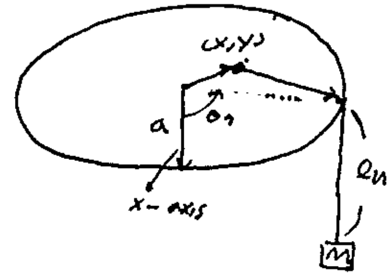
Ph 205

PROBLEM SET 5

2. We will use elementary method to solve this problem. Referreing to the right figure, the vector from the central mass M to the edge of the circle where the string passes through is given by,



$$\vec{r}_n \equiv x_n\hat{x} + y_n\hat{y} = (a\cos\theta_n - x)\hat{x} + (a\sin\theta_n - y)\hat{y} \qquad (n = 1 \sim N)$$

Thus, in Cartesian coordinate the force components can be written as

$$F_{nx} = T_n \frac{x_n}{\sqrt{x_n^2+y_n^2}}$$

$$F_{ny} = T_n \frac{y_n}{\sqrt{x_n^2+y_n^2}}$$

in terms of tension T_n of n-th string. Since the length of each string is constant in time, we have constraints,

$$\sqrt{x_n^2+y_n^2} + \ell_n = \ell_0 : \text{constant.}$$

Now, Newton's 3rd law gives us

$$M\ddot{\ell}_n = Mg - T_n$$

$$M\ddot{x} = \sum_n F_{nx} \quad , \quad \sum_n F_{ny} = M\ddot{y}$$

Combining these two equations, we get,

$$M\ddot{x} = \sum_n (Mg - M\ddot{\ell}_n)x_n/\sqrt{x_n^2+y_n^2} \quad , \quad M\ddot{y} = \sum_n (Mg - M\ddot{\ell}_n)y_n/\sqrt{x_n^2+y_n^2}$$

and,

$$\dot{\ell}_n = -\frac{x_n\dot{x}_n + y_n\dot{y}_n}{\sqrt{x_n^2+y_n^2}}$$

$$\ddot{\ell}_n = -\frac{x_n\ddot{x}_n + y_n\ddot{y}_n + \dot{x}_n^2 + \dot{y}_n^2}{\sqrt{x_n^2+y_n^2}} + \frac{(x_n\dot{x}_n + y_n\dot{y}_n)^2}{(x_n^2+y_n^2)^{3/2}}$$

We retain only linear order term in x and y (which correspond to first non-vanishing terms to get,

$$\ddot{x} = \sum_n (g\cos\theta_n - \cos^2\theta_n\,\ddot{x} - \cos\theta_n\sin\theta_n\,\ddot{y} + \frac{g}{a}y\cos\theta_n\sin\theta_n - \frac{g}{a}\sin^2\theta_n\,x)$$

$$\ddot{y} = \sum_n (g\sin\theta_n - \cos\theta_n\sin\theta_n\,\ddot{x} - \sin^2\theta_n\,\ddot{y} + \frac{g}{a}x\cos\theta_n\sin\theta_n - \frac{g}{a}\cos^2\theta_n\,y)$$

From the geometry of this problem, $\theta_n = \frac{2\pi}{N}n$, we have,

$$\sum_n \cos\theta_n = \sum_n \sin\theta_n = 0 = \sum_n \sin\theta_n\cos\theta_n$$

Additionally we can show that (see comments)

$$\sum_n \sin^2\theta_n = \sum_n \cos^2\theta_n = \frac{N}{2}.$$

for n>2 and $\sum_n \sin^2\theta = 0, \sum_n \cos^2\theta_n = 2$ for n=2. (n; integer)

Thus, the equations of motion becomes simply,

$$(1+\frac{N}{2})\ddot{x} + \frac{g}{a}\frac{N}{2}x = 0$$

$$(1+\frac{N}{2})\ddot{y} + \frac{g}{a}\frac{N}{2}y = 0$$

for n>2 and

$$\ddot{x} = 0$$

$$\ddot{y} + \frac{2g}{a}y = 0$$

for n=2. Consequently, the angular frequently for n>2 case is, $\omega = \sqrt{\frac{ng}{a(n+2)}}$ (same frequency for x and y direction) N=2 case, we find $\omega = 0$ (longitudinal oscillation) and $\omega = \sqrt{(}$ (transverse oscillation).

Some comments.

1. Calculation of $\sum_{n=1}^{N} f(\sin\theta_n, \cos\theta_n)$, $\theta_n = kn$ → constant

One trick is to change, $\sin\theta_n = \frac{e^{i\theta_n} - e^{-i\theta_n}}{2i}$ and $\cos\theta_n = \frac{e^{i\theta_n} + e^{-i\theta_n}}{2}$. Then, for example,

$$\sum_{n=1}^{N} \cos^2\frac{2\pi n}{N} = \frac{1}{4}\sum_{n=1}^{N}\left(\exp(i2kn) + 2 + \exp(-i2kn)\right)$$

We immediately find that resulting series is nothing but a simple geometric series

$$S = r + r^2 + r^3 + \cdots r^N$$

$$\therefore \sum_{n=1}^{N} \cos^2\frac{2\pi n}{N} = \begin{cases} \frac{N}{2} & n \neq 2 \\ 2 & n = 2. \end{cases}$$

with $r = e^{2ik}$, ... Thus, Using the formula,

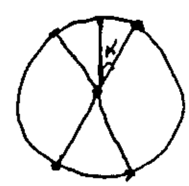
$$S = \frac{r - r^{N+1}}{1 - r} \text{ for } r \neq 1, \quad S = N \text{ for } r = 1,$$

which can be easily proved, we get, $(k = \frac{2\pi}{N})$

$$\sum_{n=1}^{N} e^{2ikn} = r\,\frac{1 - \exp(\frac{4\pi}{N}\cdot N)}{1 - \exp(\frac{4\pi}{N})} = 0 \text{ for } \exp(\tfrac{4\pi}{N}) \neq 1, \text{ i.e., } N \neq 2. \quad \sum_{n=1}^{2} e^{2ikn} = 2.$$

Sometimes changing trigonometric functions into exponential functions simplifies calculations. At least you don't have to memorize trigonometric function formulas.

2. Almost all students solved this problem assuming simple one dimensional motion. However, it must be proved that for small oscillation the effective potential is spherically symmetric. In fact, the central force nature of this problem is quite non-trivial consequence of the symmetry of this problem. For example, consider the case depicted right. In this case, $(\alpha \neq 45°)$



not equally spaced.

$$\sum_n \cos\theta_n = \sum_n \sin\theta_n = \sum_n \cos\theta_n \sin\theta_n = 0$$

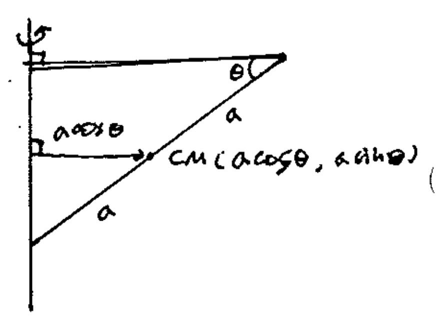
However, we find that,

$$\sum_n \cos^2\theta_n = 4\cos^2\alpha \neq \sum_n \sin^2\theta_n = 4\sin^2\alpha.$$

Thus, we hace two frequencies rather than a single one.

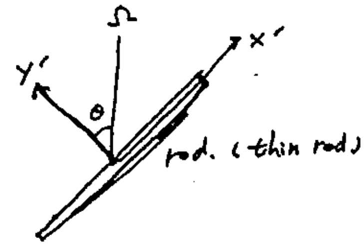
The system we've considered does not have enough symmetries, whereas the origianl problem has degenerate frequency due to the higher symmetry. In general, if there are n-coordinates in small oscillation problem, n different frequencies are possible. However in the presence of enough symmetries, some of them are degenerate. (Think about the reason why I choose 4-masses example rather than seemingly simple 3-masses example!)

3. The kinetic energy of rod can be decomposed into center of mass motion and the rotation about the center of mass. From the right figure the CM translation energy can be written as $(I_{cm} = \frac{1}{3}ma^2)$



$$\frac{1}{2}ma^2\dot\theta^2 + \frac{1}{2}ma^2\cos^2\theta\,\Omega^2$$

since the rotation direction and the plane containing gizmo are perpendicular. The rotational energy can



be written as,

$$\frac{1}{2}\left(\frac{ma^2}{3}\right)\dot{\theta}^2 + \frac{1}{2}\left(\frac{1}{3}ma^2\right)(\Omega\cos\theta)^2$$

where we projected our angular momentum Ω into two axes x' and y' and we used the fact $I_{x'}=0$ and $I_{y'}=\frac{1}{3}ma^2$. Thus, the total Lagrangian can be written as,

$$L = \frac{2}{3}ma^2\dot{\theta}^2 + \frac{2}{3}ma^2\cos^2\theta\,\Omega^2 + mga\sin\theta$$

The Euler-Lagrange equation gives,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0 \Rightarrow \frac{4}{3}ma^2\ddot{\theta} = -\frac{4}{3}ma^2\cos\theta\sin\theta + mga\cos\theta$$

Thus, the effective potential and effective mass can be obtained to give

$$m_{eff} = \frac{4}{3}ma^2$$

$$V_{eff} = -\int \text{force}\, d\theta = -\frac{1}{3}ma^2\cos 2\theta - mga\sin\theta$$

The equilibrium location is determined by requiring,

$$\frac{d}{d\theta}V_{eff}\Big|_{\theta=\theta_0} = \cos\theta_0\left(mga - \frac{4}{3}ma^2\sin\theta_0\right) = 0 \Rightarrow \theta_0 = \frac{\pi}{2} \text{ or } \theta_0 = \sin^{-1}\frac{3g}{4a\Omega^2}$$

The effective spring constants are,

$$k_{eff}^2 = \frac{d^2}{d\theta^2}V_{eff}\Big|_{\theta=\theta_0} = \frac{4}{3}ma^2\left(-\sin\theta_0\left(\sin\theta_0\,\Omega^2 - \frac{3g}{4a}\right) + \cos\theta_0\cos\theta_0\,\Omega^2\right)$$

Notice that for $\Omega^2 < \frac{3g}{4a}$, $\frac{\pi}{2}$ is stable equilibrium (corresponding $k_{eff}^2>0$) and for $\Omega^2 > \frac{3g}{4a}$ $\sin^{-1}\frac{3g}{4a\Omega^2}$ is stable eqilibrium, which is physically reasonable. The small oscillation frequency is,

$$\omega^2 = \frac{3g}{4a} - \Omega^2 = \frac{3g}{4a}\left(1 - \frac{4a\Omega^2}{3g}\right) \quad \text{for } \theta_0 = \frac{\pi}{2}$$

$$\omega^2 = \cos^2\theta_0\,\Omega^2 = \Omega^2\left(1 - \left(\frac{3g}{4a\Omega^2}\right)^2\right) \quad \text{for } \theta_0 = \sin^{-1}\frac{3g}{4a\Omega^2}$$

5. The Lagrangian can be written as,

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{c}{r^\lambda}$$

Thus, Euler-Lagrange equation gives,

$$m\ddot{r} - mr\dot{\theta}^2 + \lambda\frac{c}{r^{\lambda+1}} = 0 \Rightarrow m\ddot{r} = \frac{L^2}{mr^3} - \lambda\frac{c}{r^{\lambda+1}}$$

For an exact circular orbit, RHS should vanish since there is no radial motion. Thus, we have,

$$\frac{L^2}{mr_0^3} - \lambda\frac{c}{r_0^{\lambda+1}} = 0 \Rightarrow r_0^{\lambda-2} = \frac{m\lambda c}{L^2}$$

(cf. Notice that we should require $c>0$ for $\lambda>0$ and $c<0$ for $\lambda<0$, to have circular orbit.)

Now, we consider the small oscillation about $r = r_0$. Putting $r=r_0+\Delta r$, we get

$$m\Delta\ddot{r} = \frac{d}{dr}\left(\frac{L^2}{mr^3} - \lambda\frac{c}{r^{\lambda+1}}\right)\Big|_{r=r_0}\Delta r \Rightarrow \Delta\ddot{r} = \frac{L^2}{m^2 r_0^4}(\lambda-2)\Delta r$$ factor

up to the leading order in Δr. To have a stable equilibrium, we should have negative term in RHS. Thus, we get

$$\lambda < 2$$

Writing $L = mr_0^2\Omega$, the above equation reduces to

$$\Delta\ddot{r} = -\omega^2 \Delta r$$

where $\omega^2 = (2-\lambda)\Omega$. By properly choosing phase, we get the solution,

$$r = r_0 (1 + \epsilon \cos\omega t)$$

as is familiar. From the angular momentum conservation,

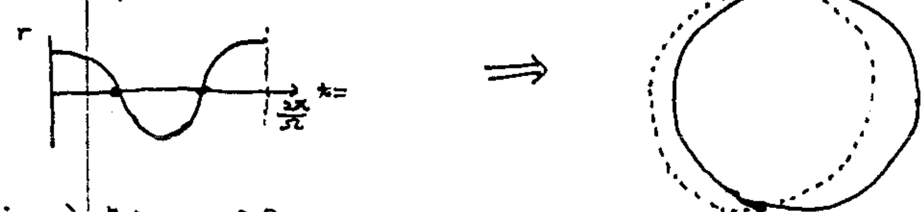
$$m r_0^2 \Omega = \text{constant} = m r^2 \dot\theta \Rightarrow \dot\theta = \frac{r_0^2\Omega}{r^2} \approx \Omega(1 - 2\epsilon\cos\omega t) \quad \text{(leading term in Taylor expansion)}$$

we get,

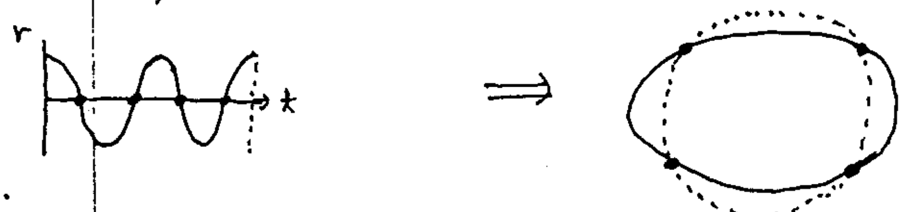
$$\theta = \Omega t - \frac{2\epsilon\Omega}{\omega}\sin\omega t = \Omega t - \frac{2\epsilon}{\sqrt{2-\lambda}}\sin\omega t$$

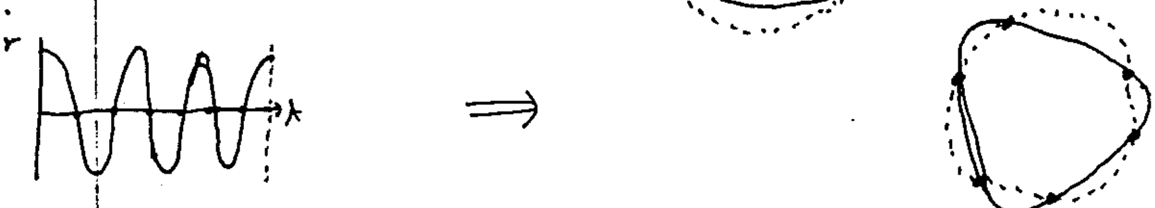
The graph for some closed orbit cases are (---- unperturbed, — perturbed)

$\lambda = 1, \ \omega = \Omega$



$\lambda = -2, \ \omega = 2\Omega$





Comment : In attractive central force case, they should concave toward the center as shown in lecture notes.

7. (a) The equation of conic section in general is given by (Goldstein, p. 96 Eq.(3-56))

$$\frac{1}{r} = C(1 + e\cos\theta) \qquad e = 1 \text{ in this case.}$$

From the figure, $r = p$ at $\theta = 0$. Thus, C is $C = \frac{p}{2}$. The angle at which the parabola meets a circle of radius a satisfies,

$$\frac{1}{a} = \frac{1}{2p}(1 + \cos\theta), \qquad \cos\theta = \frac{2p - a}{a}$$

(b) From the note, the eccentricity is given by,

$$\epsilon = \sqrt{1 + \frac{2EL^2}{m\alpha^2}}$$

Thus $\epsilon = 1$ gives $E = 0$. (cf. Hyperbola denotes a particle having asymptotically free, i.e. positive energy. The ellipse denotes a bounded particle which has negative energy. The parabola case, consequently has zero energy.)

(c) As the particle moves around the perihelion point, the position vector from the sun to the particle is perpendicular to the velocity at that time, Thus,

$$L = mr^2\dot{\theta} = mrv\big|_{r=p}$$

From the energy conservation with the total energy 0, we calculate v.

$$0 = -\frac{GMm}{r} + \frac{1}{2}mv^2\Big|_{r=p} \Rightarrow \qquad v = \sqrt{2\frac{GM}{r}}\Big|_{r=p}$$

Thus,

$$L = m\sqrt{2GMp}.$$

(d) From the angular momentum conservation, we have,

$$mr^2\dot{\theta} = L_0 \text{ ; constant.} \quad \Rightarrow \quad dt = \frac{mr^2}{L_0}d\theta$$

Using the trajectory equation derived in (a), we get $\left(\frac{1}{r} = \frac{1}{2p}(1+\cos\theta)\right)$

$$\frac{dr}{r^2} = \frac{\sin\theta}{2p}d\theta$$

(cf. integration in r variable is easier than integration in variable.)

Thus,

$$dt = \frac{m}{L_0}\frac{2p}{\sin\theta}dr = \frac{m}{L_0}\frac{r}{\sqrt{r/p-1}}dr$$

Thus, the time during which the comet stays inside the orbit of earch is

$$T = 2\cdot\frac{m}{L_0}p\int_p^a \frac{(r/p-1)+1}{\sqrt{r/p-1}}dr \quad \text{(factor 2 : } r\to\theta \text{ is two-to-one mapping!)}$$

$$= \frac{2m}{L_0}p\sqrt{r/p-1}\left(\frac{2p}{3}(r/p-1)+2p\right)\Big|_p^a = \frac{2}{3}\sqrt{2}\sqrt{\frac{a^3}{GM}}\sqrt{1-\frac{p}{a}}\left(1+\frac{2p}{a}\right)$$

where we used L_p derived in (c). To maximize time, we take,

$$\frac{dT}{dp} = 0 \quad \Rightarrow \quad \frac{1}{\sqrt{1-p/a}}\left(-\frac{1}{2} - p/a + 2 - 2p/a\right) = 0 \quad , \ p = \frac{1}{2}a$$

Thus, the maximum time is,

$$T\Big|_{p=\frac{a}{2}} = \frac{2}{3}\sqrt{\frac{2a^3}{GM}}\frac{1}{\sqrt{2}}\cdot 2 \approx 6.7\times10^6 \text{ sec} \simeq 11 \text{ weeks.}$$

8. From the note, we write the down the orbit equation,

$$\frac{d^2}{d\theta^2}u + u = -\frac{M}{L^2u^2}F\left(\frac{1}{u}\right)$$

We are given $F = -\frac{C}{r^3}$. Thus,

$$\frac{d^2}{d\theta^2}u + u = \frac{MC}{L^2}u \quad \Rightarrow \quad \frac{d^2}{d\theta^2}u + \left(1 - \frac{MC}{L^2}\right)u = 0 \qquad (1)$$

We define

$$\beta^2 = \left|1 - \frac{MC}{L^2}\right|$$

Thus, i) if $1-\frac{MC}{L^2}>0$, then (1) becomes,

$$\frac{d^2}{d\theta^2}u + \beta^2 u = 0$$

which have the well-known solution,

$$u = \frac{1}{r_0}\cos\beta(\theta-\theta_0) \quad \Rightarrow \quad \frac{1}{r} = \frac{1}{r_0}\cos\beta\theta \quad (\text{we set } \theta_0 = 0)$$

ii) if , then (1) becomes,

$$1 - \frac{MC}{L^2} = 0 \qquad \frac{d^2}{d\theta^2}u = 0$$

which can be trivially solved to yield,

$$u = \frac{1}{r_0} + A\theta \quad \Rightarrow \quad \frac{1}{r} = \frac{1}{r_0} + A\theta$$

iii) If $1-\frac{mc}{L^2}<0$, then (1) becomes,

$$\frac{d^2}{d\theta^2}u-\beta^2 u=0$$

which can be solved to give,

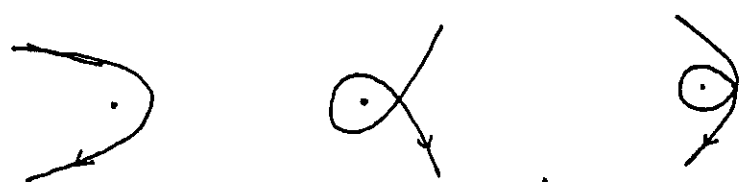
$$u=Ae^{-\beta r}+Be^{\beta r} \Rightarrow \frac{1}{r}=\frac{1}{r_0}e^{\pm\beta r}$$

In case (ii), if A=0, we have $r = r_0$;constant, i.e., circular orbit. As we've shown in problem 5, this is a unstable orbit. In this case, from the condition in (ii), we get

$$L^2 = Mc \Rightarrow m^2 r_0^4 \dot{\theta}^2 = mc$$

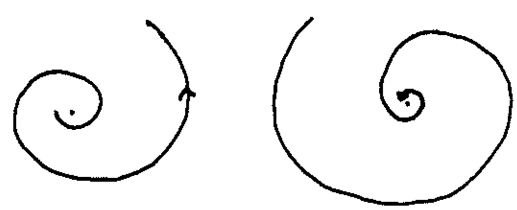
(cf. You might have tried to obtain $r=r(c)$ and $\dot{\theta}=\dot{\theta}(c)$. However, the above relation is the maximal information.) One cautionary comment is that to have three classes of solutions, we should require $c>0$ since $m>0$ and $L^2>0$, obviously. Thus, in case (i) $\beta^2<1$. In case (iii), there is no restriction in β^2. We list some typical graphs.

case i)



case ii)

case iii)



9. From the orbit equation

$$\frac{d^2}{d\theta^2}u+u=-\frac{m}{L^2u^2}F(u^{-1})$$

and $F=-\frac{A}{r^2}-\frac{B}{r^3}$, we get,

$$\frac{d^2}{d\theta^2}u+u=+\frac{m}{L^2u^2}(Au^2+Bu^3) \Rightarrow \frac{d^2}{d\theta^2}u+\beta^2u=\frac{mA}{L^2} \quad \text{where } \beta^2=1-\frac{mB}{L^2}$$

This is a simple linear oscillator equation with a constant inhomogeneous term. The solution can trivially be obtained to give,

$$\frac{1}{r}=u=\frac{mA}{L^2\beta^2}+\text{constant}\cos\beta\theta \qquad (1)$$

The above equation can be rewritten as,

$$\frac{1}{r}=\frac{1+\epsilon\cos\beta\theta}{a(1-\epsilon^2)} \qquad (2)$$

The angle required to complete one radial oscillation is $\frac{2\pi}{\beta}$ whereas the angle required for one orbital oscillation (If we can say some orbit is precessing!) is 2π . Thus, the perihelin advances angle,

$$\Delta\theta = \frac{2\pi}{\beta} - 2\pi = 2\pi\,\frac{1-\beta}{\beta}$$

per one circular motion. If we assume ϵ is not near 1, we can assert that $\beta \approx 1$ when $\eta = \frac{B}{Aa} \ll 1$ because,

$$\frac{BM}{L^2} = \frac{B}{aA(1-\epsilon^2)}\beta^2 = \frac{\eta}{1-\epsilon^2}\beta^2 \quad \text{(from comparing (1) \& (2))}$$

$$\beta = \sqrt{1 - BM/L^2} \Rightarrow \beta^2 = 1 - \frac{\eta}{1-\epsilon^2}\beta^2 \Rightarrow \beta^2 = 1/\left(1+\frac{\eta}{1-\epsilon^2}\right) \Rightarrow \beta \approx 1 - \frac{1}{2}\frac{\eta}{1-\epsilon^2}$$

Thus,

$$\Delta\theta = 2\pi\,\frac{1-\beta}{\beta} \approx 2\pi\cdot\frac{1}{2}\,\frac{\eta}{1-\epsilon^2} = \frac{\pi\eta}{1-\epsilon^2}$$

We use the information, $\epsilon = 0.206$ and the period of orbital motion of mercury is 0.24 year. Setting $\Delta\theta = 4.7\times10^{-7}$ (rad) which corresponds to 40'' per century gives,

$$\eta = \frac{1}{\pi}\cdot(1-\epsilon^2)\,\Delta\theta = 1.4\times10^{-7}.$$

10. (a) We represent the given potential as,

$$V = -\frac{A}{r} - \frac{B}{r^3}$$

where $A = GM_1M_2$, $B = A\cdot\frac{1}{5}\eta R^2$. $(\theta = \frac{\pi}{2})$

Thus, the effective potential given by adding centrifugal potential is,

$$V_{eff} = \frac{L^2}{2m r^2} - \frac{A}{r} - \frac{B}{r^3}$$

The circular motion radius is determined by,

$$\frac{d}{dr}V_{eff}\Big|_{r=r_0} = 0 \Rightarrow \frac{1}{r_0^3} = \frac{L^2}{A m_2 r_0^4} - \frac{3B}{A r_0^6} = \frac{\Omega^2}{GM_1} - \frac{3}{5}\frac{\eta R^2}{r_0^6} \quad (1)$$

The effective spring constant is given by,

$$k_{eff} = \frac{d^2}{dr^2}V_{eff}\Big|_{r=r_0} = \frac{3L^2}{m_2 r_0^4} - 2\frac{A}{r_0^3} - 12\frac{B}{r_0^5} = \frac{L^2}{M_2 r_0^4} - 6\frac{B}{r_0^5} \quad (L = M_2 r_0^2\Omega^2)$$

where we used (1). Thus, the small radial oscillation frequency is,

$$\omega = \sqrt{\frac{k_{eff}}{m}} = \Omega\sqrt{1 - \frac{6B}{\Omega^2 M_2 r_0^5}} = \Omega\sqrt{1 - \frac{6}{5}\frac{GM_1\eta R^2}{\Omega^2 r_0^5}}$$

During one ~~orbital~~ radial motion the time lapse is $\frac{2\pi}{\omega}$. Thus, The phase $\frac{2\pi}{\omega}\Omega$ is passed in ~~radial~~ orbital oscillation. That gives precession angle per one orbital motion as,

$$\Delta\theta = \left(\frac{2\pi}{\omega}\right)\Omega - 2\pi = 2\pi\left(\left(\sqrt{1 - \frac{6}{5}\frac{GM_1\eta R^2}{\Omega r_0^5}}\right)^{-1} - 1\right) = \underline{6\pi\frac{B}{\Omega^2 M_2 r_0^5}} = \frac{6\pi}{5}\frac{GM_1\eta R^2}{\Omega^2 r_0^5}$$

Setting $\Delta\theta = 4.7\times10^{-7}$ (from problem 9) gives

$$\eta = \frac{5}{6\pi}\cdot\frac{\Omega^2 r_0^5\,\Delta\theta}{GM_1R^2} \approx 1\times10^{-3}. \quad \text{(using } M_1 \,\&\, R \text{ given in this problem)}$$

(b) Directly integrating the given force

$$F = -\frac{GM_1M_2}{r^2}\left(1 + \frac{3L^2}{(M_2 r c)^2}\right)$$

gives the potential, $(F = -\nabla V)$

$$V = -\frac{GM_1M_2}{r} - GM_1M_2\cdot\frac{L^2}{M_2^2c^2}\,\frac{1}{r^3}$$

Thus, we set $A = GM_1M_2$ and $B = A\cdot\frac{L^2}{M_2^2c^2}$. Then formula we derived in (a) becomes,

$$\Delta\theta = \frac{6\pi}{c^2}GM_1\frac{1}{r_0} = \frac{24\pi^3 r_0^2}{T^2c^2} \sim 5\times10^{-7} \quad (\text{similar to } 4.7\times10^{-7})$$

$(4\pi^2 r_0^3)$