING $\vec{F} = m\vec{a}$, WE CAN LEARN A LOT ABOUT THE NATURE OF THE WAVES FROM TWO ADDITIONAL RESTRICTIONS.

CONSERVATION OF MATTER

CONSIDER A SMALL VOLUME $dV_{OL} = dx dy dz$ OF WATER AT EQUILIBRIUM. THE DISPLACED WATER OCCUPIES A DISTORTED CUBE, BUT THE VOLUME MUST BE THE SAME, SINCE THE DENSITY IS CONSTANT.

IF A BOX HAS EDGES WITH VECTORS $\vec{a}, \vec{b}, \vec{c}$, THE VOLUME IS

$$\vec{a} \cdot \vec{b} \times \vec{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

THE EDGE ORIGINALLY DESCRIBED BY $d\vec{x} = (dx, 0, 0)$ BECOMES $d\vec{x} + \vec{s}(x+dx, y, z) - \vec{s}(x, y, z) \sim \left(\left(1 + \frac{\partial s_x}{\partial x}\right) dx, \frac{\partial s_y}{\partial x} dx, \frac{\partial s_z}{\partial x} dx \right)$

SIMILARLY $(0, dy, 0) \rightarrow \left(\frac{\partial s_x}{\partial y} dy, \left(1 + \frac{\partial s_y}{\partial y}\right) dy, \frac{\partial s_z}{\partial y} dy \right)$

$(0, 0, dz) \rightarrow \left(\frac{\partial s_x}{\partial z} dz, \frac{\partial s_y}{\partial z} dz, \left(1 + \frac{\partial s_z}{\partial z}\right) dz \right)$

AND SO $dx dy dz \rightarrow \begin{vmatrix} 1 + \frac{\partial s_x}{\partial x} & \frac{\partial s_y}{\partial x} & \frac{\partial s_z}{\partial x} \\ \frac{\partial s_x}{\partial y} & 1 + \frac{\partial s_y}{\partial y} & \frac{\partial s_z}{\partial y} \\ \frac{\partial s_x}{\partial z} & \frac{\partial s_y}{\partial z} & 1 + \frac{\partial s_z}{\partial z} \end{vmatrix} dx dy dz$

$\sim dx dy dz \left(1 + \frac{\partial s_x}{\partial x} + \frac{\partial s_y}{\partial y} + \frac{\partial s_z}{\partial z} \right) + O\left(\frac{\partial s}{\partial x}\right)^2 + \dots$

SO FOR DISPLACEMENTS WHICH HAVE REASONABLY SMALL DERIVATIVES,

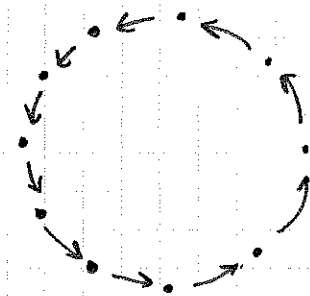
WE MUST HAVE $\boxed{\nabla \cdot \vec{s} = \frac{\partial s_x}{\partial x} + \frac{\partial s_y}{\partial y} + \frac{\partial s_z}{\partial z} = 0}$

(THE DISPLACEMENTS THEMSELVES NEED NOT BE SMALL.)

NO WHIRL POOLS

WE ADD ANOTHER RESTRICTION;
NO WHIRLPOOL MOTION

WE REQUIRE $\oint \vec{s} \cdot d\vec{l} = 0$
ANY LOOP



NOT ALLOWED

PHOTOGRAPH OF THE DISPLACEMENTS AT SOME TIME t

THUS OUR TREATMENT EXCLUDES MOTIONS WHICH ARE IN FACT PHYSICALLY POSSIBLE — SUCH AS A ROTATING BUCKET OF WATER, WATER SPOUTS AND MAELSTROMS, AND A WHOLE CLASS OF WAVES — VORTEX WAVES (= SMOKE RINGS).

THE RELATION $\oint_{\text{LOOP}} \vec{s} \cdot d\vec{l} = 0$ IS LIKE THAT FOR A CONSERVATIVE FORCE. HENCE A KIND OF 'POTENTIAL' MUST EXIST, $\phi(x, y, z, t)$ SUCH THAT $\vec{s} = \vec{\nabla}\phi$. ϕ IS A SCALAR FUNCTION.

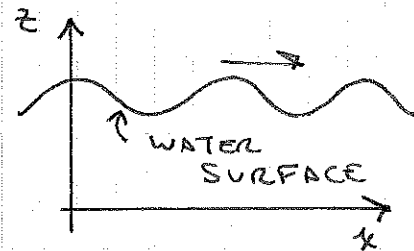
ALSO WE LEARN AT ONCE THAT $\boxed{\vec{\nabla} \times \vec{s} = 0}$ (STOKES' THEOREM)

$$\text{i.e. } \frac{\partial s_y}{\partial z} - \frac{\partial s_z}{\partial y} = \frac{\partial s_z}{\partial x} - \frac{\partial s_x}{\partial z} = \frac{\partial s_x}{\partial y} - \frac{\partial s_y}{\partial x} = 0$$

THE FORM OF THE WAVES

OUR RESTRICTIONS, $\vec{\nabla} \cdot \vec{s} = 0$, $\vec{\nabla} \times \vec{s} = 0$, COMBINED WITH THE BOUNDARY CONDITIONS DISCUSSED BELOW ALMOST COMPLETELY DETERMINE THE FORM OF THE WAVES — WITHOUT ANY CONSIDERATION OF $\vec{F} = m\vec{a}$!

WE CONSIDER MOTION ONLY IN THE x - z PLANE, WHERE z IS VERTICAL, x & y ARE HORIZONTAL. LET $z=0$ BE THE EQUILIBRIUM SURFACE.



WE ALSO TAKE $s_y = 0$ EVERYWHERE \Rightarrow PLANE WAVE MOTION.

IN THE SPIRIT OF FOURIER WE SEEK SOLUTIONS WHICH ARE SINE WAVES. SUPPOSE THE VERTICAL MOTION TAKES THE FORM OF A TRAVELLING SINE WAVE

$$s_z = F(z) \sin(kx - \omega t)$$

AN OBSERVER READILY SEES ONLY THE WAVE AT THE SURFACE $z=0$, BUT WE ALLOW THE POSSIBILITY THAT THE SIZE OF THE WAVE DISPLACEMENT VARIES WITH DEPTH.

IN GENERAL, $s_x = G(x, z, t)$.

BUT G IS NOT ARBITRARY, GIVEN s_z !

$$\begin{aligned} \vec{\nabla} \cdot \vec{s} = 0 &\Rightarrow \frac{\partial s_x}{\partial x} = -\frac{\partial s_z}{\partial z} \Rightarrow \frac{\partial G}{\partial x} = -F'(z) \sin(kx - \omega t) \\ &\Rightarrow G = \frac{F'(z)}{k} \cos(kx - \omega t) \end{aligned}$$

$$\nabla \times \vec{S} = 0 \Rightarrow \frac{\partial S_x}{\partial z} = \frac{\partial S_z}{\partial x} \Rightarrow \frac{F''}{K} \cos(kx - \omega t) = KF \cos(kx - \omega t)$$

$$\text{OR } F'' = K^2 F \Rightarrow F = A e^{Kz} + B e^{-Kz}$$

$$\text{AND } S_x = (A e^{Kz} - B e^{-Kz}) \omega (kx - \omega t)$$

$$S_z = (A e^{Kz} + B e^{-Kz}) \sin(kx - \omega t)$$

BOUNDARY CONDITION AT THE BOTTOM

LET $z = -h$ BE THE BOTTOM OF THE OCEAN. CERTAINLY WE MUST HAVE $S_z(x, y, -h, t) = 0$ AS OUR BOUNDARY CONDITION!

$$\text{HENCE } A e^{-Kh} + B e^{Kh} = 0 \Rightarrow B = -A e^{2Kh}$$

AND SO

$S_x = C \cosh K(z+h) \omega (kx - \omega t)$ $S_z = C \sinh K(z+h) \sin(kx - \omega t)$	$C = 2A e^{-Kh}$
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MOTION OF A MOLECULE

A FAMOUS GEOMETRICAL RELATION CAN NOW BE OBSERVED. THE MOTION OF A MOLECULE INITIALLY AT DEPTH z IS AN

ELLIPSE :

$$\frac{S_x^2}{\cosh^2 K(z+h)} + \frac{S_z^2}{\sinh^2 K(z+h)} = C^2$$

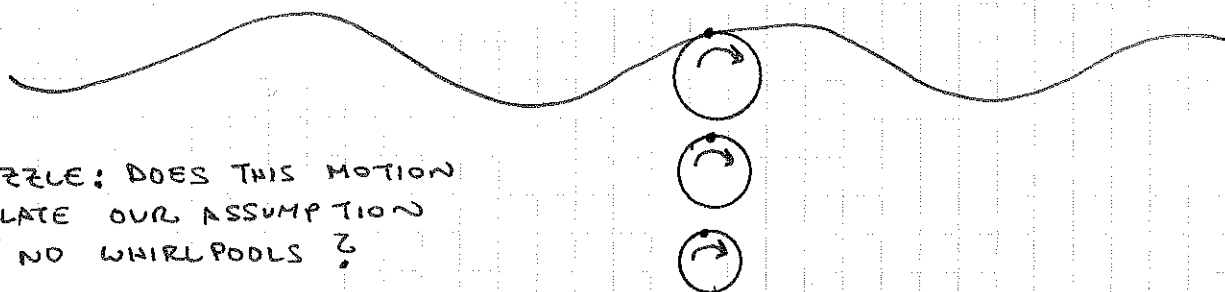
SOME SPECIAL CASES ARE NOTE WORTHY:

a) DEEP WATER ($h \rightarrow \infty$)

$$\cosh^2 K(z+h) \sim \sinh^2 K(z+h) \sim \text{CONSTANT} \cdot e^{2Kz}$$

$$\text{SO } S_x^2 + S_z^2 = \text{CONSTANT} \cdot e^{2Kz}$$

THE MOTION IS A CIRCLE, WHOSE RADIUS DECREASES WITH DEPTH.



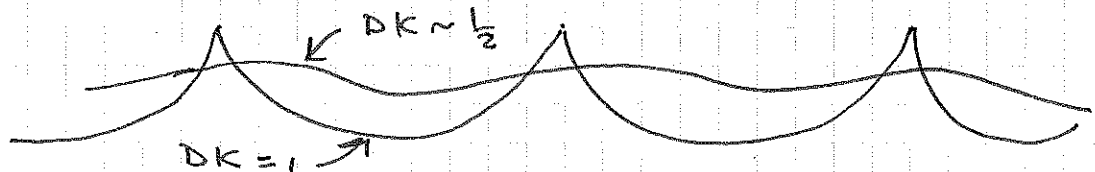
PUZZLE: DOES THIS MOTION VIOLATE OUR ASSUMPTION OF NO WHIRLPOOLS?

AT THE SURFACE $z=0$, SAY AT TIME $t=0$, THE SHAPE IS

$$R_x = x + D \omega D K x = \frac{1}{K} (Kx + DK \cos Kx) \sim x + DK \cos \alpha \quad [\alpha = Kx]$$

$$R_z = D \sin Kx = \frac{1}{K} DK \sin Kx \sim DK \sin \alpha$$

THIS IS THE PATTERN OF A POINT AT RADIUS DK ON A ROLLING WHEEL OF RADIUS 1.



THE EXTREME CASE $DK=1$ IS A CYCLOID. FOR $DK \ll 1$, THE SURFACE LOOKS LIKE A SINE WAVE.

b) SHALLOW WATER ($h \rightarrow 0, Kh \ll 1$)

THE WAVELENGTH $\lambda = \frac{2\pi}{K}$ IS MUCH LONGER THAN THE DEPTH OF THE WATER.

TIDEN $\sin K(z+h)$ IS ALWAYS VERY SMALL COMPARED TO $\cos K(z+h) \sim 1$

\Rightarrow THE WATER MOVES MAINLY HORIZONTALLY, SLOWLY BACK AND FORTH



THE HORIZONTAL MOTION IS LARGE NEAR THE BOTTOM, WHICH IS POSSIBLE IF WE IGNORE FRICTION.

ALL OF THE ABOVE RESULTS COME JUST FROM THE CONSERVATION OF MATTER AND THE ABSENCE OF WHIRLPOOLS. WE HAVEN'T USED $\vec{F} = m\vec{a}$ YET. THIS IS NEEDED TO DETERMINE THE VELOCITY OF THE WAVES.

WE CAN GET SOME INSIGHT INTO THE VELOCITY OF THE WAVES FROM DIMENSIONAL ANALYSIS!

DIMENSIONAL ANALYSIS

IF WE CAN FIND THE DISPERSION RELATION $\omega = \omega(k)$, WE IMMEDIATELY KNOW THE WAVE VELOCITIES:

$$v_{\text{PHASE}} = \frac{\omega}{k} \quad v_{\text{GROUP}} = \frac{d\omega}{dk}$$

WE CAN LEARN A LOT FROM DIMENSIONAL ANALYSIS. ω MIGHT DEPEND ON:

k OF COURSE
 g = GRAVITY
 ρ = DENSITY
 h = DEPTH
 C = WAVE AMPLITUDE

WE SHALL ASSUME ω DOESN'T DEPEND ON THE AMPLITUDE C , AS IS TYPICAL IN SMALL OSCILLATORY MOTION

a) DEEP WATER. h IS SO BIG THAT THE WAVE ON THE SURFACE DOESN'T KNOW ABOUT THE BOTTOM, SO ω CAN'T DEPEND ON h .

$$\text{THEN } \omega \sim k^\alpha g^\beta \rho^\gamma$$

$$\text{DIMENSIONALLY: } \frac{1}{s} = \left(\frac{1}{L}\right)^\alpha \left(\frac{L}{s^2}\right)^\beta \left(\frac{M}{L^3}\right)^\gamma$$

$$\text{HENCE } \gamma = 0, \beta = \frac{1}{2} \text{ AND } \alpha = \frac{1}{2}$$

$$\Rightarrow \omega = \sqrt{gk} \quad (\text{UP TO A CONSTANT FACTOR})$$

$$v_p = \sqrt{\frac{g}{k}} \quad v_g = \frac{1}{2} \sqrt{\frac{g}{k}}$$

AND $v_p = 2v_g$ NO MATTER WHAT THE CONSTANT IS!

THE VELOCITY DEPENDS ON THE WAVELENGTH. LONG WAVES TRAVEL FAST!

b) SHALLOW WATER. NOW WE CAN'T NEGLECT h

$$\omega \sim k^\alpha g^\beta \rho^\gamma h^\delta$$

$$\frac{1}{s} \sim \left(\frac{1}{L}\right)^\alpha \left(\frac{L}{s^2}\right)^\beta \left(\frac{M}{L^3}\right)^\gamma L^\delta$$

$$\Rightarrow \gamma = 0, \beta = \frac{1}{2} \text{ AND } \alpha - \delta = \frac{1}{2}$$

$$\omega = \sqrt{gk} (kh)^\delta$$

WE KNOW $\delta \neq 0$. $\delta = 27$ IS UNLIKELY

AN INSPIRED GUESS IS $\delta = 1/2$

$$w = k \sqrt{gh} \quad v_p = v_g = \sqrt{gh} \Rightarrow \text{NO DISPERSION}$$

TIDE EQUATION OF MOTION

WE NOW USE $\bar{F} = m\bar{a}$ ON A SMALL VOLUME OF WATER WHOSE EQUILIBRIUM COORDINATE IS $\bar{r} = (x, y, z)$. RECALL THAT THE ACTUAL POSITION IS $\bar{R} = \bar{r} + \bar{s}$.

THE VOLUME ELEMENT IS $d\text{vol} = dx dy dz$

THEN $m\bar{a} = \rho d\text{vol} \ddot{\bar{s}}$, SINCE \bar{r} IS CONSTANT IN TIME.

\bar{F} INCLUDES ONLY GRAVITY, AND WATER PRESSURE (AN EFFECT OF GRAVITY). WE IGNORE VISCOSITY, AIR PRESSURE, ETC.

THE FORCE OF GRAVITY IS OF COURSE $-\rho g d\text{vol} \hat{z}$

IF THE PRESSURE IS UNIFORM, THERE IS NO NET PRESSURE FORCE. THE FORCE DEPENDS ON THE PRESSURE GRADIENT.

$$F_x = d\text{AREA}_x (P(R_x) - P(R_x + dR_x)) = -\frac{\partial P}{\partial R_x} dR_x d\text{AREA}_x = -\frac{\partial P}{\partial R_x} d\text{vol}$$

[NOT $F_x = -\frac{\partial P}{\partial x} d\text{vol}$. THE PRESSURE FORCE MUST INCLUDE THE EFFECT OF DISTORTIONS OF $d\text{vol}$ DUE TO THE DISPLACEMENT \bar{s}]

$$\text{ALTOGETHER} \quad \rho \ddot{\bar{s}} = -\frac{\partial P}{\partial \bar{R}} - \rho g \hat{z}$$

WE CANNOT SIMPLY WRITE $P = \rho g \cdot \text{HEIGHT}$ — WHICH IS TRUE ONLY IF THE WATER IS NOT MOVING VERTICALLY! IF THE WATER WERE FALLING VERTICALLY, $v_z = gt$ AND $P = 0$ EVERYWHERE.

THE TERM $\frac{\partial P}{\partial \bar{R}}$ IS SOMEWHAT CLUMSY. IT WOULD BE NICER TO HAVE $\frac{\partial P}{\partial \bar{r}}$ INSTEAD ($\Rightarrow \frac{\partial P}{\partial x_i}$ IN COMPONENTS)

$$\text{BUT } \frac{\partial P}{\partial x_i} = \sum_j \frac{\partial P}{\partial R_j} \frac{\partial R_j}{\partial x_i} = \sum_j \frac{\partial P}{\partial R_j} (\delta_{ij} + \frac{\partial s_j}{\partial x_i})$$

IN COMPONENT FORM, THE EQUATION OF MOTION IS

$$\ddot{S}_j = -\frac{1}{\rho} \frac{\partial P}{\partial R_j} - g \delta_{j3}$$

MULTIPLY BY $\frac{\partial R_j}{\partial x_i} = \delta_{ij} + \frac{\partial S_j}{\partial x_i}$ AND SUM OVER j

$$\ddot{S}_i + \sum_j \ddot{S}_j \frac{\partial S_j}{\partial x_i} = -\frac{1}{\rho} \sum_j \frac{\partial P}{\partial R_j} \frac{\partial R_j}{\partial x_i} - g \left(\delta_{i3} + \frac{\partial S_3}{\partial x_i} \right)$$

$$\text{OR } \boxed{\ddot{S}_i + \sum_j \ddot{S}_j \frac{\partial S_j}{\partial x_i} = -\frac{\partial}{\partial x_i} \left(\frac{P}{\rho} + gz + gS_3 \right)}$$

THE EQUATION OF MOTION

UNFORTUNATELY, THIS IS NON-LINEAR

ULTIMATELY TO GET A SIMPLE RESULT, WE WILL HAVE TO IGNORE THE NON-LINEAR TERM. BUT WE CAN LEARN A BIT ABOUT THE NON-LINEAR BEHAVIOR BY DEFERRING THIS STEP FOR A WHILE.

RECALL THE FORM OF OUR TRIAL SOLUTIONS:

$$S_x = C \cosh k(z+h) \cos(kx - \omega t), \quad S_y = 0, \quad S_z = C \sinh k(z+h) \sin(kx - \omega t)$$

$$\text{THUS } \ddot{S}_j = -\omega^2 S_j \quad \text{OF COURSE}$$

$$\text{ALSO } S_i = \frac{\partial}{\partial x_i} \left(\frac{C}{k} \cosh k(z+h) \sin(kx - \omega t) \right)$$

$$[\text{REMEMBER: } \nabla \times \bar{S} = 0 \Rightarrow \bar{S} = \nabla \phi \text{ FOR SOME FUNCTION } \phi]$$

$$\text{ALSO } \sum_j \ddot{S}_j \frac{\partial S_j}{\partial x_i} = -\omega^2 \sum_j S_j \frac{\partial S_j}{\partial x_i} = -\frac{\omega^2}{2} \frac{\partial}{\partial x_i} \sum_j S_j^2$$

$$\begin{aligned} \text{AND } \sum_j S_j^2 &= C^2 \cosh^2 k(z+h) \cos^2(kx - \omega t) + C^2 \sinh^2 k(z+h) \sin^2(kx - \omega t) \\ &= C^2 (\cosh^2 k(z+h) - \sin^2(kx - \omega t)) \end{aligned}$$

ALL TERMS IN THE EQUATION OF MOTION HAVE NOW BEEN EXPRESSED AS DERIVATIVES WITH RESPECT TO x_i , SO WE CAN INTEGRATE ONCE:

$$\begin{aligned} \omega^2 \left[\frac{C}{k} \cosh k(z+h) \sin(kx - \omega t) + \frac{C^2}{2} (\cosh^2 k(z+h) - \sin^2(kx - \omega t)) \right] \\ = \frac{P(x, z, t)}{\rho} + gz + gC \sinh k(z+h) \sin(kx - \omega t) + \text{CONST.} \end{aligned}$$

AT THE SURFACE, THE PRESSURE MUST BE A CONSTANT (= ZERO IGNORING AIR PRESSURE). PUTTING $z=0$, $P=0$ AND THE CONSTANT OF INTEGRATION TO ZERO,

$$\omega^2 \left[\frac{c}{k} \cosh kh \sin(kx - \omega t) + \frac{c^2}{2} (\cosh^2 kh - \sin^2(kx - \omega t)) \right] \\ = g C \sinh kh \sin(kx - \omega t)$$

THIS CANNOT BE TRUE IN GENERAL, BUT IF $CK \ll 1$ IT'S O.K.

THUS OUR SOLUTION IS RESTRICTED TO WAVES WHOSE HEIGHT IS MUCH LESS THAN A WAVELENGTH

THEN $\omega = \sqrt{gk \tanh kh}$

a) DEEP WATER $h \rightarrow \infty$ $\tanh kh \rightarrow 1$

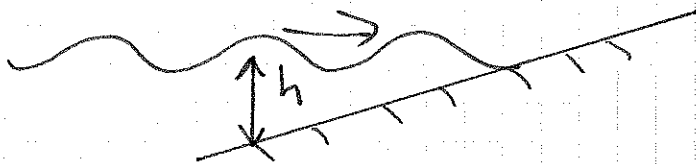
$$\omega = \sqrt{gk}$$

b) SHALLOW WATER $kh \ll 1$, $\tanh kh \sim kh$

$$\omega = k \sqrt{gh}$$
 AS GUESSED BEFORE.

SURF'S UP

WHAT HAPPENS WHEN THE DISTANCE TO THE BOTTOM IS DECREASING?



THE WAVES BREAK! THAT IS, THEY TOPPLE OVER IN THE DIRECTION OF MOTION. THIS IS CERTAINLY A NON-LINEAR EFFECT.

WE CAN OFFER 2 PSEUDO-EXPLANATIONS OF THIS EFFECT:

a) SINCE $v \sim \sqrt{gh}$ FOR SMALL h , v DECREASES AS THE WAVE NEARS THE SHORE. THE FRONT PART OF THE WAVE SLOWS DOWN FIRST, AND THE BACK OF THE WAVE OVERRUNS IT.

b) LET'S RE-EXAMINE THE NON-LINEAR RELATION FOR THE WAVE AT THE SURFACE (P. 266) IN THE CASE $kh \ll 1$.

$$\Rightarrow \cosh kh \approx 1, \sinh kh \approx kh$$

THEN FOR $P = \text{CONSTANT}$ WE MUST HAVE

$$\frac{\omega^2 C}{K} \sin(Kx - \omega t) - \frac{\omega^2 C^2}{2} \sin^2(Kx - \omega t) = g C K h \sin(Kx - \omega t)$$

$$\text{OR } \frac{\omega^2}{K^2} = v_p^2 = gh + \frac{\omega^2 C}{2K} \sin(Kx - \omega t)$$

NOTE THAT $S_z |_{\text{SURFACE}} \approx C K h \sin(Kx - \omega t)$

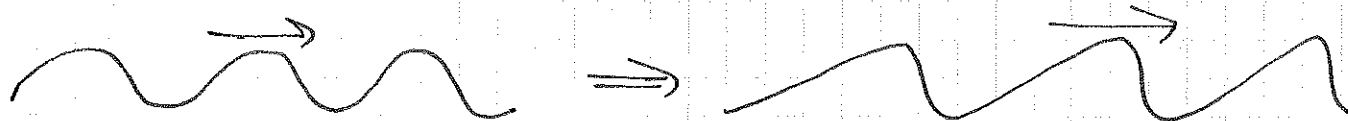
$$\text{SO } v_p^2 = gh + \frac{\omega^2 S_z}{2K^2 h} = gh + \frac{v_p^2 S_z}{2h}$$

$$\text{AND } v_p^2 = gh / \left(1 - \frac{S_z}{2h}\right) \approx gh \left(1 + \frac{S_z}{2h}\right)$$

THUS THE PHASE VELOCITY (AND GROUP VELOCITY!) IS GREATER ON A CREST THAN IN A VALLEY. AGAIN THE TOP OF THE WAVE WILL OVERRUN THE BOTTOM.

NOTE THAT THIS EFFECT OCCURS EVEN IF h IS CONSTANT, BUT WHEN THE AMPLITUDE S_z IS NOT INFINITESIMAL COMPARED TO h .

HENCE THE NON-LINEAR EQUATION OF MOTION IMPLIES THAT PURE SINE WAVES DISTORT AS THEY MOVE FORWARD (IN SHALLOW WATER) — IN PARTICULAR THE WAVE BECOMES STEEPER IN FRONT.



THUS WAVES BREAK ON WATER EVEN WHEN THE BOTTOM IS FLAT AND THE SHORE IS FAR AWAY...

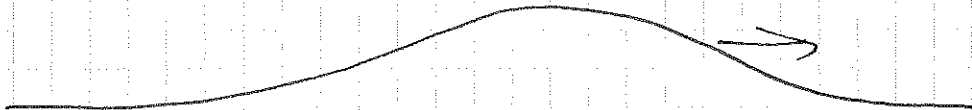
BY A MATHEMATICAL MIRACLE, IF YOU LET $h \rightarrow \infty$, THE NON-LINEAR TERMS ON P. 266 GO TO A CONSTANT (SUPPOSING $C e^{Kh} \rightarrow \text{CONST}$ AS $h \rightarrow \infty$) — AND PURE SINE WAVES PROPAGATE UNDISTORTED ON DEEP WATER.

c) WE MIGHT ALSO SUPPOSE THAT FRICTION AT THE BOTTOM CANNOT BE COMPLETELY NEGLECTED. THIS WOULD RETARD THE WATER IN THE VALLEYS MORE THAN IN THE CRESTS...

SOLITONS[SCIENTIFIC AMERICAN, FEB. 1978
STOKER, 'WATER WAVES']

WE MIGHT SUPPOSE THAT THE NON-LINEAR TERM CAUSES DISTORTIONS IN ALL WATER WAVES IF h IS SMALL. AMAZINGLY THIS IS NOT THE CASE - AS WAS DISCOVERED BY SCOTT RUSSELL IN 1834 BY OBSERVING WAVES IN A BARGE CANAL.

THERE IS A KIND OF BELL-SHAPED PULSE



WHICH PROPAGATES UNDIMINISHED AND UNDISTORTED, SATISFYING THE NON-LINEAR EQUATION OF MOTION (TO 2ND ORDER)

THE EQUATION OF THE SURFACE IS

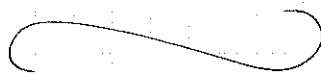
$$z = A \operatorname{sech}^2 \left[\sqrt{\frac{3A}{4h^3}} (x-ct) \right]$$

WHICH MOVES WITH VELOCITY $c^2 = g(h+A)$

THIS IS THE SOLITON OR SOLITARY WAVE SOLUTION.

IN RECENT YEARS (KRUSKAL, 1967 AT PRINCETON!) IT HAS BEEN SHOWN THAT 2 COLLIDING SOLITONS SCATTER DUE TO THE NON-LINEAR TERMS, BUT RE-EMERGE AS SOLITONS. (LINEAR WAVES JUST PASS THRU ONE ANOTHER). HENCE THESE NON-LINEAR SOLUTIONS ARE VERY MUCH LIKE PARTICLES, WHICH HAS CAUSED A LOT OF EXCITEMENT IN VIEW OF CERTAIN CLAIMS OF QUANTUM MECHANICS.

SEE PHYSICS TODAY, NOV 1980, P. 20 FOR A SATELLITE PHOTOGRAPH OF SOME GIANT SOLITONS.

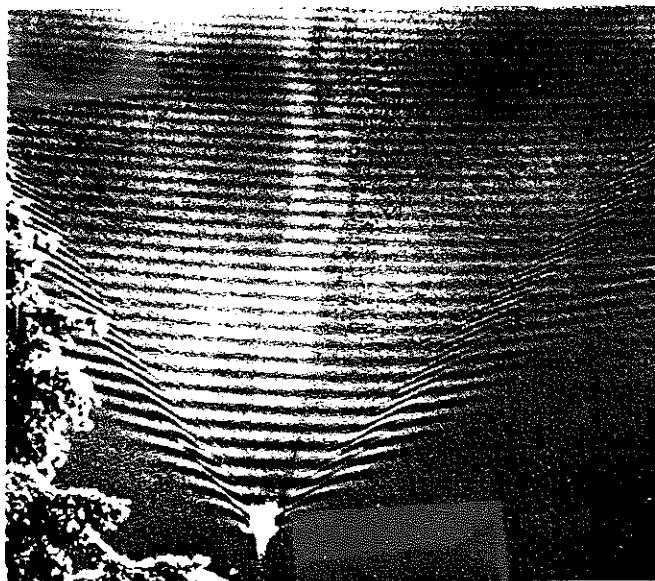


THE WAKE OF A BOAT

[SEE F.S. CRAWFORD, AM. J. PHYS 52, 782 (1984)]

THE WAKE OF A BOAT HAS A COMPLICATED STRUCTURE →

THERE IS A V-SHAPED DISTURBANCE WHICH EXTENDS BACKWARDS FROM THE TIP OF THE BOAT - WHICH IS SOMETIMES CALLED A 'SHOCK WAVE' (ALTHOUGH THIS IS ONLY PARTLY CORRECT, AS WE WILL SEE).



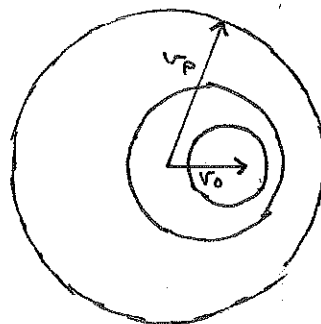
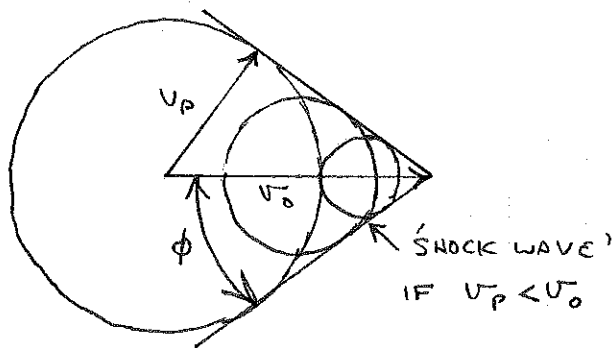
THE HALF ANGLE OF THIS DISTURBANCE IS ABOUT 20° , AND IS INDEPENDENT OF THE SPEED OF THE BOAT! COMPARE THE WAKE OF A DUCK ON LAKE CARNEGIE!

WE WILL SHOW HOW THE PROMINENT FEATURES OF THE WAKE FOLLOW FROM THE DISPERSION RELATION FOR WATER WAVES IN DEEP WATER

$$\omega = \sqrt{gk} \quad (\text{DEEP WATER})$$

$$\Rightarrow v_p = \frac{\omega}{k} = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}} \quad ; \quad v_g = \frac{d\omega}{dk} = \frac{1}{2} v_p$$

BUT FIRST, WE EXAMINE THE 'SHOCK WAVE' EFFECT WHICH OCCURS IN CASE $v_p = \text{CONSTANT}$ (I.E., NO DISPERSION)



JUST CIRCLES IF $v_p > v_0$

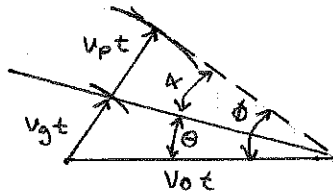
AS THE BOAT MOVES WITH VELOCITY v_0 IT GENERATES A SUCCESSION OF CIRCULAR WAVES. IF $v_0 > v_p$, THESE CIRCLES INTERFERE CONSTRUCTIVELY TO GIVE A STRONG WAVE AT ANGLE ϕ , WHERE

$$\sin \phi = \frac{v_p}{v_0}$$

FOR HIGHER-SPEED BOATS, THIS WOULD GIVE SMALLER ANGLES, ϕ , UNLIKE THE ACTUAL OBSERVATIONS!

BUT FOR REAL WATER, v_p VARIES WITH WAVELENGTH, AND THE GROUP VELOCITY OBEYS $v_g = \frac{1}{2} v_p$ (IF DEPTH $h \gg \lambda$)

CONSIDER AGAIN WHAT HAPPENS TO WAVES OF A GIVEN WAVELENGTH λ GENERATED BY THE BOAT.



AGAIN, WHILE THE BOAT MOVES DISTANCE $v_0 t$, A WAVE OF LENGTH λ MOVES BY $v_p t$. BUT THE BOAT GENERATES WAVES OF A WIDE SPECTRUM OF WAVELENGTHS. THESE DIFFERENT WAVES INTERFERE TO PRODUCE THE OBSERVED PATTERN. RECALL THAT THE GROUP VELOCITY DESCRIBES THE PROPAGATION OF THE INTERFERENCE PATTERN, WHILE THE PHASE VELOCITY DESCRIBES ITS COMPONENT WAVES

HENCE THE DISTURBANCE CAUSED BY THE BOAT AT $t=0$ PROPAGATES ONLY A DISTANCE $v_g t$, SO FAR AS THE EYE WILL READILY SEE. THIS WILL LEAD TO AN APPARENT 'SHOCK WAVE' AT ANGLE $\theta(\lambda)$ RATHER THAN $\phi(\lambda)$ AS SKETCHED. THESE SHOCK WAVES WILL BE PROMINENT ONLY IF MANY DIFFERENT WAVELENGTHS λ ALL CONTRIBUTE TO THE SAME θ

i.e. $\frac{d\theta}{d\lambda} = 0$ AT THE MOST PROMINENT θ .

SINCE ϕ IS A MONOTONIC FUNCTION OF λ , IT IS SUFFICIENT TO REQUIRE $\frac{d\theta}{d\phi} = 0$

NOW $\theta = \phi - \alpha$, AND $\tan \alpha = \frac{1}{2} \tan \phi$ FROM THE FIGURE ABOVE.

TAKING DERIVATIVES, WE FIND $\tan \phi = \sqrt{2} \Rightarrow \phi = 54.7^\circ$

$\tan \alpha = \sqrt{2}/2$

$\Rightarrow \theta = 19.5^\circ$ INDEPENDENT OF v_0 !

THE FAVORED ANGLE, $\phi = 54.7^\circ$, LEADS TO A FAVORED WAVELENGTH (FROM P 269)

$$\lambda = \frac{2\pi}{g} \sin^2 \phi v_0^2 \sim v_0^2$$

THIS IS THE SEPARATION OF THE RIDGES SEEN ALONG THE 'SHOCK WAVE',

FURTHER, THESE RIDGES MAKE THE CHARACTERISTIC ANGLE $\phi = 54.7^\circ$ TO THE MOTION OF THE BOAT...



Q UOD
E RUM
D UCKY