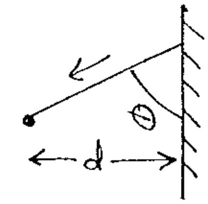
PH 205 SET 3

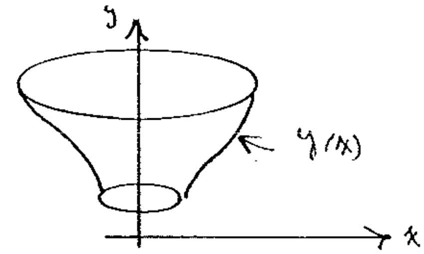
DUE: TUES OCT 11, 1988
MAXIMUM RECORDED SCORE = 70 POINTS

(1) WE WISH TO SLIDE THINGS DOWN A STRAIGHT, FRICTIONLESS CHUTE WHICH BEGINS AT A WALL AND ENDS AT A POINT DISTANCE d FROM THE WALL. AT WHAT ANGLE Θ SHOULD THE CHUTE BE PLACED TO MINIMIZE THE TIME OF DESCENT?



IF THE CHUTE COULD BE BENT INTO A CYCLOID, THE DESCENT WOULD BE THE FASTEST POSSIBLE. IT IS AGREEABLE TO OUR INTUITION THAT A CUSP OF THE CYCLOID MUST BE AT THE WALL, AND THE END POINT BE AT THE VERY BOTTOM OF THE CYCLOID. SEE SEC. 3-11 OF WEINSTOCK FOR THE DERIVATION.

(2) A CURVE $y = y(x)$ IS ROTATED ABOUT THE <u>Y-AXIS</u> TO FORM A SURFACE OF REVOLUTION. WHAT IS THE FORM OF y TO PRODUCE THE MINIMUM SURFACE AREA?

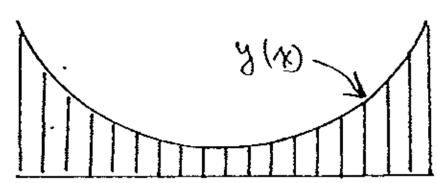


WE EXPECT $x = A\cosh\left(\frac{y-B}{A}\right)$

HOWEVER, DO <u>NOT</u> WRITE $x = x(y)$ AND COPY THE DERIVATION FROM THE NOTES. USE $y = y(x)$

(3) a) <u>HANGING ROPE</u>. A ROPE OF UNIFORM DENSITY (MASS/LENGTH) IS HUNG FROM TWO FIXED POINTS. USE ELEMENTARY METHODS TO EXAMINE THE CONDITIONS OF STATIC EQUILIBRIUM OF A SECTION OF ROPE dx LONG. (x = HORIZONTAL) FIRST LOOK AT F_x, THEN F_y TO SHOW $y''/\sqrt{1+y'^2} = \text{CONST}$. $\Rightarrow$ $y(x)$ IS A <u>CATENARY</u>

b) <u>SUSPENSION BRIDGE</u>. A MASSLESS CABLE IS STRUNG BETWEEN TWO FIXED POINTS, AND VERTICAL MASSLESS CABLES ARE ATTACHED TO IT TO CARRY A UNIFORM HORIZONTAL LOAD - THE BRIDGE.



WHAT IS THE SHAPE OF THE SUSPENDED CABLE SO THAT THE TENSIONS ARE THE SAME IN ALL THE VERTICAL CABLES? ASSUME AN INFINITE NUMBER OF EVENLY SPACED VERTICAL CABLES.

USE ELEMENTARY METHODS TO SHOW THAT $y'' = \text{CONST}$

$\Rightarrow$ <u>PARABOLA</u>

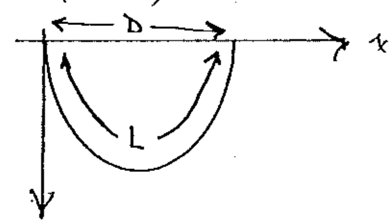
③ c) <u>OPTIONAL</u> USE THE CALCULUS OF VARIATIONS TO DERIVE THE PARABOLIC SHAPE OF THE SUSPENSION BRIDGE. THIS IS QUITE TRICKY BECAUSE DURING THE VARIATION WE SHOULD NOT REGARD THE LOAD AS UNIFORM IN x ....

CAN YOU FIND AN APPROPRIATE MODIFICATION OF THE EULER-LAGRANGE TECHNIQUE?

④ a) USE THE CALCULUS OF VARIATIONS TO WORK ③ a)

LET L = LENGTH OF ROPE, AND D = SEPARATION OF THE TWO FIXED POINTS, WHICH ARE AT THE SAME HEIGHT.

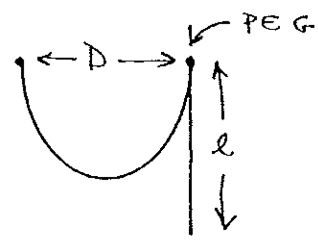
"MINIMIZE THE POTENTIAL ENERGY" J & J BERNOULLI (1696)



ANS: $y = C\left[\cosh\left(\frac{x - D/2}{C}\right) - \cosh\frac{D}{2C}\right]$

WHERE $\frac{L}{2C} = \sinh\frac{D}{2C}$

b) THE LENGTH L OF THE ROPE IS FIXED, BUT AT ONE END THE ROPE MERELY HANGS OVER A PEG. WHAT IS THE LENGTH ℓ OF THE VERTICAL SEGMENT?



WE WANT TO MINIMIZE $V = \int f\,dx + F(\ell)$

SUBJECT TO $L = \int g\,dx + G(\ell) = \text{CONST.}$

WITHOUT PROOF, WE CLAIM THE TECHNIQUE IS TO FORM

$$\left.\begin{aligned} f^* &= f + \lambda g \\ F^* &= F + \lambda G \end{aligned}\right\} \text{ SAME } \lambda$$

THEN MINIMIZE $I^* = \int f^*\,dx + F^*(\ell)$

YOU SHOULD FIND $\lambda = \ell$

AND $\ell = C\cosh\frac{D}{2C}$ WITH $\frac{L-\ell}{2C} = \sinh\frac{D}{2C}$

⑤ <u>GEODESICS ON A SPHERE</u> (FULL CREDIT FOR WORKING EITHER PART a) OR PART b))

FIND THE CURVE ON THE SURFACE OF A SPHERE WHICH HAS THE SHORTEST DISTANCE BETWEEN 2 POINTS.

a) ONE APPROACH IS TO REDUCE THE PROBLEM TO 2 DIMENSIONS. PARAMETERIZE THE SURFACE BY 2 INDEPENDENT COORDINATES (u, v).

THEN $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$ IS THE SURFACE.

WE NEED THE ARC LENGTH ALONG THE SURFACE: $ds^2 = dx^2 + dy^2 + dz^2$

WE CAN SUBSTITUTE THE EQUATIONS OF THE SURFACE TO FIND FUNCTIONS $P(u,v)$, $Q(u,v)$ & $R(u,v)$ SUCH THAT

$$ds^2 = P\,du^2 + 2Q\,du\,dv + R\,dv^2$$

A ONE-DIMENSIONAL CURVE ON THE SURFACE CAN BE WRITTEN $v = v(u)$, SO $ds = \sqrt{P + 2Qv' + Rv'^2}\,du$

WE WANT TO MINIMIZE $\int ds$ ON A SPHERE.

TRY PARAMETERS a = RADIUS = CONS, $u = \phi$, $v = \theta$.

WITH THE HELP OF AN INTEGRAL TABLE, SUCH AS GRADSHTEYN & RYZHIK 2.599.6

YOU SHOULD EVENTUALLY FIND A SOLUTION LIKE

$$x \sin C_2 + y \cos C_2 - \frac{z}{\sqrt{\frac{a^2}{C_1^2} - 1}} = 0$$

THIS IS THE EQUATION OF A PLANE PASSING THRU THE CENTER OF THE SPHERE. THE INTERSECTION OF SUCH A PLANE WITH THE SURFACE IS A GREAT CIRCLE WHICH IS THE GEODESIC ON A SPHERE.

b) WE STAY IN 3 DIMENSIONS AND REGARD THE SURFACE AS A CONSTRAINT. LET $x = x(t)$, $y = y(t)$, $z = z(t)$ BE OUR DESIRED CURVE, AND LET $g(x,y,z) = 0$ BE THE EQUATION OF THE SURFACE.

THEN $ds = f\,dt$ = ARC LENGTH WITH $f = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$

USE THE CALCULUS OF VARIATION TO SHOW

$$\frac{\frac{d}{dt}\left(\frac{\dot{x}}{f}\right)}{\partial g/\partial x} = \frac{\frac{d}{dt}\left(\dot{y}/f\right)}{\partial g/\partial y} = \frac{\frac{d}{dt}\left(\dot{z}/f\right)}{\partial g/\partial z} \Rightarrow \frac{\frac{d^2x}{ds^2}}{\partial g/\partial x} = \frac{\frac{d^2y}{ds^2}}{\partial g/\partial y} = \frac{\frac{d^2z}{ds^2}}{\partial g/\partial z}$$

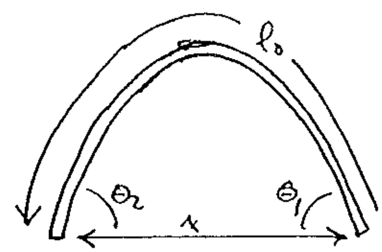
BY REFERING TO P.12 OF THE NOTES WE SEE THAT THE NUMERATORS ARE COMPONENTS OF A VECTOR $\bar{n}$ NORMAL TO THE CURVE $\left[\hat{n} \sim \frac{d\hat{s}}{ds}\right]$

THE DENOMINATORS ARE COMPONENTS OF $\bar{\nabla} g$ WHICH IS NORMAL TO THE SURFACE. THE EQUALITY OF THE RATIOS CONFIRMS THE GEOMETRICAL PICTURE OF THE LAGRANGE MULTIPLIER GIVEN IN THE NOTES.

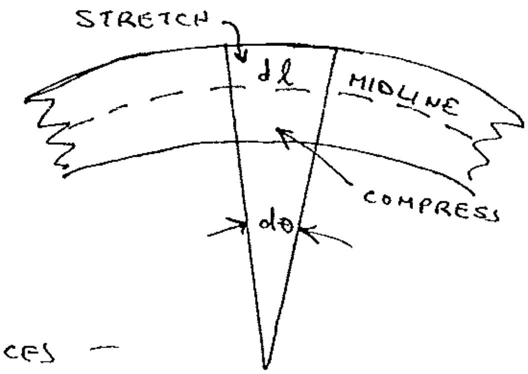
FOR A SPHERE USE THE LEFT SET OF EQUATIONS TO FIND THE GEODESIC CURVE.

A TRICK IS TO WRITE $\dot{f}/f$ TWO WAYS, YIELDING EXPRESSIONS WHICH CAN BE INTEGRATED YIELDING LOGARITHMS. REARRANGE & INTEGRATE AGAIN TO SHOW $x + Ay + Bz = 0$ AS IN PART a).

(6) MEANDERS A FLEXIBLE METAL TAPE IS BENT BY FIXING THE POSITIONS AND SLOPES OF ITS END POINTS. THE TAPE HAS LENGTH l_0, THICKNESS h_0 AND SPRING CONSTANT K FOR STRETCHING ALONG ITS LENGTH.



IN THE BEND SUPPOSE THE LENGTH ALONG THE MIDLINE REMAINS l_0, BUT ABOVE THE MIDLINE THE TAPE IS STRETCHED, WHILE BELOW THE MIDLINE IT IS COMPRESSED.



THE TAPE WILL ASSUME WHATEVER SHAPE REQUIRES THE LEAST WORK OF DEFORMATION. THAT IS, THE STORED POTENTIAL ENERGY WILL BE A MINIMUM.

DIVIDE THE WEDGE SHOWN ABOVE INTO SLICES — EACH A TINY SPRING. DETERMINE THE SPRING CONSTANT OF EACH SLICE, AND INTEGRATE TO SHOW

$$\text{P.E.} = \frac{1}{24} K l_0 h_0^2 \int_0^{l_0} \left(\frac{d\theta}{dl}\right)^2 dl$$

WE WISH TO MINIMIZE THIS SUBJECT TO A SUITABLE CONSTRAINT. JUST LENGTH $= l_0$ WONT HELP. INSTEAD, CONSIDER THE DISTANCE BETWEEN THE END POINTS,

$$x = \int_0^{l_0} \cos\theta \, dl = \text{const.}$$

THE OTHER CONSTRAINTS, θ_1, θ_2 FIXED CAN BE APPLIED ONCE THE GENERAL CLASS OF SHAPES IS KNOWN.

USE THE CALCULUS OF VARIATIONS TO FIND $\theta(l)$. IF θ_1 SMALL SHOW $\theta \sim \theta_1 \cos(\omega l)$ $\omega =$ CONST.

SKETCH 2 OR 3 PERIODS OF THE CURVE FOR $\theta_1 \sim 120°$

THIS KIND OF CURVE APPEARS ON MAPS — AS THE SHAPE OF MEANDERING RIVERS. SEE SCIENTIFIC AMERICAN, JUNE 1966.

(7) COMPONENTS OF ACCELERATION IN A NON-CARTESIAN COORDINATE SYSTEM

SUPPOSE WE USE A COORD SYSTEM (q_1, q_2, q_3) TO DESCRIBE THE POSITION OF A POINT PARTICLE.

SUPPOSE ALSO THAT WE CAN WRITE DOWN THE LINE ELEMENT:

$$ds^2 = ds_1^2 + ds_2^2 + ds_3^2 \quad \text{WHERE} \quad ds_i = f_i(q_1, q_2, q_3)\, dq_i$$

THE VELOCITY v IS JUST $v = ds/dt$, SO THE KINETIC ENERGY IS

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \left(\frac{ds}{dt}\right)^2 = \frac{1}{2} m \left[f_1^2 \left(\frac{dq_1}{dt}\right)^2 + f_2^2 \left(\frac{dq_2}{dt}\right)^2 + f_3^2 \left(\frac{dq_3}{dt}\right)^2 \right]$$

LAGRANGE'S EQUATIONS GIVE US A QUICK WAY OF WORKING OUT THE COMPONENTS OF ACCELERATION IN THIS COORD. SYSTEM.

NOW $$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j$$

WHERE Q_j OBEYS $\sum_j Q_j \, \delta q_j = \bar{F} \cdot \delta \bar{r}$

IF WE DECOMPOSE VECTORS $\bar{F}$ AND $\delta \bar{r}$ IN OUR NEW COORD SYSTEM,

$$\bar{F} = F_1 \hat{q}_1 + F_2 \hat{q}_2 + F_3 \hat{q}_3 \quad \text{AND} \quad \delta \bar{r} = \delta r_1 \hat{q}_1 + \delta r_2 \hat{q}_2 + \delta r_3 \hat{q}_3$$

BUT BY DEFINITION OF THE LINE ELEMENT, $\delta r_j = \delta s_j = f_j \, \delta q_j$

THUS $$\sum_j Q_j \, \delta q_j = \sum_j F_j f_j \, \delta q_j$$

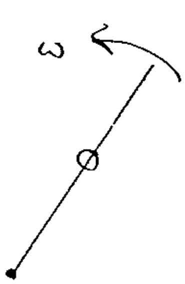
SO $Q_j = f_j F_j$ RELATES GENERALISED FORCE TO ORDINARY FORCE

BUT NEWTON SAYS $F_j = m a_j$, SO THE COMPONENT OF ACCELERATION IN THE $\hat{q}_j$ DIRECTION IS JUST

$$a_j = \frac{Q_j}{m f_j} = \frac{\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}}{m f_j}$$

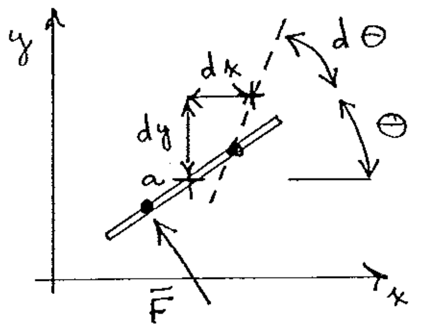
USE THESE TRICKS TO CALCULATE THE a_j IN CYLINDRICAL AND SPHERICAL COORD. SYSTEMS. COMPARE TO THE RESULTS GIVEN ON pp 9 & 10 OF THE NOTES.

(8) FIND THE CONSTRAINT FORCE IN OUR EXAMPLE OF A BEAD SLIDING WITHOUT FRICTION ON A WIRE WHICH IS CONSTRAINED TO ROTATE IN A PLANE WITH CONSTANT ANGULAR VELOCITY ω.



USE BOTH ELEMENTARY METHODS, AND THE METHOD OF LAGRANGE MULTIPLIERS.

(9) THE RETURN OF THE ONE-LEGGED ICE SKATER



$+$ = C.M.

$\bullet$ = POINT OF APPLICATION OF THE CONSTRAINT FORCE

SUPPOSE THE CONSTRAINT FORCE IS APPLIED AT DISTANCE a FROM THE C.M.

ALTHOUGH $\bar{F}\cdot\overline{\text{SKATE}}=0$, $\bar{F}\cdot \bar{v}_{cm}\neq 0$

$\Rightarrow v_{cm}$ NOT CONSTANT

ALSO, $\bar{F}$ NOW EXERTS A TORQUE ABOUT THE C.M., SO $\dot{\Theta}$ WILL NOT BE CONSTANT. BUT $\bar{F}$ SHOULD DO NO NET WORK, SO THE KINETIC ENERGY SHOULD REMAIN CONSTANT. NOW IF v_{cm} DECREASES, $\dot{\Theta}$ CAN INCREASE KEEPING K.E. CONSTANT. THE POSSIBILITY EXISTS THAT THE SKATER CAN COME TO A STOP, BUT BE ROTATING MUCH FASTER THAN INITIALLY!

BEGIN BY USING x, y OF THE CM, AND Θ AS VARIABLES. USE THE METHOD OF LAGRANGE MULTIPLIERS TO FIND THE EQUATIONS OF MOTION. ESTABLISH THE CONSTRAINT RELATION WHICH NOW INVOLVES dx, dy AND $d\Theta$.

NOTE: THE EQUATIONS OF MOTION ARE ALSO READILY FOUND BY ELEMENTARY METHODS.

WE GIVE SOME HINTS OF A METHOD TO INTEGRATE THE EQUATIONS; REPLACE $\dot{x}$ AND $\dot{y}$ BY FUNCTIONS OF v AND Θ WHERE

v = VELOCITY OF POINT OF APPLICATION OF $\bar{F}$, <u>NOT</u> v_{cm}

WRITE $I = mb^2$ = MOMENT OF INERTIA, AND LET $K^2 = 1 + b^2/a^2$

ON ELIMINATING λ ETC YOU SHOULD FIND

$$K^2 a\,\ddot{\Theta} + v\,\dot{\Theta} = 0 \qquad \text{AND} \qquad \dot{v} = a\,\dot{\Theta}^2$$

AS THE EQUATIONS OF MOTION.

DIFFERENTIATE & COMBINE, MULTIPLY BY $\ddot{\Theta}/\dot{\Theta}$ TO FIND

$$K^2 \frac{d}{dt}\left(\frac{\ddot{\Theta}}{\dot{\Theta}}\right)^2 = -\frac{d}{dt}\dot{\Theta}^2$$

INTEGRATING A 3RD TIME

INTEGRATE (TWICE) TO SHOW $\omega = \dot{\Theta} = \dfrac{c}{\cosh ct/K} = c\cos\dfrac{\Theta}{K}$

c = CONSTANT. HENCE $v = caK\tanh\dfrac{ct}{K} = caK\sin\Theta/K$

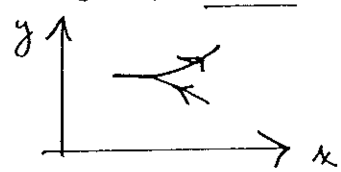
SKETCH $v(t)$ AND $\dot{\Theta}(t)$ TO NOTE THEY TAKE ON ASYMPTOTIC VALUES. LIKEWISE $\tan\dfrac{\Theta}{K} = \sinh\dfrac{ct}{K}$ SHOWS Θ TAKES ON ASYMPTOTIC VALUES.

THE BEHAVIOR NEAR $t=0$ CAN BE EXPLORED BY EVALUATING

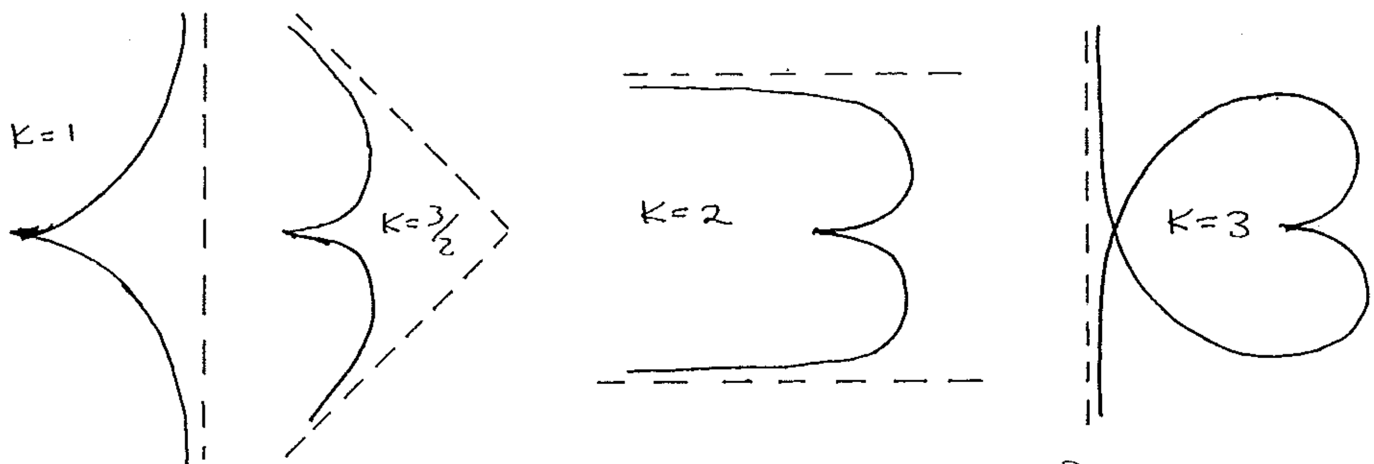
$$\frac{dx}{d\theta}, \frac{dy}{d\theta} \quad \frac{dx^2}{d\theta^2}, \frac{d^2y}{d\theta^2} \quad \text{ETC.}$$

SHOW THAT AT $t=0$ $\frac{dx}{d\theta} = \frac{dy}{d\theta} = \frac{d^2y}{d\theta^2} = 0$ WHILE $\frac{d^2x}{d\theta^2} \neq 0$, $\frac{d^3y}{d\theta^3} \neq 0$

TO THE ENLIGHTENED, THIS IMPLIES A CUSP AT $t=0$, WITH THE POINT ALONG THE x-AXIS



SOMMERFELD (MECHANICS P 252) SKETCHES SOME ORBITS FOR VARIOUS K

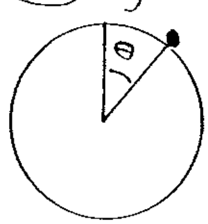


CAN REAL ICE SKATERS PRODUCE THESE SHAPES?

YOU SHOULD BE ABLE TO VERIFY THAT KINETIC ENERGY $= \frac{M}{2} c^2 a^2 k^2$

AND $F = \lambda = \frac{M}{2} \frac{c^2 b^2}{aK} \sin \frac{2\theta}{K}$

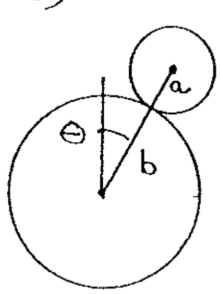
(10) a)



A PARTICLE STARTS FROM REST AT THE TOP OF A FRICTIONLESS SPHERE AND SLIDES DOWN. WHEN THE NORMAL FORCE VANISHES IT FLIES OFF. USE THE METHOD OF LAGRANGE MULTIPLIERS TO SHOW THIS OCCURS WHEN $\cos\theta = 2/3$

(THIS WAS ON THE Ph 103 LEARNING GUIDE - SANS LAGRANGE!)

b)



A SPHERE OF RADIUS a ROLLS WITHOUT SLIPPING DOWN A FIXED SPHERE OF RADIUS b. SHOW BY ANY METHOD THAT THE UPPER SPHERE FALLS OFF WHEN $\cos\theta = 10/17$.

BE VERY CAREFUL WHEN ESTABLISHING THE CONSTRAINT BETWEEN θ AND THE ANGLE OF ROTATION OF THE UPPER SPHERE.