Ph 205 Lecture 3

Development of a General Method

In principle, $\bar{F} = m\bar{a}$ is enough to solve any mechanics problem. But in practice it's helpful to use a few tricks, such as the conservation laws. In problems involving several masses we could identify all forces and then solve $\bar{F} = m\bar{a}$ for each mass individually. But often some of the forces don't directly affect the motion, but rather serve to maintain some constant relation between various masses. Examples are the normal force of contact between two objects, tensions in strings or rods, etc. We will call these forces of constraint.

Perhaps you have noticed how you may solve for these forces in many problems, only to have them eliminated from the equations before the final step. Our search for a general method will involve a systematic elimination of these constraint forces from the equations of motion — reducing the application of $\bar{F} = m\bar{a}$ to a bare minimum. In addition, we seek guidlines as to when we should use the conservation laws.

Altogether this is an ambitious program, and a single method which can do all these things will not look quite as simple as "$\bar{F} = m\bar{a}$". There is another disadvantage to any general method: it tends to replace insight by cookbook type instructions. The use of $\bar{F} = m\bar{a}$ by an expert always succeeds, and usually gives the satisfaction that the problem is really 'understood'! Our general method will not appear so intuitive. But it also works, and features of it have been carried over into quantum mechanics where $\bar{F} = m\bar{a}$ faltered.

The development of the general method occupied the time of the best scientists for a century after Newton. We will try to compress 100 years of experience into 4 lectures.

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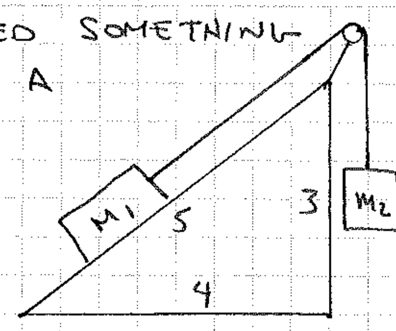
STEVINUS

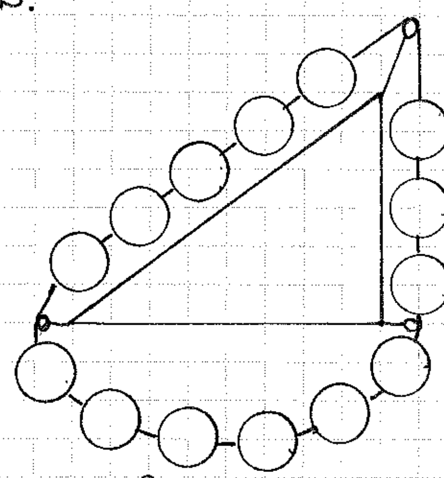
IN THE 1600'S THERE WERE OTHER APPROACHES TO MECHANICS THAN THAT OF NEWTON. SOME OF THE INSIGHTS OF THESE ALTERNATE METHODS WERE INCORPORATED INTO THE FINAL METHOD OF LAGRANGE. IN PARTICULAR, THE PRINCIPLE OF VIRTUAL WORK PLAYED A LARGE ROLE. WE EXAMINE IT IN MORE DETAIL.

IN 1586 STEVINUS (A BELGIAN) INTRODUCED SOMETHING LIKE THE PRINCIPLE OF VIRTUAL WORK TO SOLVE A THEN DIFFICULT STATICS PROBLEM:
HOW SHOULD M_1 AND M_2 BE RELATED FOR STATIC EQUILIBRIUM?



THIS IS EASY FOR US NOW BECAUSE WE KNOW THAT FORCE IS A VECTOR, BUT THIS WAS NOT COMMONLY APPRECIATED PRIOR TO NEWTON.

STEVINUS IMAGINED A CHAIN OF EQUAL-MASS BALLS STRUNG OVER THE WEDGE AS SHOWN. THEN 5 BALLS FIT ALONG THE HYPOTENUSE WHILE 3 BALLS HANG VERTICALLY.

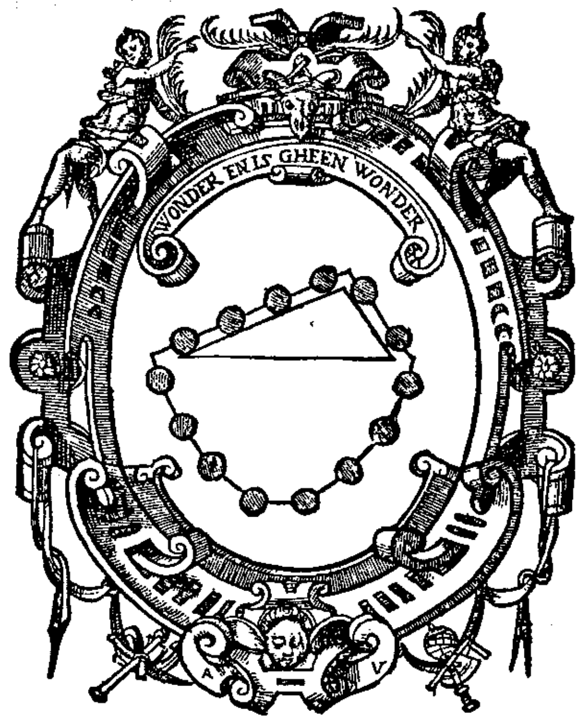


HE THEN NOTED THAT IT WAS 'SELF-EVIDENT' THAT THE CHAIN WOULD NOT MOVE. WHICH WAY WOULD IT MOVE? AND IF IT DID MOVE WE WOULD HAVE A PERPETUAL MOTION MACHINE!

IF THE CHAIN DOESN'T MOVE, WE CAN SURELY REMOVE THE LOWER 6 BALLS, LEAVING THE UPPER 8 IN EQUILIBRIUM. HENCE THE ANSWER TO THE ORIGINAL QUESTION IS

$$\frac{M_1}{M_2} = \frac{5}{3}$$

STEVINUS HAS THE ABOVE DIAGRAM CARVED ON HIS TOMBSTONE.



THE ABOVE IS THE FRONTISPIECE OF STEVINUS' BOOK ON MECHANICS. 'WONDER EN IS GHEEN WONDER' CAN BE TRANSLATED IN THE WORDS OF PRINCETONIAN J.A. WHEELER AS 'MAGIC WITHOUT MAGIC.'

THE PRINCIPLE OF VIRTUAL WORK (CONT'D)

STEVINUS ALSO SAID: 'UT SPATIUM AGENTIS AD SPATIUM PATIENTIS, SIT POTENTIA PATIENTIS AD POTENTIAM AGENTIS.' LATIN LOVERS WILL RECOGNIZE THIS AS THE FIRST STATEMENT OF THE PRINCIPLE OF VIRTUAL WORK.

A FIRST STEP IN AVOIDING UNNECESSARY CONSIDERATION OF CONSTRAINT FORCES IS THE STRONG FORM OF THE PRINCIPLE OF VIRTUAL WORK: THE FORCES OF CONSTRAINT DO NO NET WORK IN A VIRTUAL DISPLACEMENT.

WE WILL EVENTUALLY SHOW HOW ONE CAN ARRIVE AT A GENERAL METHOD STARTING FROM THIS PRINCIPLE. APPARENTLY NO COMPLETELY GENERAL PROOF OF THIS STATEMENT HAS BEEN GIVEN BY STARTING FROM NEWTON'S LAWS. INDEED, IF FRICTION IS INVOLVED ONE MUST BE VERY CAREFUL. IF YOU WISH, REGARD THE PRINCIPLE AS THE 4RTH LAW OF MECHANICS.

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FOR A SYSTEM IN STATIC EQUILIBRIUM IT IS CERTAINLY TRUE THAT $\delta W = \sum_i \bar{F}_i \cdot \delta \bar{r}_i = 0$ FOR SMALL $\delta \bar{r}_i$

WE NOW DECOMPOSE $\bar{F}_i = \bar{F}_i^e + \bar{F}_i^c$

WHERE e LABELS EXTERNAL, AND c LABELS CONSTRAINT FORCES. THE INTERNAL FORCES HOLDING A BODY TOGETHER ARE CERTAINLY CONSTRAINT FORCES.

BY THE STRONG PRINCIPLE OF VIRTUAL WORK

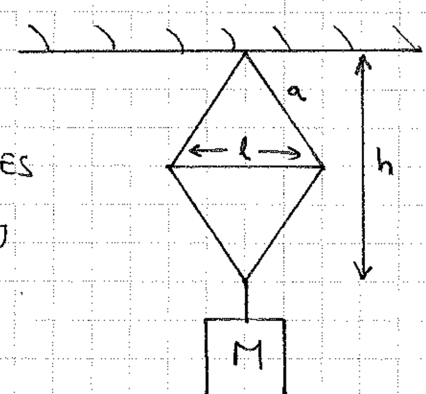
$$\sum_i \bar{F}_i^c \cdot \delta \bar{r}_i = 0$$

HENCE WE ALSO HAVE $\sum_i \bar{F}_i^e \cdot \delta \bar{r}_i = 0$

WHICH CAN BE USED TO FIND RELATIONS AMONG THE EXTERNAL FORCES WHICH MAINTAIN EQUILIBRIUM.

IF WE WISH TO LEARN ABOUT A PARTICULAR FORCE OF CONSTRAINT, WE COULD ALWAYS REGARD IT AS AN EXTERNAL FORCE.

<u>EXAMPLE:</u> FIND THE COMPRESSION, C, IN THE HORIZONTAL STRUT.

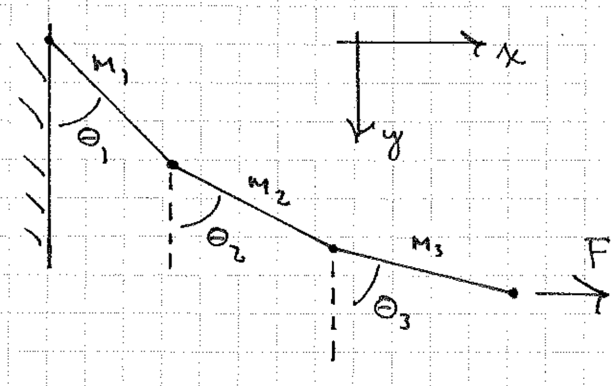


IMAGINE THE STRUT LENGTH INCREASES FROM l TO $dl + l$, WHICH RAISES THE WEIGHT BY dh.

THEN $\delta W = C\,dl + Mg\,dh = 0$

$$C = -Mg\frac{dh}{dl} = Mg\frac{l}{h} \quad \text{USING} \quad h = 2\sqrt{a^2 - l^2/4}$$

<u>EXAMPLE:</u> A HORIZONTAL FORCE F PULLS ON A SYSTEM OF 3 RODS AS SHOWN. WHAT ARE THE ANGLES θ_1, θ_2 & θ_3 AT EQUILIBRIUM?



THIS IS NOT TOO HARD BY $\bar{F}=m\bar{a}$. SPLIT THE PROBLEM INTO 3 SINGLE ROD PROBLEMS, USE $\sum \bar{F}=0$, $\sum \bar{N}=0$ IN EACH CASE TO FIND THE ANGLES. NOTE THAT YOU MUST INTRODUCE THE CONSTRAINT FORCES AT THE HINGES TO DO THIS.

BUT WE CAN SOLVE IT ALL WITH ONE APPLICATION OF THE PRINCIPLE OF VIRTUAL WORK. SUPPOSE WE MAKE THE VIRTUAL DISPLACEMENTS $\theta_i \to \theta_i + d\theta_i$

THEN THE HEIGHT OF THE C.M. OF EACH ROD CHANGES $\Rightarrow$ GRAVITY DOES WORK; AND THE FAR END OF ROD 3 MOVES HORIZONTALLY $\Rightarrow$ F DOES WORK. THE 'EXTERNAL' FORCE ON THE LEFTMOST PIVOT DOES NO WORK IN THIS SET OF DISPLACEMENTS.

$$\delta W = m_1 g\, dy_1 + m_2 g\, dy_2 + m_3 g\, dy_3 + F\, dx = 0$$

$y_1 = \frac{l_1}{2}\cos\theta_1$ [y IS POSITIVE DOWNWARDS]

$y_2 = l_1\cos\theta_1 + \frac{l_2}{2}\cos\theta_2$

$y_3 = l_1\cos\theta_1 + l_2\cos\theta_2 + \frac{l_3}{3}\cos\theta_3$

$x = l_1\sin\theta_1 + l_2\sin\theta_2 + l_3\sin\theta_3$

so
$$\begin{aligned}\delta W = & -m_1 g \frac{l_1}{2}\sin\theta_1\, d\theta_1 \\ & - m_2 g\, l_1 \sin\theta_1\, d\theta_1 - m_2 g\frac{l_2}{2}\sin\theta_2\, d\theta_2 \\ & - m_3 g\, l_1\sin\theta_1\, d\theta_1 - m_3 g\, l_2 \sin\theta_2\, d\theta_2 - m_3 g \frac{l_3}{2}\sin\theta_3\, d\theta_3 \\ & + F l_1\cos\theta_1\, d\theta_1 + F l_2 \cos\theta_2\, d\theta_2 + F l_3\cos\theta_3\, d\theta_3 \\ = & \, 0\end{aligned}$$

SINCE $d\theta_1$, $d\theta_2$ AND $d\theta_3$ ARE INDEPENDENT,

$$\tan\theta_1 = \frac{2F}{(m_1+2m_2+2m_3)g} \qquad \tan\theta_2 = \frac{2F}{(m_2+2m_3)g} \qquad \tan\theta_3 = \frac{2F}{m_3 g}$$

IT STILL TAKES EFFORT, BUT THERE IS A CERTAIN ELEGANCE TO THE APPROACH.

D'ALEMBERT'S PRINCIPLE

THE PRINCIPLE OF VIRTUAL WORK SEEMS TO OFFER A GENERAL METHOD OF SOLUTION TO STATICS PROBLEMS. THIS LED PEOPLE TO TRY TO REDUCE DYNAMICAL PROBLEMS TO STATICS PROBLEMS BY CLEVER USE OF THIS PRINCIPLE.

IN 1743 D'ALEMBERT SUGGESTED THE FOLLOWING APPROACH:

FROM NEWTON WE KNOW THAT $m_i \bar{a}_i = \bar{F}_i$

AGAIN WE WRITE $\bar{F}_i = \bar{F}_i^{\,e} + \bar{F}_i^{\,c}$

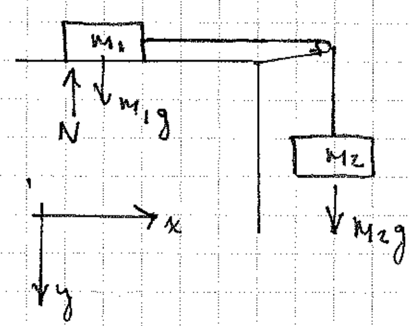
SO $m_i \bar{a}_i - \bar{F}_i^{\,e} = \bar{F}_i^{\,c}$

WE THEN NOTE WITH D'ALEMBERT THAT THE CONSTRAINT FORCES DO NOT IMPART MOTION TO THE SYSTEM, BUT RATHER WOULD HOLD THE SYSTEM IN STATIC EQUILIBRIUM IN THE ABSENCE OF ANY EXTERNAL FORCES.

HENCE THE QUANTITIES $m_i \bar{a}_i - \bar{F}_i^{\,e}$ ARE EQUIVALENT TO A SET OF FORCES WHICH WOULD HOLD THE SYSTEM IN STATIC EQUILIBRIUM. THIS IS <u>D'ALEMBERT'S PRINCIPLE</u>

WE GIVE SOME EXAMPLES TO SHOW THAT THIS SOMEWHAT PECULIAR POINT OF VIEW IN FACT MAKES SENSE.

<u>EXAMPLE 1</u> MASS m_1 IS CONNECTED TO MASS m_2 BY A STRING AS SHOWN.



m_1 DOES NOT MOVE VERTICALLY SO OUR APPRECIATION OF STATICS QUICKLY TELLS US THAT $N = m_1 g$.

TO SOLVE FOR THE MOTION, WE CONSIDER THE TENSION IN THE STRING AS A CONSTRAINT FORCE, SO THAT $m_2 g$ IS THE ONLY RELEVANT EXTERNAL FORCE.

DUE TO THE CONSTRAINT OF THE STRING, $a_{1x} = a_{2y}$

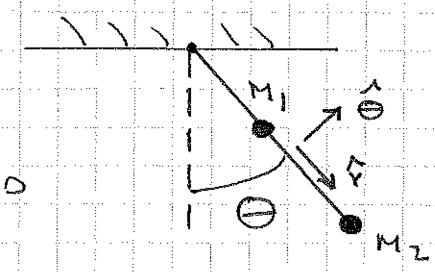
D'ALEMBERT TELLS US TO WRITE $m_1 a_{1x} - F_{1x}^{e} = m_1 a_{1x} \equiv f_1$

$$m_2 a_{2y} - F_{2y}^{e} = m_2 a_{2y} - m_2 g \equiv f_2$$

THEN WE REQUIRE THAT f_1 AND f_2 COULD HOLD THE SYSTEM IN EQUILIBRIUM. NOTING THAT THESE ARE TO BE THOUGHT OF AS TENSIONS IN THE STRING FOR A CASE OF NO MOTION, SURELY $f_1 = -f_2$

SETTING $a_{1x} = a_{2y} = a$, WE GET $(m_1 + m_2) a = m_2 g$

AS EXPECTED.



EXAMPLE 2: A MASSLESS ROD IS HUNG FROM A PIVOT AT ONE END, AND TWO MASSES M_1 AND M_2 ARE ATTACHED AT DISTANCES r_1 AND r_2 FROM THE PIVOT

WHAT LENGTH SIMPLE PENDULUM WOULD HAVE THE SAME PERIOD OF OSCILLATION?

HISTORICALLY, THIS PROBLEM WAS A FAVORITE EXAMPLE OF THE NEED FOR A METHOD. PAMPHLETS AND COUNTER-PAMPHLETS WERE WRITTEN ABOUT IT CIRCA 1700.

THIS PROBLEM IS FAIRLY SIMPLE FOR OUR 'ELEMENTARY' METHOD. JUST USE $N = I\alpha$

$$I = M_1 r_1^2 + M_2 r_2^2 \qquad N = -M_1 g r_1 \sin\theta - M_2 g r_2 \sin\theta$$

$$\text{so} \quad \alpha = \ddot{\theta} = -\left(\frac{M_1 r_1 + M_2 r_2}{M_1 r_1^2 + M_2 r_2^2}\right) g \sin\theta$$

BUT THE EQUATION FOR A SIMPLE PENDULUM OF LENGTH l IS

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

$$\therefore \; l_{\text{EFFECTIVE}} = \frac{M_1 r_1^2 + M_2 r_2^2}{M_1 r_1 + M_2 r_2}$$

THE POINT AT l_{eff} FROM THE PIVOT IS SOMETIMES CALLED THE CENTER OF OSCILLATION.

NOW WE TRY D'ALEMBERT'S METHOD.

$$M_1 \bar{a}_1 - M_1 \bar{g} = \bar{f}_1 \qquad M_2 \bar{a}_2 - M_2 \bar{g} = \bar{f}_2$$

FOR STATIC EQUILIBRIUM, WE NEED A 3RD CONSTRAINT FORCE, THE FORCE OF THE PIVOT ON THE ROD, $\bar{f}_p$. THEN $\bar{f}_1 + \bar{f}_2 + \bar{f}_p = 0$. WE AVOID THE QUESTION OF $\bar{f}_p$ BY USING THE OTHER CONDITION OF STATIC EQUILIBRIUM: $\sum_i \bar{N}_i = 0$ FOR TORQUES CALCULATED ABOUT THE PIVOT. (A USEFUL TRICK!)

HENCE $\quad \bar{r}_1 \times \bar{f}_1 + \bar{r}_2 \times \bar{f}_2 = 0$

HENCE $m_1 \bar{r}_1 \times \bar{a}_1 - m_2 \bar{r}_1 \times \bar{g} + m_2 \bar{r}_2 \times \bar{a}_2 - m_2 \bar{r}_2 \times \bar{g} = 0$

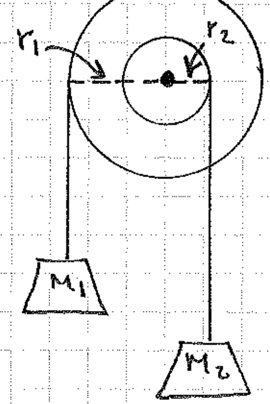
IN POLAR COORDS, $\bar{a} = r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r}$ FOR FIXED r (p. 9)

NOTING $\bar{r} \times \hat{r} = 0$, WE HAVE

$$m_1 r_1^2 \ddot{\theta} + m_1 g r_1 \sin\theta + m_2 r_2^2 \ddot{\theta} + m_2 g r_2 \sin\theta = 0$$

SO $$\ddot{\theta} = -\left(\frac{m_1 r_1 + m_2 r_2}{m_1 r_1^2 + m_2 r_2^2}\right) g \sin\theta \quad \text{AS BEFORE}$$

EXAMPLE 3. THE PULLEY IS MASSLESS. FIND THE ACCELERATIONS.



D'ALEMBERT SAYS:

$$m_1 a_1 - m_1 g = f_1$$
$$m_2 a_2 - m_2 g = f_2$$

FOR STATIC EQUILIBRIUM, $r_1 f_1 = r_2 f_2$ — WHICH WE COULD INFER FROM THE PRINCIPLE OF VIRTUAL WORK.

WE MUST ALSO NOTE THAT $\alpha = \frac{a_1}{r_1} = -\frac{a_2}{r_2}$ = ANGULAR ACCELERATION

THEN $r_1 m_1 a_1 - r_1 m_1 g = r_2 m_2 \left(-\frac{r_2}{r_1} a_1\right) - r_2 m_2 g$

SO $a_1 = r_1 g \left(\frac{m_1 r_1 - m_2 r_2}{m_1 r_1^2 + m_2 r_2^2}\right) \qquad a_2 = -\frac{r_2}{r_1} a_1 = -r_2 g \frac{m_1 r_1 - m_2 r_2}{m_1 r_1^2 + m_2 r_2^2}$

WE STILL HAD TO INSERT THE CONSTRAINT RELATION $\frac{a_1}{r_1} = -\frac{a_2}{r_2}$ BY HAND!

EXAMPLE 4. MASS M_1 SLIDES DOWN THE WEDGE M_2 WHICH IS FREE TO SLIDE ON A HORIZONTAL SURFACE. THERE IS NO FRICTION, FIND THE ACCELERATION.

THIS PROBLEM CAN BE DONE BY $\bar{F} = m\bar{a}$, OR BY D'ALEMBERT'S PRINCIPLE, BUT IT IS NOT EASY!

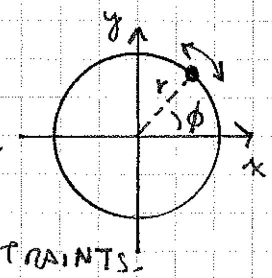
CONSTRAINTS AND GENERALIZED COORDINATES

WE NOW CONSIDER THE INSIGHT OF LAGRANGE (1788) IN BRIEF, HE SUGGESTS THAT THE MOST EFFICIENT WAY TO FIND THE EQUATIONS OF MOTION OF A SYSTEM IS NOT TO INTRODUCE ANY UNNECESSARY VARIABLES.

IN A PROBLEM OF n PARTICLES WE DON'T WANT TO SOLVE ALL $3n$ EQUATIONS, $\bar{F} = m\bar{a}$, IF WE CAN AVOID IT. TYPICALLY WE DON'T KNOW ALL THE $\bar{F}_i$ BECAUSE MANY OF THEM ARE INTERNAL FORCES WHICH MAINTAIN VARIOUS MECHANICAL CONSTRAINTS ON THE POSSIBLE MOTION.

BECAUSE OF THE CONSTRAINTS THE NUMBER OF INDEPENDENT COORDINATES NEEDED TO DESCRIBE THE SYSTEM IS LESS THAN $3N$. OUR GOAL WILL BE A METHOD TO FIND ONLY THE EQUATIONS OF MOTION OF THE INDEPENDENT VARIABLES.

IN THIS ATTEMPT WE WILL OFTEN BE LED TO CONSIDER COORDINATES OTHER THAN THE CARTESIAN COORDINATES OF A PARTICLE. FOR EXAMPLE, THE MOTION OF A BEAD WHICH SLIDES ON A FIXED CIRCULAR HOOP HAS ONLY ONE DEGREE OF FREEDOM = 1 INDEPENDENT VARIABLE.



THERE ARE TWO RELATIONS AMONG THE 3 CARTESIAN COORDINATES, WHICH RELATIONS EXPRESS THE CONSTRAINTS: $z = 0$, $x^2 + y^2 = r^2$. IT SEEMS APPROPRIATE TO USE A POLAR COORDINATE SYSTEM. THEN ϕ IS THE ONLY INDEPENDENT VARIABLE.

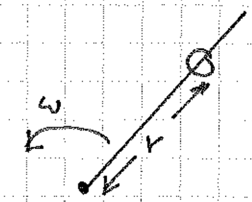
THUS WE CONSIDER THE GENERALISED COORDINATES — ANY SET OF INDEPENDENT VARIABLES SUFFICIENT TO COMPLETELY SPECIFY THE POSITION OF THE SYSTEM. WE LABEL THESE

$$q_1 \cdots q_K \qquad K \leq 3N$$

THEN EACH $\quad \bar{r}_i = \bar{r}_i(q_1, \ldots q_K, t)$

THE q_j NEED NOT HAVE DIMENSIONS OF LENGTH - OFTEN THEY ARE ANGLE VARIABLES. IT IS SOMETIMES NECESSARY TO INCLUDE THE TIME t EXPLICITY IN THE TRANSFORMATION EQUATIONS $\bar{r}_i = \bar{r}_i(q_j, t)$

EXAMPLE: A BEAD SLIDES ON A STRAIGHT WIRE WHICH ROTATES IN A PLANE WITH CONSTANT ANGULAR VELOCITY ω. THEN



$x = r\cos\omega t$, $y = r\sin\omega t$, $z = 0$

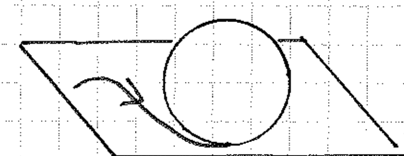
r IS THE SINGLE GENERALISED COORDINATE, BUT $x = x(r,t)$ ....

ACTUALLY IT IS NOT POSSIBLE TO FIND A SET OF INDEPENDENT GENERALISED COORDINATES WHERE $K < 3N$ FOR ALL PROBLEMS. THE COORDINATES WILL BE INDEPENDENT ONLY IF THE CONSTRAINTS CAN BE WRITTEN

$$f_m(\bar{r}_1 \dots \bar{r}_n) = 0 \qquad m = 1 \dots 3n-K$$

SUCH CONSTRAINTS ARE CALLED <u>HOLONOMOUS</u> BY JARGON LOVERS (A NAME INVENTED BY HERTZ IN 1894). IT IS ONLY FOR SYSTEMS WITH HOLONOMOUS CONSTRAINTS THAT WE WILL FIND A COMPLETELY GENERAL METHOD.

AN EXAMPLE OF A SYSTEM WITH <u>NON-HOLONOMOUS</u> CONSTRAINTS IS A SPHERE ROLLING WITHOUT SLIPPING ON A PLANE.



THE C.M. POSITION OBEYS A HOLONOMOUS CONSTRAINT: $z_{cm} = z_0$. BUT THE REQUIREMENT THAT THE SPHERE ROLL WITHOUT SLIPPING CANNOT BE REDUCED TO A SIMPLE RELATION AMONG THE 5 OTHER COORDINATES NEEDED TO SPECIFY THE SPHERE'S POSITION. WE SHALL BE ABLE TO GIVE SPECIAL METHODS FOR SOME CLASSES OF NON-HOLONOMOUS CONSTRAINT PROBLEMS — WHEN THE CONSTRAINTS CAN BE WRITTEN IN TERMS OF VELOCITIES RATHER THAN POSITIONS.

ANOTHER NON-HOLONOMOUS PROBLEM IS ICE SKATING.

FOR NOW, WE RESTRICT OURSELVES TO HOLONOMOUS SYSTEMS FOR WHICH THE GENERALISED COORDINATES q_j ARE INDEPENDENT. WE CALL $\dot{q}_j = \dfrac{dq_j}{dt}$ = <u>GENERALIZED VELOCITIES</u>

AGAIN, THESE MAY NOT HAVE THE DIMENSIONS OF VELOCITY.

LAGRANGE'S METHOD

IN 1788 LAGRANGE PUBLISHED HIS FAMOUS EQUATIONS WHICH APPEAR TO REDUCE THE TASK OF FINDING THE EQUATIONS OF MOTION TO A MERE MATHEMATICAL PROCEDURE. TO EMPHASIZE THIS, HIS TREATISE HAD NO DIAGRAMS.

HIS APPROACH IS TO COMBINE THE PRINCIPLE OF VIRTUAL WORK, D'ALEMBERT'S METHOD, AND THE USE OF INDEPENDENT, GENERALISED COORDINATES.

WE BEGIN WITH D'ALEMBERT'S PRINCIPLE

$$m_i \bar{a}_i - \bar{F}_i^e = \bar{F}_i^c$$

AND CONSIDER THE VIRTUAL WORK OF THE WHOLE SYSTEM IN A SMALL DISPLACEMENT:

$$\sum_i \left(m_i \bar{a}_i - \bar{F}_i^e\right) \cdot \delta \bar{r}_i = \sum_i \bar{F}_i^c \cdot \delta \bar{r}_i$$

IF THE DISPLACEMENTS ARE CONSISTENT WITH THE CONSTRAINTS ON THE SYSTEM, THEN THE RIGHT HAND SIDE VANISHES

$$\sum_i \left(m_i \bar{a}_i - \bar{F}_i^e\right) \cdot \delta \bar{r}_i = 0$$

THE TRICK OF LAGRANGE IS TO REARRANGE THIS INTO

$$\sum_j (\text{SOMETHING})_j \, \delta q_j = 0$$

WHERE THE q_j ARE THE INDEPENDENT GENERALISED COORDINATES. THEN SINCE EACH q_j CAN BE VARIED INDEPENDENTLY OF THE OTHERS, EACH $(\text{SOMETHING})_j = 0$ INDIVIDUALLY. THIS WILL GIVE US THE DESIRED EQUATIONS OF MOTION OF THE GENERALISED COORDINATES.

FIRST, SINCE $\bar{r}_i = \bar{r}_i(q_1 \ldots q_k, t)$

$$\delta \bar{r}_i = \sum_j \frac{\partial \bar{r}_i}{\partial q_j} \delta q_j + \frac{\partial \bar{r}_i}{\partial t} \delta t$$

WE AVOID THE COMPLICATION OF THE LAST TERM BY CONSIDERING ONLY DISPLACEMENTS AT A FIXED TIME

$$\Rightarrow \delta t = 0$$

THEN $\sum_i m_i \bar{a}_i \cdot \delta \bar{r}_i = \sum_j \sum_i m_i \ddot{\bar{r}}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j} \delta q_j$

WE NOW DO SOMETHING SIMILAR TO OUR DISCUSSION OF WORK AND ENERGY WHERE WE SAW THAT $m_i \ddot{\bar{r}}_i \cdot \delta \bar{r}_i = \delta\left(\frac{1}{2} m_i v_i^2\right)$ IF THE DISPLACEMENTS $\delta \bar{r}_i$ WERE THE ACTUAL DISPLACEMENTS IN TIME DUE TO THE MOTION OF THE SYSTEM. IN THE PRESENT CASE WE HOLD $\delta t = 0$, BUT WE STILL TRY TO BRING THE KINETIC ENERGY INTO THE ABOVE EXPRESSION SOMEHOW.

WE NEED A USEFUL RELATION:

$$\bar{v}_i = \frac{d}{dt} \bar{r}_i(q_j, t) = \frac{\partial \bar{r}_i}{\partial t} + \sum_j \frac{\partial \bar{r}_i}{\partial q_j} \dot{q}_j \qquad \dot{q}_j = \frac{dq_j}{dt}$$

THUS $\underline{\frac{\partial \bar{V}_i}{\partial \dot{q}_j} = \frac{\partial \bar{r}_i}{\partial q_j}}$ I.E. THE VELOCITY DEPENDS ON THE GENERALISED VELOCITIES IN THE SAME WAY THAT THE POSITION DEPENDS ON THE GENERALISED COORDINATES.

$$m_i \ddot{\bar{r}}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j} = m_i \dot{\bar{v}}_i \cdot \frac{\partial \bar{v}_i}{\partial \dot{q}_j} = m_i \frac{d}{dt}\left(\bar{v}_i \cdot \frac{\partial \bar{v}_i}{\partial \dot{q}_j}\right) - m_i \bar{v}_i \cdot \frac{d}{dt} \frac{\partial \bar{v}_i}{\partial \dot{q}_j}$$

$$= \frac{d}{dt} \frac{\partial}{\partial \dot{q}_j} \left(\frac{1}{2} m_i v_i^2\right) - m_i \bar{v}_i \cdot \frac{d}{dt} \frac{\partial \bar{r}_i}{\partial q_j}$$

$$= \frac{d}{dt} \frac{\partial T_i}{\partial \dot{q}_j} - m_i \bar{v}_i \cdot \frac{\partial}{\partial q_j} \underbrace{\frac{d \bar{r}_i}{dt}}_{\bar{v}_i} = \underline{\frac{d}{dt} \frac{\partial T_i}{\partial \dot{q}_j} - \frac{\partial T_i}{\partial q_j}}$$

WHERE $T_i = \frac{1}{2} m_i v_i^2$. THIS TRANSFORMATION IS ONE OF THE MAJOR DISCOVERIES OF MATHEMATICAL PHYSICS IN THE 1700'S.

IF $T = \sum_i T_i$, D'ALEMBERT'S PRINCIPLE CAN NOW BE PUT IN THE FORM

$$\sum_j \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}\right) \delta q_j = \sum_i \bar{F}_i^e \cdot \delta \bar{r}_i$$

WE NOW TRANSFORM THE RIGHT HAND SIDE:

$$\sum_i \bar{F}_i^e \cdot \delta \bar{r}_i = \sum_j \sum_i \bar{F}_i^e \cdot \frac{\partial \bar{r}_i}{\partial q_j} \delta q_j \equiv \sum_j Q_j \delta q_j$$

WHERE $Q_j = \sum_i \bar{F}_i^e \cdot \frac{\partial \bar{r}_i}{\partial q_j} \equiv j$TH GENERALISED FORCE

Q_j NEED NOT HAVE THE DIMENSIONS OF FORCE, BUT BY DEFINITION, $Q_j \delta q_j$ HAS THE DIMENSIONS OF WORK.

(IF q_j IS AN ANGLE VARIABLE, Q_j IS SOMETHING LIKE TORQUE)

ie $N d\theta = dW$

THUS $$\sum_j \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j \right) \delta q_j = 0$$

IF THE q_j ARE INDEPENDENT $\Leftrightarrow$ HOLONOMOUS CONSTRAINTS, THEN WE COULD SET ALL THE $\delta q_j = 0$ EXCEPT ONE. HENCE

$$\boxed{\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j} \qquad j = 1 \ldots k$$

THESE ARE LAGRANGE'S EQUATIONS IN TERMS OF THE GENERALISED FORCES.

IF THE EXTERNAL FORCES ARE CONSERVATIVE, WE SAW THAT WE COULD CONSTRUCT A POTENTIAL ENERGY FUNCTION

$$V = V(\bar{r}_1 \ldots \bar{r}_n) \qquad \text{[NO EXPLICIT TIME DEPENDENCE]}$$

WHERE $\bar{F}_i^e = -\bar{\nabla}_i V \qquad \left[F_i^e \big|_x = -\frac{\partial}{\partial x_i} V \right]$

THEN $$Q_j = \sum_i \bar{F}_i^e \cdot \frac{\partial \bar{r}_i}{\partial q_j} = -\sum_i \bar{\nabla}_i V \cdot \frac{\partial \bar{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

THE FORM OF THIS RELATION ADDS TO THE ELEGANCE OF OUR DEFINITION OF GENERALISED FORCE.

[NOTE $\frac{\partial V}{\partial q_j} = \sum_i \left(\frac{\partial V}{\partial x_i} \frac{\partial x_i}{\partial q_j} + \frac{\partial V}{\partial y_i} \frac{\partial y_i}{\partial q_j} + \frac{\partial V}{\partial z_i} \frac{\partial z_i}{\partial q_j} \right) = \sum_i \bar{\nabla}_i V \cdot \frac{\partial \bar{r}_i}{\partial q_j}$]

FURTHERMORE, V DOES NOT DEPEND ON $\dot{q}_j$

SO $$\frac{\partial V}{\partial \dot{q}_j} = 0$$

WE DEFINE $\boxed{L = T - V}$ = THE <u>LAGRANGIAN</u>

THEN $$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0} \qquad j = 1 \ldots k$$

THESE K EQUATIONS OF MOTION ARE THE BASIC RESULT OF LAGRANGE'S METHOD.