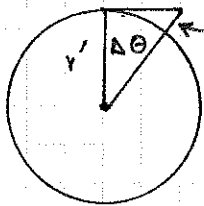


CENTRAL FORCES

IT WAS NEWTON'S INSIGHT THAT A CENTRAL FORCE ( $\vec{F}(\vec{r}) = F(r)\hat{r}$ ) SUCH AS GRAVITY COULD "HOLD" THE MOON IN ORBIT AS WELL AS CAUSE APPLES TO FALL. ACTUALLY, HE NOTED THAT THE MOON IS IN FACT "FALLING" ALL THE TIME, AND SO DEDUCED THAT  $F(r) \sim \frac{1}{r^2}$ .

THE MOON TEST. IN 1 SECOND AN APPLE FALLS  $h = \frac{1}{2} g t^2 = 16$  FEET



IF THE MOON ROTATES ABOUT THE EARTH BY  $\Delta\theta$  IN 1 SEC., THEN IT 'FALLS' BY  $h' = r'(1 - \cos \Delta\theta) \sim \frac{r' \Delta\theta^2}{2}$

NEWTON KNEW THAT  $r' \sim 60 r_E \sim 60 \times 4000 \times 5280 \sim 1.25 \times 10^9$  FT.

THE MOON'S PERIOD OF ROTATION IS  $\sim 29$  DAYS  $\sim 2.5 \times 10^6$  SEC.

$$\text{SO } \Delta\theta = \frac{2\pi}{2.5 \times 10^6} \sim 2.5 \times 10^{-6} \text{ RADIANS}$$

$$\text{HENCE } h' = \frac{(1.25 \times 10^9)(2.5 \times 10^{-6})^2}{2} \sim 4 \times 10^{-3} \text{ FEET}$$

IF  $g'$  = STRENGTH OF GRAVITY AT THE MOON, THEN  $h' = \frac{1}{2} g' t^2 \Rightarrow g' \sim 8 \times 10^{-3}$

$$\text{SO } \frac{g'}{g} = \frac{8 \times 10^{-3}}{32} \sim \frac{1}{4000} \quad \text{WHILE } \frac{r'}{r_E} \sim 60 \quad \text{SO } \frac{g'}{g} \sim \left(\frac{r_E}{r'}\right)^2$$

$$\text{OR } \underline{g \sim \frac{1}{r^2}}$$

NEWTON ALSO SHOWED THAT  $F \sim \frac{1}{r^2}$  WILL EXPLAIN KEPLER'S LAWS OF PLANETARY MOTION:

- 1) THE ORBITS ARE ELLIPSES WITH THE SUN AT A FOCUS.
- 2) THE PLANETS SWEEP OUT EQUAL AREAS IN EQUAL TIME.
- 3) THE RATIO  $T^2/a^3$  IS A CONSTANT, WHERE  $T$  = PERIOD AND  $a$  = RADIUS [FOR AN ELLIPSE, KEPLER NOTED THAT  $a$  = SEMI-MAJOR AXIS]

WE SHALL NOW DEMONSTRATE THESE RESULTS AS WELL AS MANY OTHER PROPERTIES OF MOTION UNDER A CENTRAL FORCE.

REDUCED MASS (L & L SEC 13)

ONE KIND OF CENTRAL FORCE PROBLEM IS THAT OF A MASS  $M$  SUBJECT TO A FORCE  $\vec{F} = F \hat{r}$  ABOUT A FIXED FORCE CENTER AT THE ORIGIN.

PLANETARY MOTION CONCERNS TWO MASSES  $M_1$  &  $M_2$  SUBJECT TO A FORCE  $\vec{F}(\vec{r}_1 - \vec{r}_2)$  ALONG THEIR LINE OF CENTERS.

THESE TWO PROBLEMS ARE EQUIVALENT!

TO SHOW THIS WE USE A COORDINATE SYSTEM WITH THE C.M. AS THE ORIGIN, AND SUPPOSE THE C.M. IS AT REST.

$$\text{THEN } M_1 \ddot{\vec{r}}_1 = \vec{F}(\vec{r}_1 - \vec{r}_2) \quad M_2 \ddot{\vec{r}}_2 = -\vec{F}(\vec{r}_1 - \vec{r}_2)$$

$$\text{SO } \ddot{\vec{r}}_1 - \ddot{\vec{r}}_2 = \left( \frac{1}{M_1} + \frac{1}{M_2} \right) \vec{F}(\vec{r}_1 - \vec{r}_2)$$

$$\text{OR } \frac{M_1 M_2}{M_1 + M_2} \ddot{\vec{r}} = F \hat{r} \quad \text{WHERE } \vec{r} \equiv \vec{r}_1 - \vec{r}_2$$

THUS THE PROBLEM OF TWO MASSES CAN BE REDUCED TO THAT OF A SINGLE MASS,  $\mu = \frac{M_1 M_2}{M_1 + M_2} \equiv$  REDUCED MASS,

UNDER A CENTRAL FORCE ABOUT A FIXED ORIGIN. THE MAGNITUDES OF DISTANCE AND FORCE REMAIN THE SAME IN THE EQUIVALENT AND ORIGINAL PROBLEMS.

WE VERIFY THIS FROM LAGRANGE'S POINT OF VIEW.

$$L = \frac{1}{2} M_1 \dot{\vec{r}}_1^2 + \frac{1}{2} M_2 \dot{\vec{r}}_2^2 - V(\vec{r}_1 - \vec{r}_2)$$

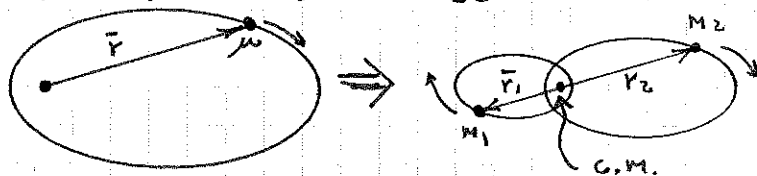
$$\text{TAKING THE C.M. AS ORIGIN, } M_1 \vec{r}_1 + M_2 \vec{r}_2 = 0$$

$$\text{AGAIN DEFINING } \vec{r} = \vec{r}_1 - \vec{r}_2 \Rightarrow \vec{r}_1 = \frac{M_2}{M_1 + M_2} \vec{r}, \quad \vec{r}_2 = -\frac{M_1}{M_1 + M_2} \vec{r}$$

$$\text{THEN } L = \frac{1}{2} \frac{M_1 M_2^2 + M_2 M_1^2}{(M_1 + M_2)^2} \dot{\vec{r}}^2 - V(r) = \frac{1}{2} \frac{M_1 M_2}{M_1 + M_2} \dot{\vec{r}}^2 - V(r)$$

AGAIN THIS IS EQUIVALENT TO A SINGLE PARTICLE PROBLEM WITH A FIXED FORCE CENTER IF WE USE EFFECTIVE MASS  $\mu = \frac{M_1 M_2}{M_1 + M_2}$

NOTE THAT THE SOLUTIONS FOR  $\vec{r}_1$  AND  $\vec{r}_2$  MUST BE INFERRERD FROM THAT FOR  $\vec{r}$ .

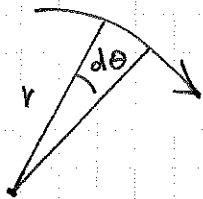


CONSERVATION LAWS ( $\vec{B} \neq 0$  SEC 4-1,  $L \neq L$  SEC 14)

WITH  $V(r) =$  POTENTIAL ENERGY CORRESPONDING TO THE CENTRAL FORCE  $\vec{F} = F(r) \hat{r}$

THEN THE ENERGY  $E = T + V$  IS CONSERVED.

SINCE TORQUE  $\vec{N} = \vec{r} \times \vec{F} = 0$ , ANGULAR MOMENTUM ABOUT THE ORIGIN (OR C.M.) IS CONSERVED.



IN POLAR COORDS  $L = m r^2 \dot{\theta} = \text{CONSTANT}$

$$\text{NOW } d\text{AREA} = \frac{1}{2} r^2 d\theta$$

$$\text{SO } \frac{d\text{AREA}}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} + \text{2ND ORDER} = \frac{L}{2m} = \text{CONSTANT}$$

HENCE KEPLER'S 2ND LAW OF EQUAL AREA IN EQUAL TIMES IS SATISFIED BY ANY CENTRAL FORCE.

CONSERVATION OF ANGULAR MOMENTUM ALSO REQUIRES THE MOTION UNDER A CENTRAL FORCE TO BE ENTIRELY IN A PLANE. (PROVE THIS TO YOURSELF.) THUS THE CENTRAL FORCE PROBLEM IS REDUCED FROM 3 DIMENSIONS TO 2.

WE MAY REDUCE IT TO A 1-DIMENSIONAL PROBLEM USING THE EFFECTIVE POTENTIAL METHOD.

$$E = \frac{1}{2} m \dot{r}^2 + V(r) = \text{CONST}$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 + V(r)$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{L^2}{2m r^2} + V(r)$$

$$\text{USING } L = m r^2 \dot{\theta}$$

$$\text{SO WE DEFINE } V_{\text{EFF}} = V(r) + \frac{L^2}{2m r^2}$$

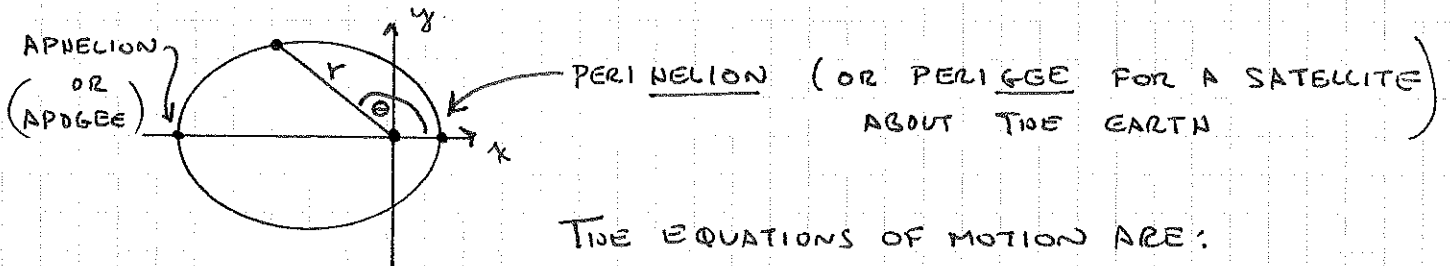
ANY CENTRAL FORCE PROBLEM CAN BE ANALYZED IN TERMS OF OSCILLATIONS ABOUT AN EQUILIBRIUM RADIUS  $r_0$  USING THIS TECHNIQUE.

THIS METHOD EMPHASIZES CIRCULAR ORBITS, WHILE KEPLER CALLS FOR ELLIPSES. CAN WE FIND ELLIPSES DIRECTLY IF  $F = \frac{-k}{r^2}$ .

ELEMENTARY DERIVATION OF PLANETARY ORBITS

NEWTONS LAW OF GRAVITATION IS  $\vec{F} = -\frac{GM_1M_2}{r^2} \hat{r}$  ( $\vec{r} = \vec{r}_1 - \vec{r}_2$ )

WE WISH TO SHOW THAT THE ORBIT IS AN ELLIPSE



THE EQUATIONS OF MOTION ARE:

$$\mu \ddot{x} = -\frac{GM_1M_2}{r^2} \cos \theta$$

$$\mu \ddot{y} = -\frac{GM_1M_2}{r^2} \sin \theta$$

$\mu = \frac{m_1 m_2}{m_1 + m_2}$  = REDUCED MASS

WE KNOW THAT ANGULAR MOMENTUM  $L = m_1 v_1 r_1 + m_2 v_2 r_2 = \mu r^2 \dot{\theta} = \text{CONST.}$

THE TRICK IS TO CONVERT  $\ddot{x} = \frac{dx}{dt} = \frac{dx}{d\theta} \frac{d\theta}{dt} = \frac{L}{\mu r^2} \frac{dx}{d\theta}$

A MIRACLE:  $y$  VANISHES FROM THE EQUATIONS

$$\frac{dx}{d\theta} = -\frac{GM_1M_2}{L} \cos \theta$$

$$\frac{dy}{d\theta} = -\frac{GM_1M_2}{L} \sin \theta$$

so  $x = A - \frac{GM_1M_2}{L} \sin \theta$

$y = B + \frac{GM_1M_2}{L} \cos \theta$

OUR INITIAL CONDITIONS (SEE FIGURE ABOVE)  $\Rightarrow A = 0$ .

PLAYING THE TRICK AGAIN

$$\dot{x} = \frac{dx}{d\theta} \dot{\theta} = \frac{L}{\mu r^2} \frac{dx}{d\theta} \Rightarrow \frac{dx}{d\theta} = -\frac{GM_1M_2}{L^2} \mu r^2 \sin \theta$$

$$\dot{y} = \frac{L}{\mu r^2} \frac{dy}{d\theta} \Rightarrow \frac{dy}{d\theta} = \frac{B \mu r^2}{L} + \frac{GM_1M_2}{L^2} \mu r^2 \cos \theta$$

$$r = r \omega \theta \Rightarrow \frac{dx}{d\theta} = \frac{dr}{d\theta} \omega \theta - r \sin \theta = -\frac{GM_1M_2}{L^2} \mu r^2 \sin \theta$$

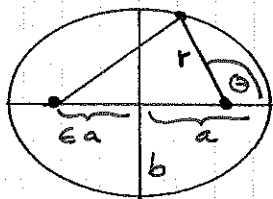
$$y = r \sin \theta \Rightarrow \frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta = \frac{B \mu r^2}{L} + \frac{GM_1M_2}{L^2} \mu r^2 \cos \theta$$

ANOTHER MIRACLE:  $\frac{dr}{d\theta}$  CAN BE ELIMINATED BY COMBINING EQUATIONS, AND WE DON'T HAVE TO INTEGRATE!

$$r = \frac{B\mu r^2 \omega \theta}{L} + \frac{GM_1 M_2 \mu r^2}{L^2}$$

$$\text{OR } \frac{1}{r} = \frac{GM_1 M_2 \mu}{L^2} + \frac{B\mu}{L} \omega \theta$$

WE RECALL THE CONSTRUCTION OF AN ELLIPSE OF SEMI-MAJOR AXIS  $a$ , AND ECCENTRICITY  $e \Rightarrow ea =$  DISTANCE FROM CENTER TO FOCUS.



THE SUM OF THE DISTANCES FROM THE TWO FOCI IS CONSTANT

$$D = 2a = 2\sqrt{b^2 + (ea)^2} \Rightarrow b = a\sqrt{1-e^2}$$

$$D = r + \sqrt{(r\cos\theta + 2ea)^2 + (r\sin\theta)^2}$$

$$\text{HENCE } (2a - r)^2 = 4a^2 - 4ar + r^2 = r^2 + 4ea r \cos\theta + 4e^2 a^2$$

$$4a^2(1-e^2) = 4ar(1 + e\cos\theta)$$

$$\frac{1}{r} = \frac{1 + e\cos\theta}{a(1-e^2)}$$

$$\text{ALSO, AREA} = \pi ab = \pi a^2 \sqrt{1-e^2}$$

COMPARING TO THE EXPRESSION FOR  $\frac{1}{r}$  ABOVE, WE SEE THAT THE ORBIT IS INDEED AN ELLIPSE WITH

$$\frac{1}{a(1-e^2)} = \frac{GM_1 M_2 \mu}{L^2} \quad \text{AND} \quad \frac{e}{a(1-e^2)} = \frac{B\mu}{L}$$

THUS WE HAVE VERIFIED KEPLER'S 1ST LAW.

TO DEMONSTRATE THE 3RD LAW, WE RECALL THE 2ND LAW:

$$\frac{d\text{AREA}}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{L}{2\mu} = \text{CONSTANT}$$

$$\text{IF } T = \text{PERIOD, THEN AREA} = \pi a^2 \sqrt{1-e^2} = T \frac{d\text{AREA}}{dt} = \frac{TL}{2\mu}$$

$$\text{SO } T^2 = 4\pi^2 a^4 (1-e^2) \frac{\mu^2}{L^2}$$

$$\text{BUT } a(1-e^2) = \frac{L^2}{GM_1 M_2 \mu}$$

$$\Rightarrow T^2 = \frac{4\pi^2 a^3}{G(M_1 + M_2)}$$

$$\text{USING } \mu = \frac{M_1 M_2}{M_1 + M_2}$$

TO THE EXTENT THAT  $\frac{M_2}{M_1} \ll 1$  ( $M_1 = M_{\text{SUN}}$ ) WE HAVE VERIFIED THE 3RD LAW. EVEN FOR JUPITER,  $\frac{M}{M_{\text{SUN}}} \sim \frac{1}{1000}$  SO THE ACCURACY IS QUITE GOOD.

NOTE THAT WE DO NOT EXPECT THE PERIOD OF THE MOON ABOUT THE EARTH TO PARTICIPATE IN THE 3RD LAW FOR THE MOTION OF PLANETS ABOUT THE SUN. KEPLER KNEW THIS ONLY BY EXPERIMENT, AND NEVER REGARDED THE 3RD LAW AS HIGHLY AS THE FIRST TWO. IT WAS NEWTON WHO ELEVATED THE 3RD LAW TO ITS PRESENT STATUS.

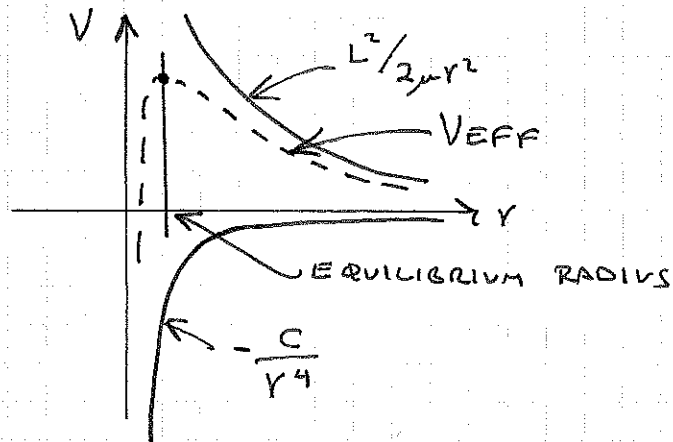
THE 3RD LAW AS GIVEN ABOVE IS EASY TO DERIVE IF YOU ASSUME CIRCULAR ORBITS. TRY IT YOURSELF!

### SOLUTION 2 - EFFECTIVE POTENTIAL METHOD

$$\text{FROM P. 94, } E = \frac{1}{2} \mu \dot{r}^2 + V_{\text{EFF}} \quad \text{WITH } V_{\text{EFF}} = V(r) + \frac{L^2}{2\mu r^2}$$

$L = \text{ANGULAR MOMENTUM} = \text{CONSTANT.}$

THIS METHOD ALLOWS ONE TO COMMENT ON THE STABILITY OF THE ORBITS. (SEE PROBLEM (5) OF THE HOMEWORK SET).



FOR EXAMPLE, IF  $V = -\frac{C}{r^4}$

THE ORBITS ARE UNSTABLE.

FOR GRAVITY,  $V = -\frac{GM_1M_2}{r}$

$$V_{\text{EFF}} = -\frac{GM_1M_2}{r} + \frac{L^2}{2\mu r^2}$$

$$\frac{dV_{\text{EFF}}}{dr} = \frac{GM_1M_2}{r^2} - \frac{L^2}{\mu r^3} \Rightarrow r_0 = \frac{L^2}{GM_1M_2\mu} = \text{EQUILIBRIUM RADIUS}$$

$$\frac{d^2V_{\text{EFF}}}{dr^2} = -\frac{2GM_1M_2}{r^3} + \frac{3L^2}{\mu r^4}$$

$$K = \left. \frac{d^2V_{\text{EFF}}}{dr^2} \right|_{r_0} = -\frac{2GM_1M_2}{r_0^3} \frac{L^2}{r_0 GM_1M_2\mu} + \frac{3L^2}{\mu r_0^4} = \frac{L^2}{\mu r_0^4} > 0 \Rightarrow \text{STABLE!}$$

THE FREQUENCY OF OSCILLATIONS ABOUT EQUILIBRIUM IS

$$\omega = \sqrt{\frac{K}{\mu}} = \frac{L}{\mu r_0^2}$$

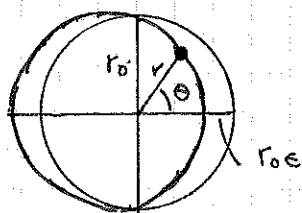
BUT THE FREQUENCY OF THE EQUILIBRIUM ROTATION IS

$$\Omega = \dot{\theta} = \frac{L}{\mu r_0^2}$$

HENCE  $\omega = \Omega \Rightarrow$  1 OSCILLATION PER REVOLUTION

$$\text{i.e. } r \approx r_0 (1 - \epsilon \cos \Omega t) \approx r_0 (1 - \epsilon \cos \theta)$$

WHICH IS ELLIPSE-LIKE



REMEMBER THAT THIS TECHNIQUE GIVES

ONLY AN APPROXIMATE SOLUTION VALID FOR SMALL OSCILLATIONS  $\Rightarrow \epsilon$  SMALL.

USING THE EXACT SOLUTION FOR  $V \propto -\frac{1}{r}$  FOUND ABOVE

$$\frac{1}{r} = \frac{1 + \epsilon \cos \theta}{r_0 (1 - \epsilon^2)} \quad \text{WE GET } r \approx r_0 (1 - \epsilon \cos \theta) \quad \text{FOR SMALL } \epsilon.$$

VERY FEW CENTRAL POTENTIALS YIELD EXACT SOLUTIONS, BUT THE EFFECTIVE POTENTIAL METHOD CAN ALWAYS BE APPLIED.

IN GENERAL,  $\omega \neq \Omega$  AND THE ORBITS PRECESS.

SEE PROBLEMS (9) AND (10) ON THE HOMEWORK SET.

NEAR THE MIDDLE OF P. 98, WE RATHER GLIBLY IDENTIFIED  $\Omega t$  WITH  $\Theta$ . HOWEVER, THIS IS STRICTLY CORRECT ONLY FOR THE EQUILIBRIUM CIRCULAR ORBIT.

IN GENERAL, WE HAVE  $L = \mu r^2 \dot{\Theta} = \text{CONSTANT ANGULAR MOMENTUM}$

OUR OSCILLATORY SOLUTION FOR  $r$  IS  $r = r_0 (1 - \epsilon \cos \Omega t)$

SO  $\dot{\Theta} = \frac{L}{\mu r^2} = \frac{L}{\mu r_0^2 (1 - \epsilon \cos \Omega t)^2} \approx \Omega (1 + 2\epsilon \cos \Omega t) + O(\epsilon^2)$

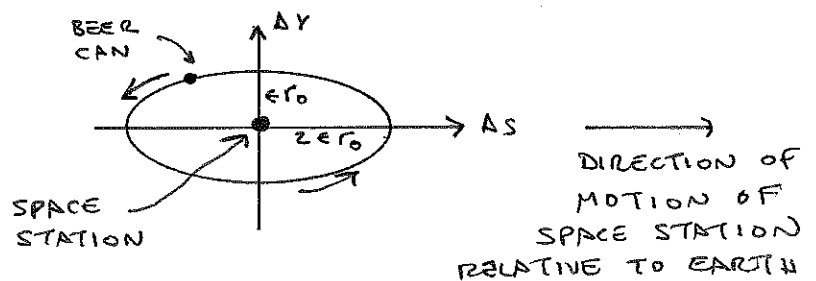
HENCE  $\left\{ \begin{array}{l} \Theta = \Omega t + 2\epsilon \sin \Omega t \\ \text{AND } r = r_0 (1 - \epsilon \cos \Omega t) \end{array} \right. \quad \text{WITH } \Omega = \frac{L}{\mu r_0^2}$

IS THE COMPLETE STATEMENT OF THE ORBIT WHICH IS A SMALL OSCILLATION ABOUT A CIRCLE.

IT IS INSTRUCTIVE TO THINK ABOUT THE BEHAVIOR OF THE FULL SOLUTION WITH AN EXAMPLE. A SPACECRAFT IS IN A CIRCULAR ORBIT ABOUT THE EARTH. AN ASTRONAUT GOES ON A SPACE WALK, AND LAUNCHES A BEER CAN WITH A SMALL VELOCITY. HOW SHOULD HE DO THIS SO THE BEER CAN APPEARS TO ORBIT AROUND THE SPACE STATION, ACCORDING TO AN OBSERVER ON THE STATION!?

CERTAINLY THE BEER CAN WILL BE IN ORBIT ABOUT THE EARTH, IN THE FORM OF A SMALL OSCILLATION ABOUT THE ORBIT OF THE SPACE STATION. SO WE WANT TO RECAST THE ABOVE SOLUTION INTO COORDINATES RELATIVE TO THE SPACE STATION. WE CHOOSE

$\Delta r = r - r_0 = -\epsilon r_0 \cos \Omega t$  AS THE RADIAL POSITION } RELATIVE TO THE STATION  
 AND  $\Delta s = r_0 (\Theta - \Omega t) = 2\epsilon r_0 \sin \Omega t$  AS THE TANGENTIAL POSITION



THE BEER CAN APPEARS TO MOVE IN AN ELLIPSE WHOSE MAJOR AXIS IS ALONG THE ORBIT OF THE SPACE STATION; THE MINOR AXIS IS 1/2 THE SIZE OF THE MAJOR AXIS.



NEAR CIRCULAR ORBIT

① RELATIVE TO THE CIRCULAR ORBIT OF RADIUS  $r_0$ ,  $\omega = \sqrt{\frac{GM}{r_0^3}}$

$r \approx r_0 (1 + \epsilon \cos \omega t)$  IS APPROXIMATE ELLIPSE FOR  $\epsilon \ll 1$

so  $\delta r = \epsilon r_0 \cos \omega t$

TO GET  $\theta$  MOTION, CONSIDER CONSERVED ANG MOM

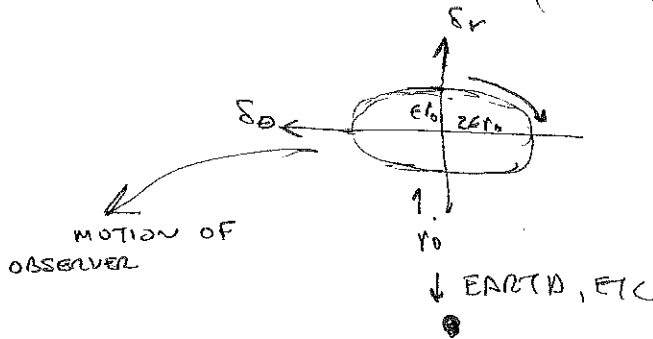
$$\frac{L}{M} = r^2 \dot{\theta} \approx r_0^2 \omega \quad \text{so} \quad \dot{\theta} = \frac{r_0^2 \omega}{r_0^2 (1 + \epsilon \cos \omega t)^2} \approx \omega (1 - 2\epsilon \cos \omega t)$$

$$\Rightarrow \theta = \omega t - 2\epsilon \sin \omega t$$

RELATIVE DISPLACEMENT IN  $\hat{\theta}$  DIR IS  $\delta_\theta = r_0(\theta - \omega t) = -2\epsilon r_0 \sin \omega t$

SO RELATIVE ORBIT, AS OBSERVED IN ROTATING COORD SYSTEM

WITH ORIGIN AT  $(r_0, \omega t)$  IS  $(\delta r, \delta_\theta) = \epsilon r_0 (\cos \omega t, -2\sin \omega t)$

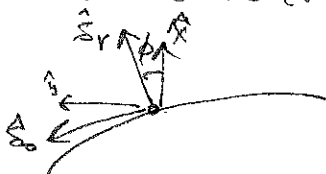


PERIOD IS  $\omega$ , SAME AS  
TROT OF OBSERVER ABOUT EARTH

RELATIVE ORBIT IS RETROGRADE  
ELLIPSE OF ASPECT RATIO  
2:1, MAJOR AXIS TANGENTIAL.

NOW CONVERT TO  $(x, y)$  AXES THAT ARE NOT ROTATING W.R.T. FIXED STARS, BUT WHOSE ORIGIN IS AT  $(r_0, \omega t)$

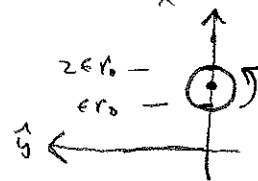
ROTATION ANGLE BETWEEN  $(x, y)$  AND  $(\delta r, \delta_\theta)$  IS  $\phi = \omega t$



$$\hat{\delta}_r = \hat{x} \cos \phi + \hat{y} \sin \phi$$

$$\hat{\delta}_\theta = -\hat{x} \sin \phi + \hat{y} \cos \phi$$

$$\begin{aligned} \vec{\delta} &= \delta_r \hat{\delta}_r + \delta_\theta \hat{\delta}_\theta = \epsilon r_0 \left( \cos \omega t (\hat{x} \cos \omega t + \hat{y} \sin \omega t) - 2 \sin \omega t (-\hat{x} \sin \omega t + \hat{y} \cos \omega t) \right) \\ &= \epsilon r_0 \left( (\cos^2 \omega t + 2 \sin^2 \omega t) \hat{x} + \sin \omega t \cos \omega t \hat{y} \right) \\ &= \epsilon r_0 \left( (1 + \sin^2 \omega t) \hat{x} + \frac{1}{2} \sin 2\omega t \hat{y} \right) \\ &= \frac{\epsilon r_0}{2} \left( (3 - \cos 2\omega t) \hat{x} + \sin 2\omega t \hat{y} \right) \end{aligned}$$



RELATIVE TO THESE AXES, THE ORBIT IS CIRCULAR WITH RADIUS  $\epsilon/2$ ,

AND CENTER OF ORBIT OFFSET BY  $\frac{3\epsilon}{2}$ ; ORBIT FREQ. IS 2 $\omega$ !

12/12/93

PERTURBING ORBITS BY AN IMPULSE  
CIRCULAR ORBIT @  $v_0$

$$\frac{GM}{r_0^3} = \Omega_0^2 \quad v_0 = \Omega_0 r_0 = \sqrt{\frac{GM}{r_0}}$$

$$\frac{L_0}{M} = \Omega_0 r_0^2 = \sqrt{GM r_0^3}$$

THEN CHANGE VELOCITY BY AN IMPULSE

a)  $\Delta v \parallel$  TO ORBIT

NEW ANG MOM IS  $r_0(v_0 - \Delta v) = \frac{L_0}{M} \left(1 - \frac{\Delta v}{v_0}\right) = \sqrt{GM r_0^3} \left(1 - \frac{\Delta v}{v_0}\right) = \sqrt{GM R_0^3}$

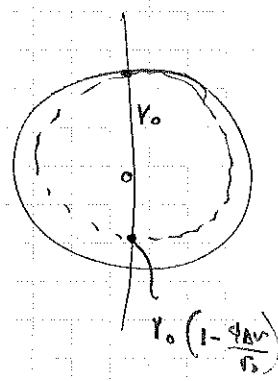
SO  $R_0 = r_0 \left(1 - \frac{2\Delta v}{v_0}\right) =$  EQUILIBRIUM RADIUS OF NEW ORBIT

1.  $r = R_0 (1 + \epsilon \cos \theta)$

WITH  $\epsilon = \frac{r_0 - R_0}{R_0} = \frac{2\Delta v}{v_0}$

AT  $\theta = \pi$   $r = R_0 (1 - \epsilon) = r_0 \left(1 - \frac{4\Delta v}{v_0}\right)$

THIS IS BIGGEST EXCURSION IS  $r$ , SO  $\Delta v_{MAX} = v_0 \cdot \frac{4\Delta v}{v_0}$



b)  $\Delta v \perp$  TO ORBIT

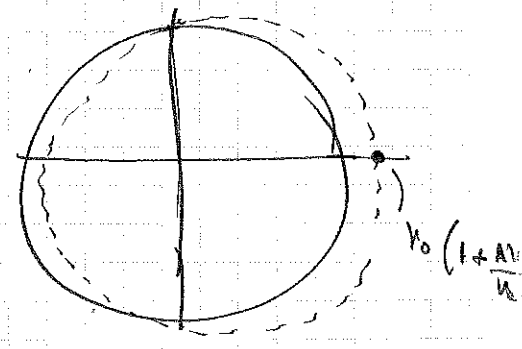
L STAYS SAME, SO  $v_0$  STAYS SAME

BUT ORBIT IS NOW AN ELLIPSE

$\Rightarrow r = r_0 (1 + \epsilon \sin \theta)$

$\Delta v \perp \dot{r} = \epsilon v_0 \dot{\theta} = \epsilon r_0 \Omega_0 = \epsilon v_0$  SO  $\epsilon = \frac{\Delta v}{v_0}$

$\Delta v = \epsilon v_0 \sqrt{\frac{GM}{r_0^3}} = \epsilon \sqrt{\frac{GM}{r_0}} = \epsilon v_0$



BACK TO a) AT  $\theta = \pi/2$

$v = R_0 = r_0 \left(1 - \frac{2\Delta v}{v_0}\right)$

SO EVEN AT  $\theta = \pi/2$  HAVE ADVANTAGE

1.  $\Delta v_a = r_0 - \left(r_0\right) \left(1 - \frac{2\Delta v}{v_0}\right) \left(1 + \frac{2\Delta v}{v_0} \cos \theta\right) = r_0 \frac{2\Delta v}{v_0} (1 - \cos \theta)$

$\Delta v_b = v_0 \frac{\Delta v}{v_0} \sin \theta$   $\Delta v_a = \Delta v_b \Rightarrow \frac{\sin \theta}{2} = 1 - \cos \theta \Rightarrow \frac{\Delta \theta}{1 - \cos \theta} = 2 = \frac{1 + \cos \theta}{1 - \cos \theta} \Rightarrow \tan \frac{\theta}{2} = 1 \Rightarrow \theta = 53^\circ$

SO IN CASE b) THE PERTURBATION GROWS MORE QUICKLY AT FIRST, BUT ULTIMATELY, CASE a) IS LARGER!

$|E|?$

$$a = \frac{\alpha}{2|E|}$$

$$b = \frac{L}{\sqrt{24|E|}}$$

$$\frac{E_0}{m} = -\frac{GM}{r_0} + \frac{1}{2}v_0^2 = -\frac{GM}{r_0} = -\frac{1}{2}v_0^2$$

$$\vec{v} \rightarrow \vec{v}_0 + \Delta\vec{v} \quad \text{if } \Delta\vec{v} \parallel \vec{v}_0 \quad v^2 = v_0^2 \left(1 - \frac{2\Delta v}{v_0}\right)$$

$$\frac{E_0}{m} \rightarrow \frac{E_0}{m} \left(1 - \frac{2\Delta v}{v_0}\right) = -\frac{GM}{r_0} + \frac{1}{2}v_0^2 - v_0^2 \left(\frac{\Delta v}{v_0}\right) = -\frac{1}{2}v_0^2 - v_0^2 \left(\frac{\Delta v}{v_0}\right) = \frac{E_0}{m} \left(1 + \frac{2\Delta v}{v_0}\right)$$

$$a \rightarrow \frac{a_0}{1 + \frac{2\Delta v}{v_0}} \approx a_0 \left(1 - \frac{2\Delta v}{v_0}\right)$$

COMPARE WITH OTHER SIDE:  $a = \frac{1}{2} \left( r_0 + r_0 \left(1 - \frac{2\Delta v}{v_0}\right) \right) = r_0 \left(1 - \frac{\Delta v}{v_0}\right)$  ✓

$$b = \frac{L}{\sqrt{24|E|}} \rightarrow \frac{L_0 \left(1 - \frac{\Delta v}{v_0}\right)}{\sqrt{E_0} \left(1 + \frac{\Delta v}{v_0}\right)} \rightarrow b_0 \left(1 - \frac{2\Delta v}{v_0}\right) = R_0 \quad \checkmark$$

if  $\Delta\vec{v} \perp \vec{v}_0$ ,  $v^2 \rightarrow v_0^2 + \Delta v^2 \approx v_0^2$  to first order

$\Rightarrow$   $a, b$  still  $a_0, b_0$ , but orbit is just offset ...

THE PTOLEMAIC SYSTEM

IT IS COMMON TO BELITTLE THE ACHIEVEMENTS OF GREEK ASTRONOMERS AS THEY USED A MODEL WHICH SEEMS MUCH TOO COMPLICATED TO BE SENSIBLE. BUT THEY DID NOT HAVE THE ADVANTAGE OF THE DATA PRESENTED IN THE FORM OF KEPLER'S LAWS, WITH ITS PARTICULAR EMPHASIS ON ELLIPTICAL ORBITS.

THE GREEKS HAD A BIAS IN FAVOR OF CIRCULAR MOTION, BUT THEY DID DO A FAIRLY GOOD JOB OF EXPLAINING THE DATA AS THEY UNDERSTOOD IT.

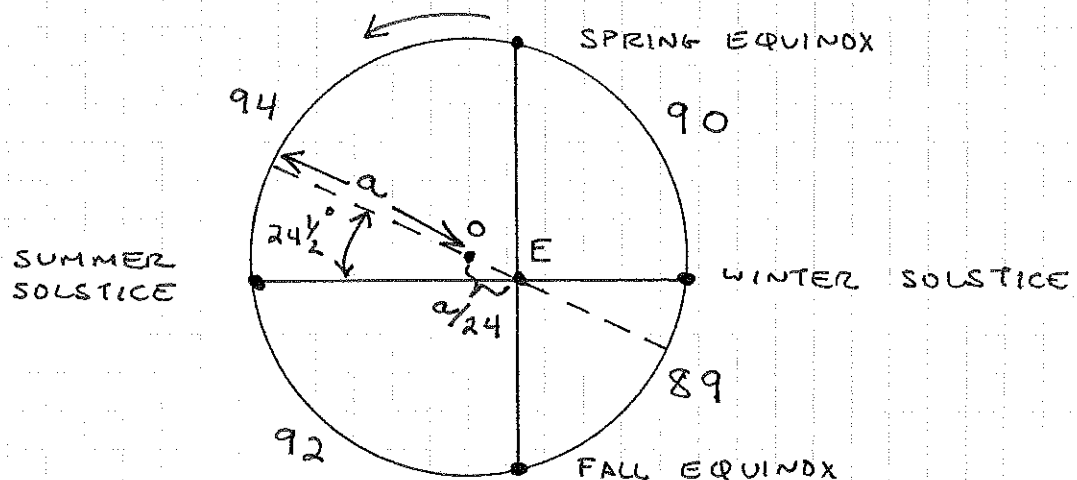
THE MOST PROMINENT PIECE OF DATA WAS THE LENGTH OF THE SEASONS:

SPRING	=	94 DAYS
SUMMER	=	92 "
FALL	=	89 "
WINTER	=	90 "

THIS CANNOT BE EXPLAINED BY UNIFORM CIRCULAR MOTION OF THE SUN ABOUT THE EARTH (OR VICE VERSA)!

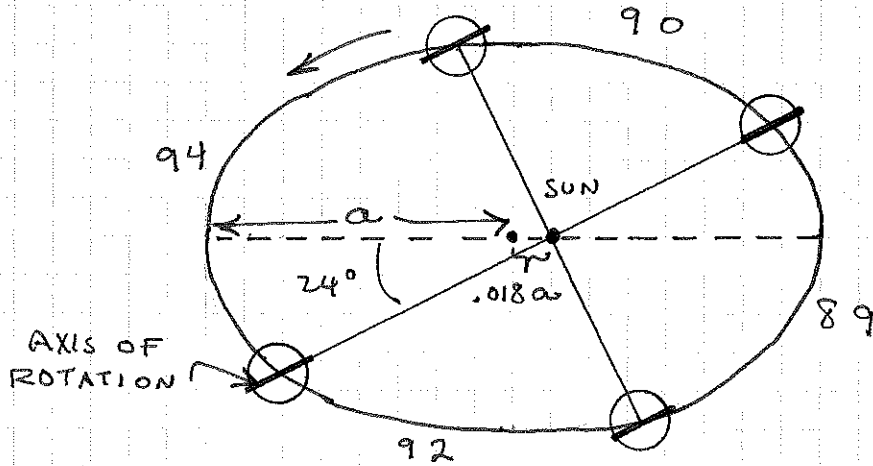
THE COMMONLY ACCEPTED GREEK EXPLANATION SEEMS TO BE DUE TO APOLLONIUS (~230 B.C.). IRONICALLY HIS OTHER FAMOUS WORK IS ON CONIC SECTIONS.

APOLLONIUS CLAIMED THE SUN MOVES WITH UNIFORM VELOCITY IN A CIRCLE ABOUT A POINT O NOT AT THE CENTER OF THE EARTH = E.



THE ECCENTRICITY OF THIS ORBIT IS ABOUT TWICE THE ACTUAL VALUE - AS MIGHT HAVE BEEN DETERMINED BY MEASUREMENT OF THE APPARENT DIAMETER OF THE SUN. THE ANGLE  $24\frac{1}{2}^\circ$  IS CORRECT TO  $\frac{1}{2}^\circ$  HOWEVER.

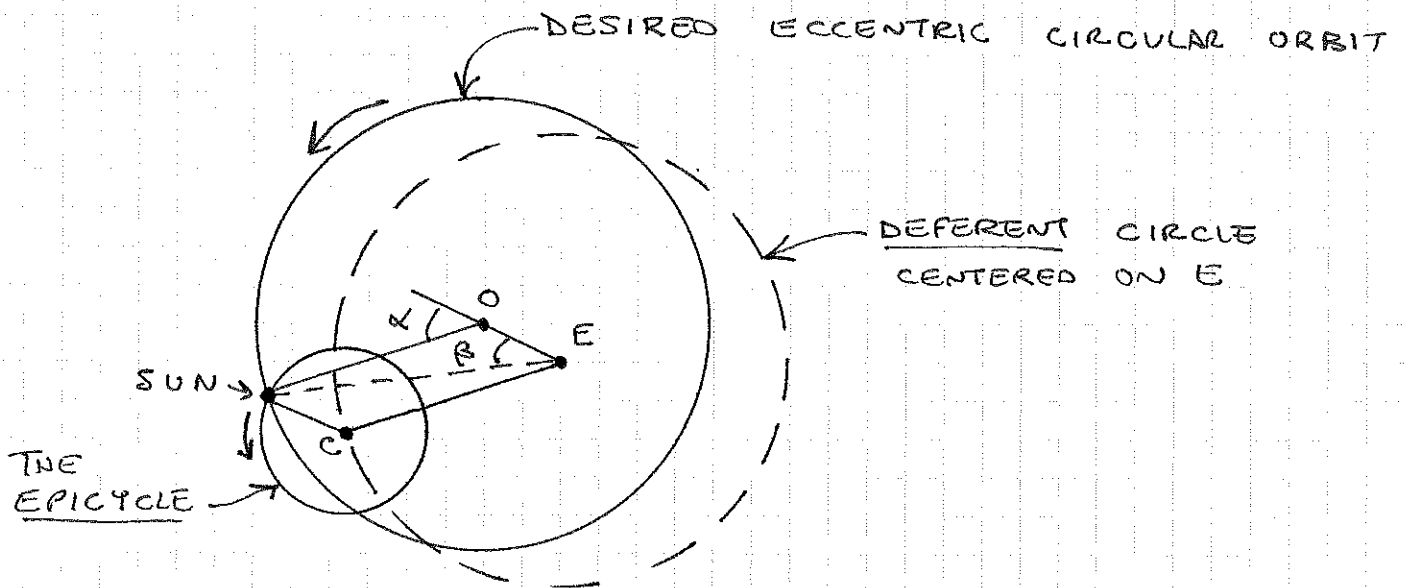
HOW DO WE EXPLAIN THE SAME PHENOMENA? THE EARTH'S AXIS OF ROTATION IS TILTED WITH RESPECT TO THE PLANE OF THE ORBIT. THE SUMMER SOLSTICE OCCURS WHEN THE AXIS POINTS TOWARDS THE SUN. (MORE PRECISELY, WHEN THE PLANE CONTAINING THE AXIS AND THE SUN IS  $\perp$  TO THE PLANE OF THE ORBIT.



IF THE EARTH IS SPHERICAL THE DIRECTION OF THE AXIS OF ROTATION CANNOT CHANGE, BY THE ARGUMENT GIVEN ON P. 21.

THIS EXPLANATION IS NOT A WHOLE LOT SIMPLER THAN APOLLONIUS', BUT IT IS MORE ACCURATE! NOTE THAT THESE MODELS SUGGEST THAT IN THE NORTHERN HEMISPHERE SUMMERS SHOULD BE COOLER (AND LONGER) AND WINTERS WARMER (BUT SHORTER) THAN THOSE IN THE SOUTHERN HEMISPHERE. THERE IS ABOUT A  $7.5\%$  DIFFERENCE IN THE SOLAR ENERGY RECEIVED DURING THE SUMMERS IN THE TWO HEMISPHERES.

APOLLONIUS ALSO NOTED THAT THE MOTION SHOWN ON P. 99 COULD BE EXPLAINED BY AN EPICYCLE AND DEFERENT



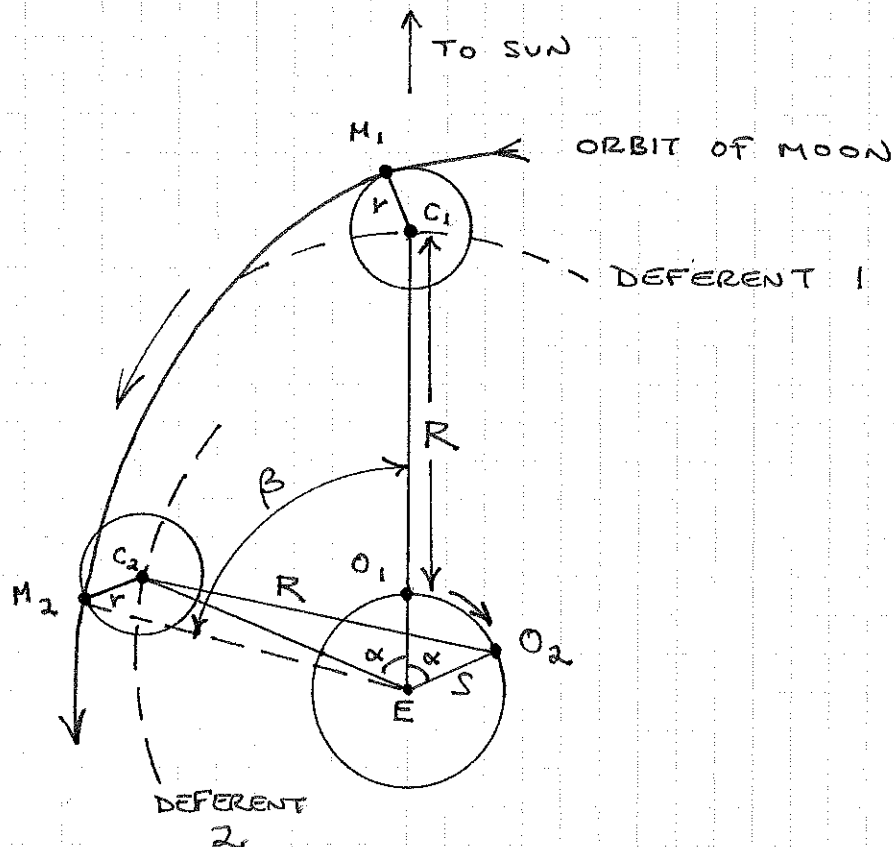
# PH 205 LECTURE 9

THE PARALLELOGRAM OECS WAS SIDE OE FIXED, AND DEFORMS WITH CONSTANT ANGULAR VELOCITY  $\dot{\alpha}$ . THE SUN MOVES WITH CONSTANT VELOCITY ON THE DESIRED ORBIT, AND ALSO ROTATES WITH UNIFORM ANGULAR VELOCITY  $\dot{\alpha}$  ABOUT POINT C - THE CENTER OF THE EPICYCLE.

ANGLE  $\alpha$  IS CALLED THE 'ANOMALY' = ANGLE OF ROTATION OF THE SUN ON THE EPICYCLE COMPARED TO THE LINE EC. NOTE THAT IF  $\alpha = 0$  ALWAYS, THE SUN WOULD MOVE IN A CIRCLE ABOUT THE EARTH WITH UNIFORM ANGULAR VELOCITY.

ANGLE  $\beta$  IS CALLED THE 'TRUE ANOMALY' - THE ACTUAL ANGLE OF THE SUN WITH RESPECT TO A FIXED LINE THRU THE EARTH.

FOR THE MOON A SIMILAR CONSTRUCTION WAS USED TO EXPLAIN THE MOTION. THE DIFFERENCE BETWEEN A LUNAR MONTH AND A SIDEREAL LUNAR MONTH WAS KNOWN, AND FIXED THE RADIUS OF THE EPICYCLE AS ABOUT  $\frac{1}{2}$  THAT OF THE DEFERENT. THIS ALLOWED A GOOD EXPLANATION OF ECLIPSES. HOWEVER, THE ANGULAR POSITION OF THE NEW MOON AND  $\frac{3}{4}$  MOON WERE NOT WELL REPRODUCED. THE FAMOUS PTOLEMY MADE A CORRECTION TO THE EPICYCLE MODEL, BASED ON OBSERVATIONS.



THE MOON  $M$  ROTATES ABOUT THE CENTER  $C$  OF THE EPICYCLE WHICH HAS RADIUS  $r$ .

THE POINT  $C$  ROTATES ABOUT POINT  $O$  WHICH IS THE CENTER OF THE DEFERENT OF RADIUS  $R$ .

POINT  $O$  IS IN RETROGRADE MOTION ABOUT  $E$  THE CENTER OF THE EARTH, ON A CIRCLE OF RADIUS  $S$ .

THE RATES OF ROTATION OF  $O$  ABOUT  $E$ , AND OF  $C$  ABOUT  $O$  ARE ESSENTIALLY EQUAL (BUT OPPOSITE) SO AS TO MAINTAIN THE EQUALITY OF THE TWO ANGLES SHOWN AS  $\alpha$ .

WITH  $R = 60$ ,  $r = 5$  AND  $S = 10$  THIS MODEL GIVES A VERY GOOD ACCOUNT OF THE ANGULAR POSITION  $\beta$  OF THE MOON WITH RESPECT TO THE EARTH AND SUN.

BUT THE MODEL DOES NOT GIVE A GOOD ACCOUNT OF THE DISTANCE FROM THE EARTH TO THE MOON (AN APPARENT DIAMETER OF THE MOON). THIS WAS KNOWN TO PTOLEMY, BUT HE MADE NO FURTHER ADVANCE.

IT WAS THIS SORT OF DIFFICULTY WHICH LED COPERNICUS TO CONSIDER ALTERNATIVE MODELS - NOT JUST THE COMPLEXITY OF THE MODEL.