

CALCULUS OF VARIATIONS

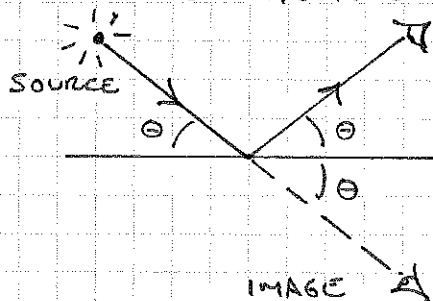
THESE NOTES WERE PREPARED FROM:

'THE SCIENCE OF MECHANICS' BY ERNST MACH, OPEN COURT PAPERBACK
'CALCULUS OF VARIATIONS' BY ROBERT WEINSTEIN, DOVER PAPERBACK
DISCUSSIONS ARE ALSO FOUND IN MECHANICS TEXTS BY MARION & BY
GOLDSTEIN. YOU SHOULD CERTAINLY READ LECTURE 19, VOL II BY FEYNMAN.

THERE IS ANOTHER METHOD OF ATTACK IN MECHANICS, BOTH USEFUL AS A MATHEMATICAL TOOL, AND AS A MEANS OF GAINING INSIGHT. THE GENERAL IDEA IS THAT WHATEVER ACTUALLY HAPPENS IS AN EXTREME CASE OF WHAT MIGHT HAPPEN. AMONG THE CONTINUUM OF POSSIBILITIES, NATURE CHOOSES THE 'MOST UNIQUE!' MATHEMATICALLY, THE METHOD IS TO FIND SOME MEASURE OF THE ENTIRE MOTION WHICH IS AT A MAXIMUM (OR MINIMUM).

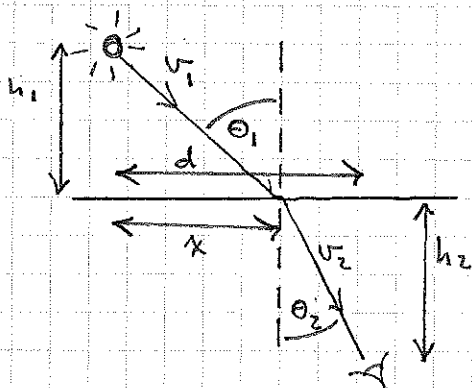
THIS VIEWPOINT HAS ANCIENT ORIGINS. THE EARLIEST EXAMPLE IS SAID TO HAVE BEEN SOLVED BY A WOMAN. ON LANDING IN WHAT IS NOW TUNISIA, DIDO WAS GRANTED BY THE NATIVES ONLY THE AMOUNT OF LAND WHICH COULD BE CONTAINED IN A BULL'S HIDE. SHE CUT IT INTO A LONG LOOP (OF FIXED LENGTH) AND ARRANGED THE LOOP IN A CIRCLE TO COVER THE MAXIMUM AREA, THUS FOUNDING CARTHAGE.

HERO DEDUCED THE LAW OF REFLECTION IN A MIRROR BY POSTULATING THAT THE PATH A LIGHT RAY TAKES IS THE SHORTEST.



THE SHORTEST PATH BETWEEN THE SOURCE AND THE IMAGE IS 'CLEARLY' A STRAIGHT LINE. HENCE EQUAL ANGLES ON REFLECTION.

THIS NOTION WAS REVIVED BY FERMAT (1640) WHO SAW THAT IT IS EQUIVALENT TO THE REQUIREMENT THAT THE LIGHT RAY TAKE THE LEAST TIME TO TRAVERSE THE ACTUAL PATH. THIS VIEW ALLOWED A DERIVATION OF SNELL'S LAW OF REFRACTION.



$$t = \frac{\sqrt{h_1^2 + x^2}}{v_1} + \frac{\sqrt{h_2^2 + (d-x)^2}}{v_2}$$

$$\frac{dt}{dx} = \frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2} = 0 \Rightarrow \frac{\sin \theta}{v} = \text{CONST.}$$

WITH $v = c/n$ THIS IS SNELL'S LAW.

NOT THE LEAST REMARKABLE FEATURE OF FERMAT'S RESULT IS THE USE OF DIFFERENTIAL CALCULUS - DEVELOPED BY FERMAT HIMSELF.

JOHN AND JAMES BERNOULLI (1696) ARE ASSOCIATED WITH TWO FAMOUS PROBLEMS. WHAT IS THE SHAPE OF A HANGING CHAIN? THEY ARGUED THAT THE SHAPE SHOULD MINIMIZE THE POTENTIAL ENERGY. WHAT IS THE SHAPE OF THE CURVE JOINING TWO POINTS SUCH THAT A PARTICLE SLIDING WITHOUT FRICTION TAKES THE LEAST TIME? - THE BRACHISTOCROME PROBLEM.

JOHN SOLVED IT BY AN APPLICATION OF FERMAT'S PRINCIPLE. JAMES USED A METHOD WHICH WAS THE BEGINNING OF THE CALCULUS OF VARIATIONS. EACH THOUGHT THE OTHER HAD STOLEN THE ANSWER AND INVENTED A 'FAKE' DERIVATION. APPARENTLY THEY REMAINED UNRECONCILED UNTIL THEIR DEATHS.

THE ATTEMPT TO DEVISE A GENERAL METHOD FOR MECHANICS WAS CONTINUED BY MAUPER TUIS (1747) AND EULER (1732) WHO SUGGESTED THAT $\int mv ds = 2 \int T dt$ SHOULD TAKE ON AN EXTREME VALUE FOR THE TRUE PATH.

LAGRANGE (1766) GREATLY CLARIFIED THE MATHEMATICAL TECHNIQUES, AND NOTED THAT IF THE POTENTIAL ENERGY V VARIES, IT IS $L = T - V$ ONE SHOULD LOOK AT. THE STATEMENT THAT $S = \int L dt \equiv$ ACTION SHOULD TAKE ON A MINIMUM (OR EXTREME) VALUE FOR THE TRUE PATH IS, HOWEVER, CALLED HAMILTON'S PRINCIPLE (1833).

TO DEVELOP THE MATHEMATICAL TECHNIQUES WE START WITH A SIMPLE CLASS OF PROBLEMS. SUPPOSE THE PATH IN THE x - y PLANE IS $y = y(x)$, AND WE WISH TO MINIMIZE SOME PROPERTY OF THE PATH

$$I = \int_{x_1}^{x_2} f(x, y, y', y'', \dots) dx$$

WE ADD THE CONSTRAINT THAT THE END POINTS OF THE PATH ARE FIXED: $y_1 = y(x_1)$ & $y_2 = y(x_2)$ ARE KNOWN.

EXAMPLES: 1) WHAT IS THE SHORTEST PATH BETWEEN TWO POINTS?

$$\text{ARC LENGTH } ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2} dx \Rightarrow f = \sqrt{1 + y'^2}$$

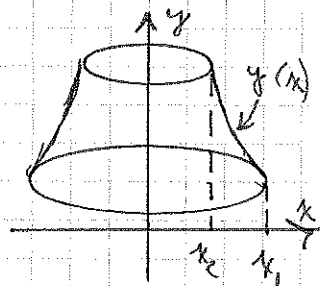
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2) WHAT IS THE SURFACE OF REVOLUTION OF MINIMUM AREA BOUNDED BY TWO DISKS?

a) THE CURVE $y = y(x)$ IS ROTATED ABOUT THE y-AXIS

THE AREA ELEMENT IS $dA = 2\pi x ds = 2\pi x \sqrt{1+y'^2} dx$

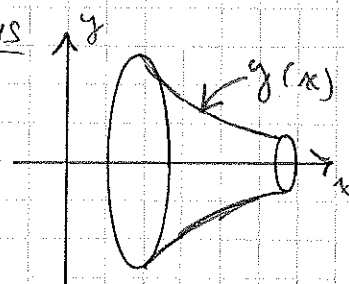
SO $f = x \sqrt{1+y'^2}$



b) THE CURVE $y = y(x)$ IS ROTATED ABOUT THE x-AXIS

THEN $dA = 2\pi y ds = 2\pi y \sqrt{1+y'^2} dx$

AND $f = y \sqrt{1+y'^2} dx$



IN MOST SIMPLE CASES, $f = f(x, y, y')$, SO WE IGNORE THE CASE OF DEPENDENCE ON y'' , y''' ETC.

IF $y(x)$ IS OUR DESIRED BUT UNKNOWN SOLUTION, WE DESCRIBE THE GENERAL POSSIBILITY BY

$$Y(x) = y(x) + \epsilon \eta(x)$$

WHERE $\eta(x_1) = \eta(x_2) = 0$

EX: $\eta = (x-x_1)(x-x_2)$

AND ϵ IS AN ARBITRARY CONSTANT.

BY CONSTRUCTION $Y(x)$ PASSES THRU THE END POINTS (x_1, y_1) & (x_2, y_2)

THE TRICK (DUE TO LAGRANGE) IS THAT WE NEED NOT SPECIFY THE EXACT FORM OF $\eta(x)$. WE WILL OBTAIN OUR SOLUTION BY MINIMIZING

$$I(\epsilon) = \int_{x_1}^{x_2} f(x, Y, Y') dx$$

WITH RESPECT TO ϵ , REQUIRING THAT THE MINIMUM OCCUR AT $\epsilon = 0$. THE FORM OF $\eta(x)$ WILL MIRACULOUSLY DROP OUT!

THIS METHOD IS WELL SUITED TO COMPUTER PROGRAMMING, WITH A MINOR VARIATION. JUST GUESS AT $y(x)$, AND GUESS AT A CORRECTION $\eta(x)$. LET THE COMPUTER FIND THE MINIMUM, SAY AT ϵ_1 . THEN YOU HAVE A NEW GUESS $y_1 = y + \epsilon_1 \eta$. IF YOU DON'T LIKE THIS, ADD A NEW CORRECTION TERM $\eta_2(x)$ AND ITERATE....

WITH $Y = y + \epsilon \eta$ WE HAVE $Y' = y' + \epsilon \eta'$, SO

$$I(\epsilon) = \int f(x, y + \epsilon \eta, y' + \epsilon \eta') dx \quad \text{SHOULD BE MAX. AT } \epsilon = 0$$

$$\left. \frac{dI}{d\epsilon} \right|_{\epsilon=0} = \int \left[\frac{\partial f(x, y, y')}{\partial y} \eta + \frac{\partial f(x, y, y')}{\partial y'} \eta' \right] dx$$

THIS LOOKS BAD, BUT WE SAVE THE DAY BY INTEGRATING BY PARTS IN THE 2ND TERM:

$$\int \frac{\partial f}{\partial y'} \eta' = \underbrace{\frac{\partial f}{\partial y'} \eta}_{=0} \Big|_{x_1}^{x_2} - \int \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \eta$$

$$\text{SO } \left. \frac{dI}{d\epsilon} \right|_{\epsilon=0} = \int \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \eta dx = 0 \quad \text{FOR A MINIMUM}$$

BUT η IS ARBITRARY. THUS THE FACTOR MUST VANISH!

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0} \quad \text{FOR } f = f(x, y, y')$$

THIS IS THE SO-CALLED EULER-LAGRANGE EQUATION

IT IS EASY TO SEE THAT IF $f = f(x, q_1, q_1', \dots, q_n, q_n')$

WHERE EACH $q_i = q_i(x)$, THEN WE GET A WHOLE FAMILY OF

EQUATIONS
$$\frac{\partial f}{\partial q_i} - \frac{d}{dx} \frac{\partial f}{\partial q_i'} = 0$$

EXAMPLES

1) SHORTEST PATH: $f = \sqrt{1+y'^2}$

NOW $\frac{\partial f}{\partial y} = 0$, SO WE HAVE $\frac{\partial f}{\partial y'} = C_1 = \text{CONSTANT}$

OR $\frac{y'}{\sqrt{1+y'^2}} = C_1 \Rightarrow y' = C_2 \Rightarrow y = C_2 x + C_3$

A STRAIGHT LINE!

2) MINIMUM SURFACE OF REVOLUTION OF $y(x)$ ABOUT THE X-AXIS

$$f = y \sqrt{1+y'^2}$$

THE E-L EQUATION LEADS TO $yy'' - y'^2 - 1 = 0$

UNPLEASANT!

WE PULL AN IMPORTANT TRICK OUT OF A HAT:

IN CASE $\frac{\partial f}{\partial x} = 0$, A FIRST INTEGRAL EXISTS. NAMELY

$$\left| y' \frac{\partial f}{\partial y'} - f = C_1 \quad \text{IF} \quad \frac{\partial f}{\partial x} = 0 \right|$$

$$\begin{aligned} \frac{d}{dx} \left(y' \frac{\partial f}{\partial y'} - f \right) &= y'' \frac{\partial f}{\partial y'} + y' \frac{d}{dx} \frac{\partial f}{\partial y'} - \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} y' - \frac{\partial f}{\partial y'} y'' \\ &= y' \left(\frac{d}{dx} \frac{\partial f}{\partial y'} - \frac{\partial f}{\partial x} \right) - \frac{\partial f}{\partial x} = -\frac{\partial f}{\partial x} \quad \text{USING E-L EQ'S.} \end{aligned}$$

IF $\frac{\partial f}{\partial x} = 0$ THE CLAIM FOLLOWS.

IN CASE OF SEVERAL VARIABLES, $f = f(x, q_1, q_1', \dots, q_n, q_n')$

$$\text{IF } \frac{\partial f}{\partial x} = 0, \text{ THEN } \left| \sum_i \left(q_i' \frac{\partial f}{\partial q_i'} \right) - f = C_1 \right|$$

IN OUR CASE $f = y \sqrt{1+y'^2}$ SO $\frac{\partial f}{\partial x} = 0$

$$\text{HENCE } y' \cdot \frac{yy'}{\sqrt{1+y'^2}} - y \sqrt{1+y'^2} = C_1$$

$$-y = C_1 \sqrt{1+y'^2}$$

$$C_1 y'^2 = y^2 - C_1^2$$

$$\frac{C_1 dy}{\sqrt{y^2 - C_1^2}} = dx$$

$$x = C \cosh^{-1} \left(\frac{y}{C} \right) + B$$

$$\text{OR } y = C \cosh \left(\frac{x-B}{C} \right)$$

FIT CONSTANTS B & C SO THAT THE CURVE GOES THRU (x_1, y_1) & (x_2, y_2) .

3) DEGENERATE SOLUTIONS (A MATHEMATICAL DIGRESSION)

THE METHOD OF THE CALCULUS OF VARIATIONS FAILS IF IT HAPPENS THAT $f = \frac{d}{dx} g(x, y)$. THEN ANY $y(x)$ WHATSOEVER IS ALLOWED.

SUCH A SITUATION WAS CLEARLY DEGENERATED.

$$f = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} y' \quad ; \quad \frac{\partial f}{\partial y'} = \frac{\partial g}{\partial y} \quad ; \quad \frac{d}{dx} \frac{\partial f}{\partial y'} = \frac{\partial^2 g}{\partial x \partial y} + \frac{\partial^2 g}{\partial y^2} y'$$

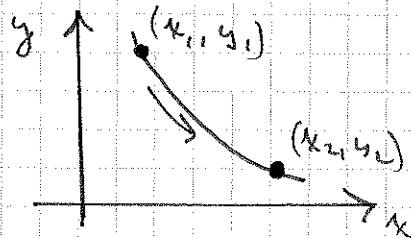
$$\frac{\partial f}{\partial y} = \frac{\partial^2 g}{\partial y \partial x} + \frac{\partial^2 g}{\partial y^2} y' \quad , \text{ AND THE E-L EQUATION IS SATISFIED!}$$

AS A COROLLARY, IF $y(x)$ SATISFIES THE E-L EQUATIONS, THEN $y(x) + \frac{d}{dx} g(x, y)$ DOES ALSO, SINCE THE EQUATIONS ARE LINEAR.

4) THE BRACHISTOCROME

THE PARTICLE STARTS FROM REST AT (x_1, y_1)

$$\text{TIME} = \int_1^2 \frac{ds}{v}$$



$$ds = \sqrt{1+y'^2} dx \quad ; \quad v^2 = 2g(y_1 - y) \Rightarrow f \sim \sqrt{\frac{1+y'^2}{y_1 - y}} \quad \text{i.e. } \sqrt{2g} t = \int f dx$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow y' \frac{\partial f}{\partial y'} - f = C \Rightarrow y' = \sqrt{1 - \frac{1}{C^2(y_1 - y)}}$$

SOME TRICK IS NEEDED! YOU MAY VERIFY THAT THE SUBSTITUTION

$$y_1 - y = \frac{1}{C^2} \sin^2 \frac{\theta}{2} \quad \text{LEADS TO}$$

$$x = x_1 + a(\theta - \sin \theta) \quad y = y_1 - a(1 - \cos \theta) \quad a \equiv \frac{1}{2C^2}$$

ANOTHER METHOD IS TO PARAMETRIZE THE PATH BY $x = x(t)$, $y = y(t)$

$$\text{THEN } ds = \sqrt{\dot{x}^2 + \dot{y}^2} dt \Rightarrow f = \sqrt{\frac{\dot{x}^2 + \dot{y}^2}{y_1 - y}}$$

$$\text{NOW THERE ARE 2 E-L EQUATIONS: } \frac{\partial f}{\partial x} - \frac{d}{dt} \frac{\partial f}{\partial \dot{x}} = 0 \quad \frac{\partial f}{\partial y} - \frac{d}{dt} \frac{\partial f}{\partial \dot{y}} = 0$$

$$\frac{\partial f}{\partial t} = 0 \text{ , SO FROM P. 55, } \quad \dot{x} \frac{\partial f}{\partial \dot{x}} + \dot{y} \frac{\partial f}{\partial \dot{y}} - f = C$$

BUT THIS TURNS OUT TO BE THE IDENTITY, $C = 0$.

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BUT $\frac{\partial f}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial \dot{x}} = C = \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \frac{1}{\sqrt{y_1 - y}}$

WE LET $\alpha =$ ANGLE OF THE TANGENT: $\tan \alpha = \dot{y} = \frac{\dot{y}}{\dot{x}} \Rightarrow \cos \alpha = \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$

THEN $y_1 - y = \frac{\cos^2 \alpha}{c^2} = \frac{1 + \cos 2\alpha}{2c^2}$ OR $y = y_1 - \frac{1}{2c^2} (1 + \cos 2\alpha)$

ALSO $\dot{x} = \cot \alpha \dot{y} = \cot \alpha \sin 2\alpha \frac{\dot{\alpha}}{c^2} = \frac{2}{c^2} \cos^2 \alpha \dot{\alpha} = \frac{1}{2c^2} (1 + \cos 2\alpha) 2\dot{\alpha}$

$\Rightarrow \alpha = \frac{1}{2c^2} (2\alpha + \sin 2\alpha) + K$

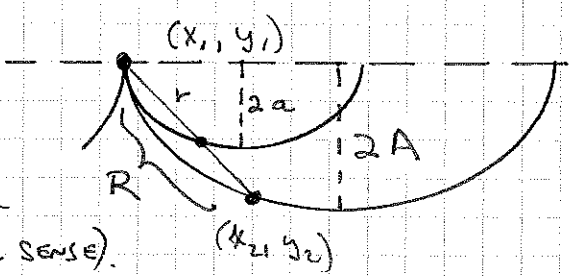
WE DEFINE $a = \frac{1}{2c^2}$ $2\alpha = \theta - \pi$ $K = \frac{\pi}{2c^2}$

THEN $\alpha = a(\theta - \sin \theta)$ $y = y_1 - a(1 - \cos \theta)$ AS BEFORE

THIS IS, OF COURSE, A CYCLOID.

NOTE THAT WE NEVER EVEN USED THE 2ND E-L EQUATION.

THE CYCLOID HAS A CUSP AT (x_1, y_1)



HOW IN PRACTICE DO WE FIND THE CYCLOID WHICH PASSES THRU (x_2, y_2) ? ALL CYCLOIDS ARE SIMILAR (IN A MATHEMATICAL SENSE).

SO DRAW ANY TRIAL CYCLOID OF PARAMETER a . IF A IS THE PARAMETER OF THE DESIRED CYCLOID THRU (x_2, y_2)

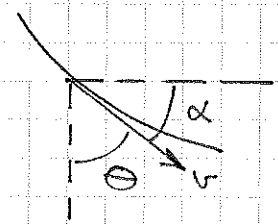
THEN $A = a \frac{R}{r}$ AS SHOWN.

IN GENERAL, (x_2, y_2) IS NOT AT THE BOTTOM OF THE CYCLOID.

JOHN BERNOULLI'S METHOD

HE SUPPOSED FERMAT'S PRINCIPLE APPLIED.

THAT IS, $\frac{\sin \theta}{v} =$ CONSTANT, AS ON P. 51.



OF COURSE, $v^2 = 2g(y_1 - y)$

AND $\sin \theta = \cos \alpha$ IN THE NOTATION USED ABOVE.

THUS $y = y_1 - \frac{\cos^2 \alpha}{c^2}$, AND THE REST FOLLOWS AS ABOVE!

5. ISOPERIMETRIC PROBLEMS

DIDO'S PROBLEM IS TO ENCLOSE THE MAXIMUM TOTAL AREA BY A CURVE OF A FIXED LENGTH. WE CANNOT USE OUR CALCULUS OF VARIATIONS IMMEDIATELY, DUE TO THE PRESENCE OF THE AUXILIARY CONSTRAINT.

HOWEVER, LAGRANGE HAS PROVIDED US WITH AN AMAZING TECHNIQUE TO HANDLE THE DIFFICULTY. IN GENERAL, IF WE WISH TO MAXIMIZE

$$F(x, y, \dots)$$

SUBJECT TO A

CONSTRAINT, $G(x, y, \dots) = \text{CONSTANT}$, WE SIMPLY CONSIDER

$$F^* = F + \lambda G \quad \text{WHERE } \lambda \text{ IS AN AS YET}$$

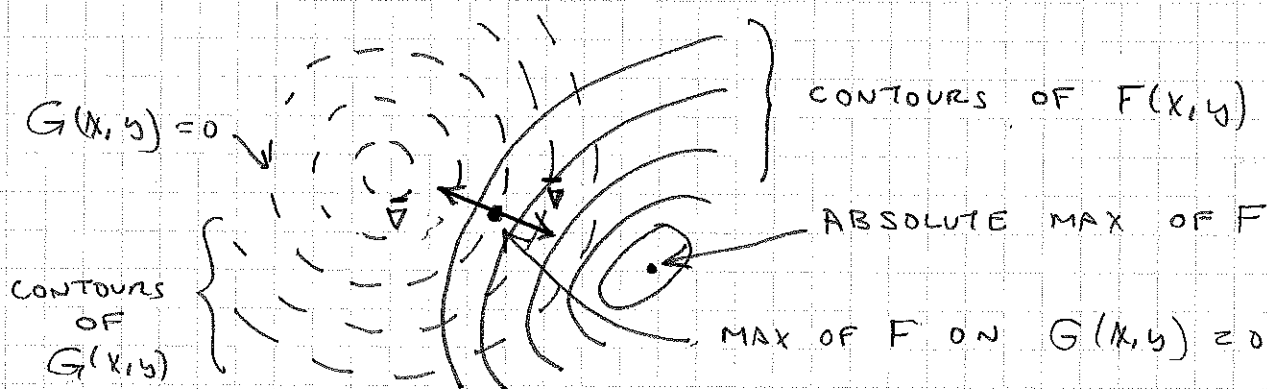
UNDETERMINED CONSTANT, THE LAGRANGE MULTIPLIER. THEN

MINIMIZE F^* BY WHATEVER PROCEDURE WOULD HAVE BEEN SUITABLE

FOR F ALONE, AND FINALLY CHOOSE λ SO THAT THE CONSTRAINT

$$G(x, y, \dots) = \text{CONST. IS SATISFIED.}$$

WE GIVE 2 JUSTIFICATIONS OF THIS METHOD. FIRST VIA PICTURES. SUPPOSE $F = F(x, y)$ AND WE WANT TO MAXIMIZE THIS ON THE CURVE $G(x, y) = 0$.



AT THE DESIRED POINT THE GRADIENTS OF F AND G ARE

$$\text{PARALLEL} \Rightarrow \nabla F = -\lambda \nabla G$$

$$\text{SO } \nabla (F + \lambda G) = 0$$

BUT THIS IS JUST THE WAY WE WOULD FIND THE ABSOLUTE MAXIMUM OF F ; I.E. FIND WHERE $\nabla F = 0$.

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WE NOW GIVE AN ALGEBRAIC DERIVATION.

WE WISH TO MAXIMIZE $F(x, y)$. IF WE MOVE A LITTLE WAY OFF THE MAXIMUM,

$$\delta F = \frac{\partial F}{\partial x} \delta x + \frac{\partial F}{\partial y} \delta y = 0$$

IF δx AND δy WERE INDEPENDENT, WE WOULD HAVE $\frac{\partial F}{\partial x} = 0 = \frac{\partial F}{\partial y}$

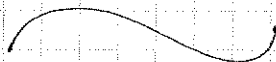
BUT WE ARE CONSTRAINED TO LIE ON THE CURVE $G(x, y) = 0$

$$\therefore \frac{\partial G}{\partial x} \delta x + \frac{\partial G}{\partial y} \delta y = 0 \quad \text{EXPRESSES THE CONSTRAINT}$$

BETWEEN δx AND δy . HENCE $\frac{\partial F / \partial x}{\partial G / \partial x} = \frac{\partial F / \partial y}{\partial G / \partial y} \equiv -\lambda$

$$\text{SO WE HAVE } \frac{\partial}{\partial x} (F + \lambda G) = 0, \quad \frac{\partial}{\partial y} (F + \lambda G) = 0 \quad \text{OR } \nabla (F + \lambda G) = 0$$

AS CLAIMED.



WE RETURN TO DIDO'S PROBLEM. WE WISH TO MAXIMIZE THE AREA INSIDE A CLOSED CURVE. IN ORDER TO USE SINGLE-VALUED FUNCTIONS, WE PARAMETERIZE THE CURVE BY

$$x = x(t), \quad y = y(t)$$

AND SUPPOSE THAT AS t INCREASES, WE MOVE ANTI-CLOCKWISE AROUND THE CURVE. THE AREA IS THEN

$$A = - \int y dx = - \int y \dot{x} dt$$

$$\text{BUT ALSO } A = \int x dy = \int x \dot{y} dt$$

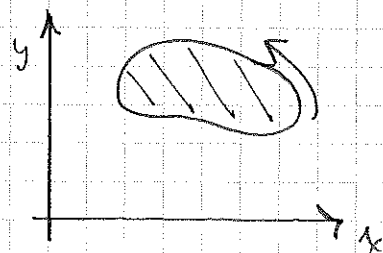
IT PROVES CONVENIENT TO COMBINE THESE TWO EXPRESSIONS

$$A = \frac{1}{2} \int (x \dot{y} - y \dot{x}) dt$$

THE CONSTRAINT IS THAT ARC LENGTH $L = \int ds = \int \sqrt{\dot{x}^2 + \dot{y}^2} dt$

IS CONSTANT. TO USE THE NOTATION INTRODUCED ABOVE:

$$A \leftrightarrow F \equiv \int f dt \quad ; \quad L \leftrightarrow G \equiv \int dt \quad ; \quad \text{THEN } F^* = \int (f + \lambda g) dt \equiv \int f^* dt$$



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SO WE HAVE $f = \frac{1}{2}(x\dot{y} - y\dot{x})$ AND $g = \sqrt{\dot{x}^2 + \dot{y}^2}$

$f^* = f + \lambda g$. WE WANT $\int f^* dt$ TO BE A MAXIMUM

SO f^* WILL OBEY THE EULER-LAGRANGE EQUATIONS:

$$\frac{\partial f^*}{\partial x} = \frac{d}{dt} \frac{\partial f^*}{\partial \dot{x}} \quad \frac{\partial f^*}{\partial y} = \frac{d}{dt} \frac{\partial f^*}{\partial \dot{y}}$$

$$\frac{\partial f^*}{\partial x} = \frac{\dot{y}}{2} = \frac{d}{dt} \frac{y}{2} \Rightarrow \frac{d}{dt} \left(\frac{y}{2} - \frac{\partial f^*}{\partial \dot{x}} \right) = 0$$

BUT $\frac{\partial f^*}{\partial \dot{x}} = \frac{-y}{2} + \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$ SO $y - \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = C_1$

$$\frac{\partial f^*}{\partial y} = -\frac{\dot{x}}{2} = \frac{d}{dt} \left(-\frac{x}{2} \right) \Rightarrow \frac{d}{dt} \left(-\frac{x}{2} - \frac{\partial f^*}{\partial \dot{y}} \right) = 0$$

$$\frac{\partial f^*}{\partial \dot{y}} = \frac{x}{2} + \frac{\lambda \dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \quad \text{SO} \quad -x - \frac{\lambda \dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = C_2$$

HENCE $(x + C_2)^2 + (y - C_1)^2 = \lambda^2$ A CIRCLE

CLEARLY $\lambda = \text{RADIUS}$, SO CHOOSE $\lambda = \frac{L}{2\pi}$ TO SATISFY THE CONSTRAINT.

HAMILTON'S PRINCIPLE

BY NOW WE SEE THAT THE CALCULUS OF VARIATIONS WAS MADE TO ORDER FOR LAGRANGE'S METHOD OF MECHANICS.

THE EQUATIONS
$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

ARE OF THE FORM OF THE EULER-LAGRANGE EQUATIONS FOR THE PROBLEM OF MINIMIZING THE ACTION $S = \int L dt.$

THUS WE COULD REPLACE NEWTON'S LAWS BY HAMILTON'S PRINCIPLE (SOMETIMES CALLED THE PRINCIPLE OF LEAST ACTION), PLUS A PRESCRIPTION OF HOW TO CALCULATE L .

AT THIS POINT WE MAKE CONTACT WITH THE TEXT OF LANDAU & LIFSHITZ, WHICH WE WILL BEGIN TO FOLLOW MORE AND MORE CLOSELY. L & L GIVE A RATHER BRILLIANT EXPOSITION OF HOW TO DEDUCE THE FORM OF THE LAGRANGIAN L OF A FREE PARTICLE FROM THE PRINCIPLE OF GALILEAN RELATIVITY (NEWTON'S 1ST LAW). THIS WAY OF THINKING IS A POWERFUL TOOL IN 'GUESSING' THE FORMS OF INTERACTIONS AMONG ELEMENTARY PARTICLES, FOR WHICH NEWTON'S LAWS NO LONGER GIVE FIRM GUIDANCE.

CONSERVATION LAWS

IT IS AMUSING AND INSTRUCTIVE TO RE-EXAMINE THE NATURE OF OUR 3 CONSERVATION LAWS OF MECHANICS FROM THE LAGRANGIAN VIEWPOINT.

ENERGY

SUPPOSE THE LAGRANGIAN L OF THE SYSTEM DOES NOT DEPEND EXPLICITLY ON TIME:
$$\frac{\partial L}{\partial t} = 0.$$

THEN FROM OUR DISCUSSION ON PSS, WE KNOW THAT

$$H = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = \sum_i q_i P_i - L = \text{CONSTANT}$$

H IS THE HAMILTONIAN OF THE SYSTEM (WHETHER OR NOT IT IS CONSTANT)

WE WISH TO SHOW THAT $H = E = \text{TOTAL ENERGY}$ IF THE CONSTRAINTS ARE TIME INDEPENDENT. I.E. IF

$$\frac{\partial \bar{T}_i(q_j)}{\partial t} = 0 \quad \text{HOLDS FOR THE COORDINATE TRANSFORMATION.}$$

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THE POTENTIAL V DOES NOT DEPEND ON THE VELOCITIES:

$$\text{so } \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$$

$$\text{NOW } T = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i \left(\sum_j \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j \right) \left(\sum_k \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k \right) \equiv \frac{1}{2} \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k$$

USING $\partial \vec{r}_i / \partial t = 0$

NOTE THAT $a_{jk} = a_{kj}$ (q_1, \dots, q_n ONLY)

$$\frac{\partial T}{\partial \dot{q}_j} = \sum_k a_{jk} \dot{q}_k \quad \text{USING } a_{jk} = a_{kj}$$

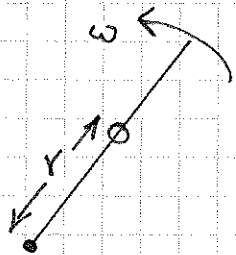
$$\text{THUS } \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} = \sum_j \sum_k a_{jk} \dot{q}_j \dot{q}_k = 2T$$

$$\text{AND } H = 2T - L = T + V = E = \text{TOTAL ENERGY.}$$

THUS CONSERVATION OF ENERGY \iff INVARIANCE OF THE LAGRANGIAN WITH RESPECT TO TIME

WHAT HAPPENS WHEN THE CONSTRAINTS DEPEND ON TIME?

EXAMPLE PROB 9 SET 2.



$$L = T = \frac{1}{2} m (\dot{r}^2 + r^2 \omega^2)$$

$$r = r_0 \cosh \omega t \quad (\text{IF } v_0 = 0)$$

$$\dot{r} = r_0 \omega \sinh \omega t$$

$$T = \frac{1}{2} m r_0^2 \omega^2 (\cosh^2 \omega t + \sinh^2 \omega t) \quad \text{NOT CONSTANT}$$

$$\begin{aligned} \text{BUT } H &= \dot{r} \frac{\partial L}{\partial \dot{r}} - L = m \dot{r}^2 - \frac{1}{2} m \dot{r}^2 - \frac{1}{2} m r^2 \omega^2 = \frac{1}{2} m (\dot{r}^2 - r^2 \omega^2) \\ &= \frac{1}{2} m r_0^2 \omega^2 (\sinh^2 \omega t - \cosh^2 \omega t) = -\frac{m r_0^2 \omega^2}{2} = \text{CONSTANT!} \end{aligned}$$

HOWEVER, H IS NOT THE ENERGY, BUT MERELY AN INITIAL CONSTANT.

WHY DOES T CHANGE? THE FORCE OF CONSTRAINT IS \perp TO THE WIRE, BUT THE WIRE IS ROTATING, SO \vec{v} HAS A COMPONENT \perp TO THE WIRE ALSO. HENCE $\frac{dT}{dt} = \vec{F} \cdot \vec{v} \neq 0$. WE ENCOURAGE YOU TO

WORK OUT THE DETAILS ON THE HOMEWORK SET. [DOES THE FACT THAT THE CONSTRAINT FORCES DO WORK INVALIDATE LAGRANGE'S METHOD HERE?]