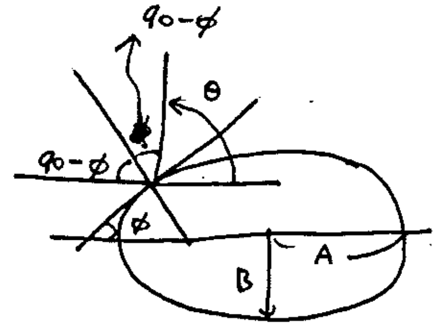
THE SOLUTION SET FOR THE PROBLEM SET 7



$d\sigma = 2\pi y\, dy.$

1. Due to the rotational symmetry about x-axis, we can consider only z=0 plane as shown right. Now,

$$\cos\theta = \cos 2\phi = \frac{\cos^2\phi - \sin^2\phi}{\cos^2\phi + \sin^2\phi} = \frac{1-\tan^2\phi}{1+\tan^2\phi} = \frac{1-(\frac{dy}{dx})^2}{1+(dy/dx)^2}$$

which gives,

$$\left(\frac{dy}{dx}\right)^2 = \frac{1-\cos\theta}{1+\cos\theta} \qquad (1)$$

and $\frac{dy}{dx} = \tan\phi$

where we used a geometric relation $\theta = 2\phi$ which is clear from the figure. The total cross section is the cross sectional area where the scattering of the incident particles occurs. Thus, from the figure, the total cross section is πB^2. On the other hand, the total cross section can be obtained from,

$$\int \frac{d\sigma}{d\cos\theta} d\cos\theta = \int_{-1}^{1} \frac{\pi a^2}{2}(1+\epsilon\cos\theta)\, d\cos\theta = \pi a^2$$

Consequently, we find that B = a. Thus, the equation of ellipsoid at z = 0 is given by

$$\frac{y^2}{a^2} + \frac{x^2}{A^2} = 1 \Rightarrow y^2 = a^2\left(1 - \frac{x^2}{A^2}\right)$$

We differentiate above equation to get,

$$y\,dy = -\frac{a^2}{A^2}x\,dx \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{y^2}\frac{a^4}{A^4} = \frac{a^2}{A^2}\frac{1-(1-x^2/A^2)}{1-x^2/A^2}$$

which yields,

$$1 - x^2/A^2 = \frac{a^2}{A^2}\frac{1}{(\frac{dy}{dx})^2 + \frac{a^2}{A^2}} \qquad (2)$$

Putting (1) into (2), we get,

$$1 - \frac{x^2}{A^2} = \frac{a^2/A^2}{(1-\cos\theta)/(1+\cos\theta) + a^2/A^2} \Rightarrow y^2 = a^2\left(1-\frac{x^2}{A^2}\right) = \frac{1+\cos\theta}{1+\frac{A^2}{a^2} + (1-\frac{A^2}{a^2})\cos\theta} \qquad (3)$$

Clearly, if $\epsilon = 0$, then we should have A = a. Thus, we can write $A = a(1+\alpha\epsilon)$ and all we have to do is determine α. Putting this into (3), we get

$$y^2 = a^2(1+\cos\theta)/(2+2\alpha\epsilon - 2\alpha\epsilon\cos\theta)$$

$$= \frac{a^2}{2}(1+\cos\theta)(1-\epsilon(1-\cos\theta)\alpha) = \frac{a^2}{2}(1+\cos\theta - \epsilon(1-\cos^2\theta)\alpha)$$

where we neglected second order term in ϵ^2 or higher. We differentiate above equation and get,

$$\frac{dy^2}{d\cos\theta} = \frac{1}{\pi}\frac{d\sigma}{d\cos\theta} = \frac{a^2}{2}(1+2\alpha\epsilon\cos\theta)$$

Comparing with the given equation,

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi a^2}{2}(1+\epsilon\cos\theta)$$

we find $\alpha = \frac{1}{2}$. Thus, $A = a\left(1+\frac{\epsilon}{2}\right)$ and B = a.

Comment: In this kind of problem, you don't have to pay attention to the second or higher order in ϵ. The reason will become clear from the more general consideration listed below. This will make the calculation less cumbersome. You might have felt that the choice of the ellipsoid is rather arbitrary. However, this choice can be justified rigorously. This is also shown in the below more general consideration.

ANOTHER METHOD. A RIGOROUS APPROACH

From the previous page, we have,

$$\cos\theta = \frac{1-(\frac{dy}{dx})^2}{1+(\frac{dy}{dx})^2}$$

We differentiate above expression with respect to y and get,

$$\frac{d\cos\theta}{dy} = -\frac{4\frac{d^2}{dx^2}y}{(1+(dy/dx)^2)^2}$$

We put this expression into

$$\frac{d\sigma}{d\cos\theta} = \frac{2\pi y\,dy}{d\cos\theta} = \frac{\pi a^2}{2}(1+\epsilon\cos\theta)$$

and get,

$$-\frac{1}{a^2}y = \frac{(1+\epsilon)+(1-\epsilon)(\frac{dy}{dx})^2}{(1+(\frac{dy}{dx})^2)^3}\frac{d^2}{dx^2}y$$

Writing $(\frac{dy}{dx})^2 = Y$ and multiplying above equation by dy we get,

$$-\frac{2}{a^2}y\,dy = \left(\frac{2\epsilon}{(1+Y)^3}+\frac{1-\epsilon}{(1+Y)^2}\right)dY$$

At y=b, dy/dx = 0 which implies Y = 0. We assume the desired function has only one point where dy/dx = 0. Then, from the total cross section considered in the previous page we have $\sigma_{tot} = \pi b^2 = \pi a^2$. Thus, we infer that b = a. We integrate above equation from a to y and get,

$$\frac{a^2-y^2}{a^2} = \frac{Y}{1+Y} + \epsilon\frac{Y}{(1+Y)^2}$$

Solving above equation for Y, we get,

$$Y = (\frac{dy}{dx})^2 = \frac{\sqrt{1+\epsilon^2-2\epsilon(1-2y^2/a^2)}+1-2y^2/a^2-\epsilon}{2y^2/a^2}$$

In fact we can directly integrate this expression. But this is unnecessarily complicated. For our purpose of small ϵ case, we expand righthand side of the above equation and get,

$$(\frac{dy}{dx})^2 = (1-\epsilon)\frac{1-y^2/a^2}{y^2/a^2}$$

which is easily ~~easily~~ integrated to yield

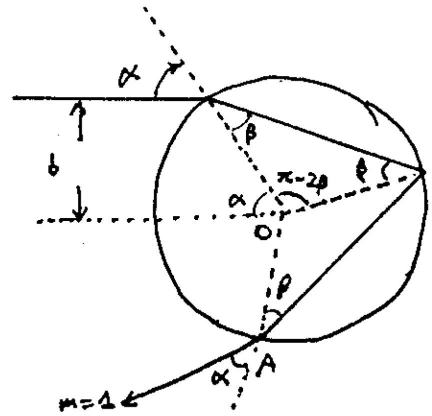
$$\mp\sqrt{1-y^2/a^2} = (x-x_0)/a\cdot\sqrt{1-\epsilon}$$

We set x_0 = 0 and get (we neglect ϵ^2 term)

$$1 = \frac{y^2}{a^2} + \frac{x^2}{a^2(1+\epsilon)}$$

This is the desired result with A = $a(1+\frac{\epsilon}{2})$ and B = a. One thing we should note is that if we include ϵ^2 effect, the result is no more an ellipsoid. This is the reason why I told you that you do not have to pay attention to ϵ^2 terms.

2. From the right figure, we find that the angle Δ is,



$$\Delta = \alpha + (\pi - 2\beta)(m+1)$$

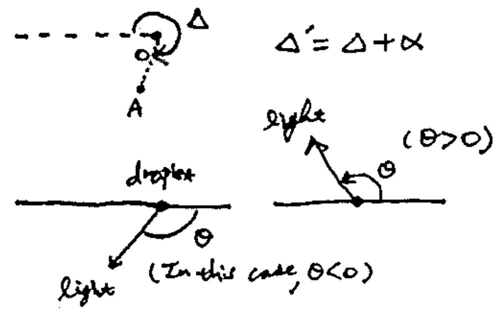
where m is the number of the internal reflection. Thus, as the ray goes out of the droplet, the angle it moved is,

$$\Delta' = \Delta + \alpha = 2\alpha + (\pi - 2\beta)(m+1)$$

Thus, the scattering angle θ is given by

$$\theta = \pi - \Delta' = 2\beta(m+1) - 2\alpha - m\pi \quad (1)$$

From the Snell's law,



$$\sin\alpha = n\sin\beta$$

we have

$$\cos\alpha\, d\alpha = n\cos\beta\, d\beta \Rightarrow \frac{d\beta}{d\alpha} = \frac{\cos\alpha}{n\cos\beta}$$

Thus, differentiating (1) with respect to α gives,

$$\frac{d\theta}{d\alpha} = 2(m+1)\frac{d\beta}{d\alpha} - 2 = 2\left((m+1)\frac{\cos\alpha}{n\cos\beta} - 1\right)$$

By requireing $\frac{d\theta}{d\alpha} = 0$, consequently, we get,

$$(m+1)\cos\alpha = n\cos\beta \Rightarrow (m+1)^2\cos^2\alpha = n^2(1-\sin^2\beta) = n^2\left(1 - \frac{1}{n^2}\sin^2\alpha\right) = n^2 - \sin^2\alpha$$

$$\therefore \sin^2\alpha = \frac{(m+1)^2 - n^2}{(m+1)^2 - 1} \quad (2)$$

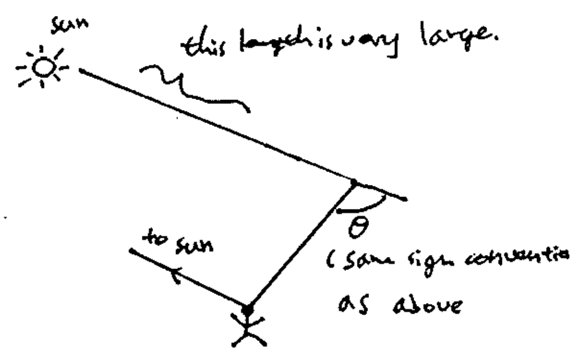
where we used Snell's law once again. In case of water, $n \sim 4/3$. Thus, we can calculate α via the above equaiton and and β via the Snell's law and θ from (1). We get,

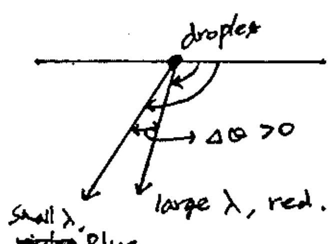
$$\alpha = 59.4°,\ \beta = 40.2°,\ \theta = -138°$$

for m = 1

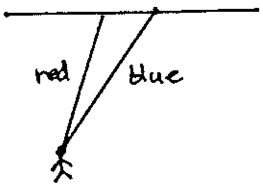
$$\alpha = 71.8°,\ \beta = 45.4°,\ \theta = 129°$$

for m = 2. The angle between the rainbow we see and the direction to the sun is depicted in the right figure. As λ gets longer, n gets smaller. As n gets smaller α gets larger as seen from (2). However, β gets larger more fast as seen from the Snell's law. As a net result, as long as $m \gtrsim 1$, we can be sure that θ increases. In case of m=1, the initial θ was a negative number. Thus, the light ray from a single droplet can be drawn as,

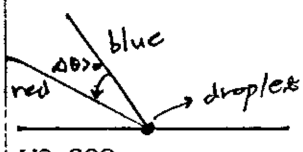




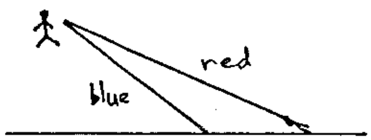
Thus, actually what we see is



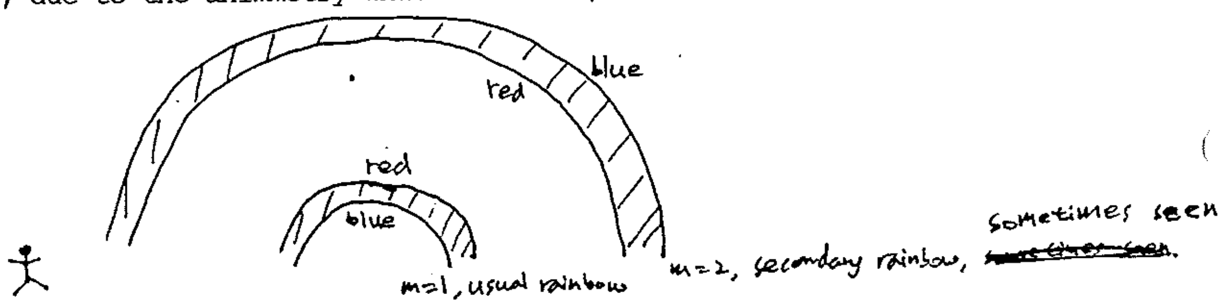
In case of m=2, the initial θ is a positive number. Thus, the light from the single droplet is,



Thus, we see,



Of course, due to the aximmetry what we see is,



comment. There can be variety of sign convention for each angular variable. Thus, one should be VERY careful. For example, the sign of θ ,... etc.

comment. Usually m=2 rainbow is not seen simply due to the fact that there is not enough droplets at that angle. (high altitude.) However, sometimes, it can be seen.

3. Newton's law along the vertical direction gives the equation of motion,

$$m\ddot{x} = -b(\dot{x}-\dot{X}) - k(x-X) - mg$$

Using $X = A\cos\omega t + X_0$, the above equatioin can be rewritten as,

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \omega_0^2(\ell + X_0) - g + \omega_0 A\cos\omega t - 2\omega\beta A\sin\omega t$$

(Notice that we are only interested in particular solutions of the above equation though it is not explicite in the statements of this problem.) The particular solution produced by the constant terms are trivially obtained to give,

$$x_c = X_0 + \ell - g/\omega_0^2$$

Thus, if we write $x = x_c + x_s$, we get,

$$\ddot{x}_s + 2\beta\dot{x}_s + \omega_0^2 x_s = \omega_0 A\cos\omega t - 2\omega\beta\sin\omega t \qquad (1)$$

Using

$$\cos\omega t = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t}) \quad \& \quad \sin\omega t = \frac{1}{2i}(e^{i\omega t} - e^{-i\omega t})$$

(1) can be written as,

$$\ddot{x}_s + 2\beta\dot{x}_s + \omega_0^2 x_s = \frac{A}{2}(\omega_0^2 + 2i\beta\omega)e^{i\omega t} + \text{complex conjugate}$$

Consequently, if we write,

$$x_s = \alpha e^{i\omega t} + \alpha^* e^{-i\omega t}$$

α satisfies,

$$(-\omega^2 + 2\beta\omega i + \omega_0^2)\alpha = \frac{A}{2}(\omega_0^2 + i2\beta\omega). \Rightarrow \alpha = \frac{A}{2}\frac{\omega_0^2 + 2\beta\omega i}{(\omega_0^2-\omega^2) + 2\beta\omega i}$$

Thus, if we use real functions to denote solution,

$$x_s = (\alpha+\alpha^*)\cos\omega t + i(\alpha-\alpha^*)\sin\omega t = 2\,\mathrm{Re}(\alpha)\cos\omega t - 2\,\mathrm{Im}(\alpha)\sin\omega t$$

$$= A\frac{\omega_0^2(\omega_0^2-\omega^2)+4\beta^2\omega^2}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}\cos\omega t + A\frac{2\beta\omega^3}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}\sin\omega t$$

The amplitude is given by,

$$\text{amplitude} = \sqrt{(\alpha+\alpha^*)^2 + (i(\alpha-\alpha^*))^2} = 2\sqrt{\alpha\alpha^*} = 2|\alpha|$$

$$= A\sqrt{\frac{\omega_0^4 + 4\beta^2\omega^2}{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}}$$

Since the average of the oscillatiing part is 0, the average height is,

$$\langle x\rangle = x_c = x_0 + l - g/\omega_0^2$$

From (1), we find the critical damping condition is $\beta=\omega$. In this case the amplitude can be written as,

$$\text{amplitude} = A\sqrt{\frac{\omega_0^4 + 4\omega_0^2\omega^2}{(\omega_0^2-\omega^2)^2 + 4\omega_0^2\omega^2}} = A\sqrt{\frac{1+4x^2}{(1-x^2)^2+4x^2}} = A\frac{\sqrt{1+4x^2}}{1+x^2}$$

where $x = \omega/\omega_0$. Thus, the requirement $\frac{d}{dx}\text{amplitude} = 0$,

$$\frac{d}{dx}\text{amplitude} = \frac{A}{(1+x^2)^2\sqrt{1+4x^2}}(2x - 4x^3) = 0$$

determines $x^2 = 1/2$. Thus, the maximum amplitude is, $A\cdot\frac{\sqrt{1+4\cdot\frac{1}{2}}}{1+\frac{1}{2}} = \frac{2\sqrt{3}}{3}A$

4. (a) If we expand F(t) in terms of Fourier series, , the general form is given by

$$F(t) = \frac{a_0}{2} + \sum_n a_n\cos n\omega t + \sum_n b_n \sin n\omega t$$

where

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} F(t)\cos n\omega t\, dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} F(t)\sin n\omega t\, dt$$

Since F(t) is even function under t -> -t and cosine function is odd function, we get $a_n = 0$. In case of b_n,

$$b_n = \frac{4}{T}\int_0^{T/2}\frac{F_0}{T}t\sin n\omega t\, dt$$

$$= \frac{4}{T}\cdot\frac{F_0}{T}\cdot\left[-\frac{t\cos n\omega t}{n\omega}\Big|_0^{T/2} + \underbrace{\int_{-T/2}^{T/2}\frac{\cos n\omega t}{n\omega}dt}_{=0}\right]$$

$$= -\frac{F_0}{n\pi}\cos n\pi = -\frac{F_0}{n\pi}\cdot(-1)^n$$

Thus,

$$F(t) = +\frac{F_0}{\pi}\left(\sin\omega t - \frac{\sin 2\omega t}{2} + \frac{\sin 3\omega t}{3} + \cdots\right)$$

(b) In this case,

$$a_n = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} F(t)\cos\omega t\, dt$$

$$= \frac{\omega}{\pi}\int_0^{\pi/\omega}\sin\omega t\cos n\omega t\, dt = \frac{\omega}{2\pi}\int_0^{\pi/\omega}(\sin(n+1)\omega t + \sin(1-n)\omega t)\, dt$$

If n = 1, $a_1 = \frac{\omega}{2\pi}\int_0^{\pi/\omega} \sin 2\omega t \, dt = 0$

If n is not 1,

$$a_n = \frac{1}{2\pi}\left[-\frac{\cos(n+1)\omega t}{n+1} - \frac{\cos(1-n)\omega t}{1-n} \right]\Bigg|_0^{\pi/\omega}$$

$$= \frac{1}{2\pi}\left(\frac{1}{n+1} - \frac{1}{n-1} - \frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} \right)$$

$$= -\frac{1}{\pi}\,\frac{1-(-1)^{n+1}}{n^2-1}.$$

From the orthogonality of each sine functions, $b_n = 0$ except for n = 1.

$$b_1 = \frac{\omega}{\pi}\int_0^{\pi/\omega} \sin^2\omega t \, dt = \frac{\omega}{\pi}\int_0^{\pi/\omega} \frac{\sin^2\omega t + \cos^2\omega t}{2}\, dt = \frac{\omega}{2\pi}\cdot\frac{\pi}{\omega} = \frac{1}{2}.$$

$$\therefore\ F(t) = \frac{1}{\pi} + \frac{1}{2}\sin\omega t - \frac{2}{3\pi}\cos 2\omega t - \frac{2}{15\pi}\cos 4\omega t + \cdots.$$

Clearly the series in (b) converges faster since each term in the series is inversely proportional to n^2 whereas the terms in (a) is inversely proportional to n.

5. (a) From the note, the general formula for x(t) is given by

$$x(t) = \int_{-\infty}^{t} dt' \frac{F(t')}{m\omega_1} e^{-\beta(t-t')} \sin\omega_1(t-t')$$

where $\omega_1^2 = \omega_0^2 - \beta^2$. We can assume F(t) = 0 for t < 0. Then, the integral becomes, is.

$$x(t) = \frac{F_0}{m\omega_1}\int_0^t dt' e^{-\beta(t-t')} \sin\omega_1(t-t') \qquad (\mathrm{Im}(Z) \text{ represents imagenary part of } Z)$$

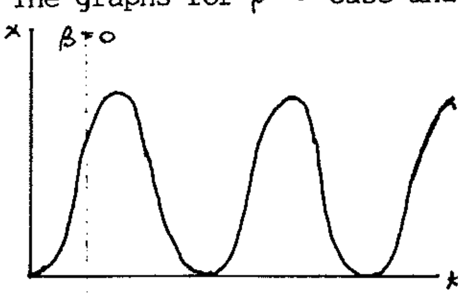
$$= \frac{F_0}{m\omega_1}\,\mathrm{Im}\int_0^t dt' e^{-(\beta-i\omega_1)(t-t')}$$

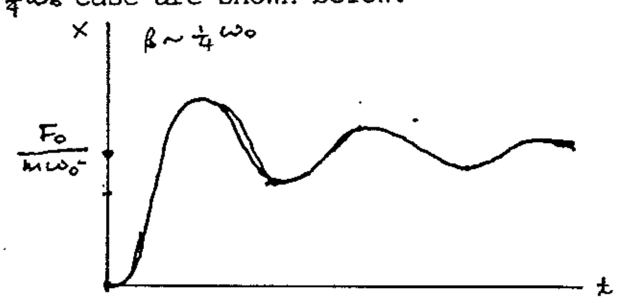
$$= \frac{F_0}{m\omega_1}\,\mathrm{Im}\left(\frac{e^{-(\beta-i\omega_1)(t-t')}}{\beta-i\omega_1}\Bigg|_0^t \right) = \frac{F_0}{m\omega_1}\,\mathrm{Im}\left(\frac{1-e^{-(\beta-i\omega_1)t}}{\beta-i\omega_1} \right)$$

$$= \frac{F_0}{m\omega_1}\,\frac{1}{\beta^2+\omega_1^2}\,\mathrm{Im}\left\{\left(-e^{-\beta t}(\cos\omega_1 t + i\sin\omega_1 t) + 1\right)(\beta + i\omega_1)\right\}$$

$$= \frac{F_0}{m\omega_0^2}\left(1 - e^{-\beta t}\cos\omega_1 t - \frac{\beta}{\omega_1}e^{-\beta t}\sin\omega_1 t\right)$$

The graphs for $\beta = 0$ case and $\beta \sim \frac{1}{4}\omega_0$ case are shown below.





The maximum overshoots are obtained by requiring,

$$\dot{x} = \frac{F_0}{m\omega_0^2}\left(\beta e^{-\beta t}\cos\omega_1 t + \omega_1 e^{-\beta t}\sin\omega_1 t + \frac{\beta^2}{\omega_1}e^{-\beta t}\sin\omega_1 t - \beta e^{-\beta t}\cos\omega_1 t\right)$$

$$= \frac{F_0}{m\omega_0^2}\left(\omega_1 + \frac{\beta^2}{\omega_1}\right)e^{-\beta t}\sin\omega_1 t = 0 \quad \Rightarrow \quad \sin\omega_1 t = 0$$

Thus, 1-st overshoot has maximum when $\omega_1 t_0 = \pi$. At this time, the the value of x is

$$x(t) = \frac{F_0}{m\omega_0^2}\left(1 + e^{-\frac{\beta}{\omega_1}\pi}\right)$$

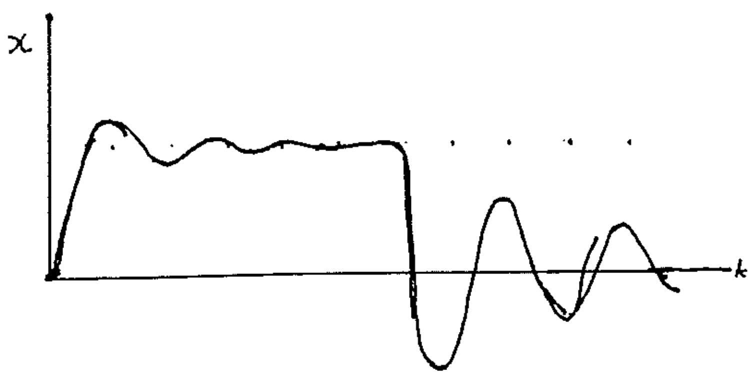
(b) This problem is exactly the same before t < T. After this time, we should integrate,

$$x(t) = \frac{F_0}{m\omega_1}\int_0^T dt'\, e^{-\beta(t-t')}\sin\omega_1(t-t')$$
$$= \frac{F_0}{m\omega_1}\,\mathrm{Im}\,\frac{e^{-(\beta-i\omega_1)(t-t')}}{\beta-i\omega_1}\Big|_0^T$$
$$= \frac{F_0}{m\omega_1}\,\mathrm{Im}\left\{\left(e^{-(\beta-i\omega_1)(t-T)} - e^{-(\beta-i\omega_1)t}\right)/(\beta - i\omega_1)\right\}$$
$$= \frac{F_0}{m\omega_1}\frac{1}{\beta^2+\omega_1^2}\,\mathrm{Im}\left\{(\beta+i\omega_1)\left(e^{-\beta(t-T)}(\cos\omega_1(t-T)+i\sin\omega_1(t-T)) - e^{-\beta t}(\cos\omega_1 t + i\sin\omega_1 t)\right)\right\}$$
$$= \frac{F_0}{m\omega_0^2}\left(e^{-\beta(t-T)}\cos\omega_1(t-T) - e^{-\beta t}\cos\omega_1 t + \frac{\beta}{\omega_1}e^{-\beta(t-T)}\sin\omega_1(t-T) - \frac{\beta}{\omega_1}e^{-\beta t}\sin\omega_1 t\right)$$

If damping has been very strong we can neglect the terms containing $e^{-\beta t}$ factor, i.e,

$$x(t) \approx \frac{F_0}{m\omega_0^2}\,e^{-\beta(t-T)}\left(\cos\omega_1(t-T) + \frac{\beta}{\omega_1}\sin\omega_1(t-T)\right) \quad \text{for } t > T.$$

which has the graph,



6. (a) The equation we should solve is,

$$m\ddot{x} + 2m\beta\dot{x} + m\omega_0^2 x = F_0 e^{-\alpha t}$$

We can guess the solution of the form x = A exp($-\alpha$t). We put this into above equation to get,

$$(\alpha^2 - 2\alpha\beta + \omega_0^2)A = F_0/m$$

Consequently,

$$x(t) = \frac{F_0/m}{\alpha^2 - 2\alpha\beta + \omega_0^2}\,e^{-\alpha t}$$

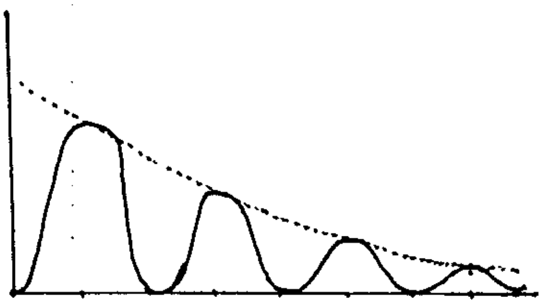
(b) For t < 0, x(t) = 0. For t>0, the Green's method gives,

$$x(t) = \int_{-\infty}^t dt'\,\frac{F(t')}{m\omega_1}e^{-\beta(t-t')}\sin\omega_1(t-t') = \frac{F_0}{m\omega_1}\int_0^t dt'\,e^{-\alpha t'}e^{-\beta(t-t')}\sin\omega_1(t-t')$$
$$= \frac{F_0}{m\omega_1}\,\mathrm{Im}\int_0^t dt'\,e^{-(\beta-i\omega_1)t}\,e^{-(\alpha-\beta+i\omega_1)t'}$$
$$= \frac{F_0}{m\omega_1}\,\mathrm{Im}\,-\frac{e^{-\alpha t'}e^{-(\beta-i\omega_1)(t-t')}}{\alpha-\beta+i\omega_1}\Big|_0^t$$
$$= \frac{F_0}{m\omega_1}\,\mathrm{Im}\,\frac{-e^{-\alpha t} + e^{-(\beta-i\omega_1)t}}{\alpha-\beta+i\omega_1}$$
$$= \frac{F_0}{m\omega_1}\frac{1}{(\alpha^2-2\alpha\beta+\beta^2+\omega_1^2)}\,\mathrm{Im}\left\{\left(-e^{-\alpha t} + e^{-\beta t}(\cos\omega_1 t + i\sin\omega_1 t)\right)(\alpha-\beta-i\omega_1)\right\}$$
$$= \frac{F_0/m}{\omega_0^2+\alpha^2-2\alpha\beta}\left(e^{-\alpha t} + e^{-\beta t}\left(\frac{\alpha-\beta}{\omega_1}\sin\omega_1 t - \cos\omega_1 t\right)\right)$$

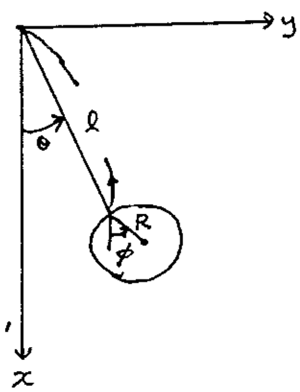
If $\alpha = \beta$,

$$x(t) = \frac{F_0/m}{\omega_0^2 - \alpha^2}\, e^{-\alpha t}\,(1 - \cos\omega_1 t).$$

Thus, the graph is



7. The kinetic energy of the hoop can be written as the sum of its internal rotational energy and the kinetic energy of CM. Since the hoop and the rod are connected by the free pivot, the rotational energy about CM is,



$$\tfrac{1}{2} I \dot\phi^2 = \tfrac{1}{2} M R^2 \dot\phi^2$$

From the above figure, the coordiante of CM are given by,

$$X = \ell\cos\theta + R\cos\phi \,,\quad Y = \ell\sin\theta + R\sin\phi$$

$$\Rightarrow\quad \dot X = -\ell\sin\theta\,\dot\theta - R\sin\phi\,\dot\phi \,,\quad \dot Y = \ell\cos\theta\,\dot\theta + R\cos\phi\,\dot\phi$$

Thus, the kinetic energy of CM is

$$\tfrac{1}{2}M(\dot x^2 + \dot y^2) = \tfrac{1}{2}M\ell^2\dot\theta^2 + \tfrac{1}{2}MR^2\dot\phi^2 + 2\ell R\dot\theta\dot\phi(\cos\theta\cos\phi - \sin\theta\sin\phi)$$

Thus, the total Lagrangian is

$$L = \tfrac{1}{2}M\ell^2\dot\theta^2 + MR^2\dot\phi^2 + M\ell R\dot\theta\dot\phi(\cos\theta\cos\phi - \sin\theta\sin\phi) + mg\ell\cos\theta + mgR\cos\phi$$

$$\approx \tfrac{1}{2}M\ell^2\dot\theta^2 + MR^2\dot\phi^2 + M\ell R\dot\theta\dot\phi + mg\ell\left(1 - \frac{\theta^2}{2}\right) + mgR\left(1 - \frac{\phi^2}{2}\right)$$

where we retained only up to second order terms in amplitudes, since we are considering small oscillation. Now, Euler-Lagrange equation gives,

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot\theta}L\right) - \frac{\partial L}{\partial\theta} = 0 \;\Rightarrow\; \ell^2\ddot\theta + \ell R\ddot\phi + g\ell\theta = 0$$

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot\phi}L\right) - \frac{\partial L}{\partial\phi} = 0 \;\Rightarrow\; 2R^2\ddot\phi + \ell R\ddot\theta + gR\phi = 0$$

Using harmonic forms $\ddot\theta = -\omega^2\theta$ and $\ddot\phi = -\omega^2\phi$, above equation can be rewritten as,

$$\begin{pmatrix} g\ell - \ell^2\omega^2 & -\ell R\omega^2 \\ -\ell R\omega^2 & -2\omega^2R^2 + gR \end{pmatrix}\begin{pmatrix}\theta \\ \phi\end{pmatrix} = 0$$

To have non-trivial solutions, we require the determinant of the above matrix to vanish. Thus,

$$\begin{vmatrix} g\ell - \ell^2\omega^2 & -\ell R\omega^2 \\ -\ell R\omega^2 & -2\omega^2R^2 + gR \end{vmatrix} = 0 \;\Rightarrow\; \omega^4 - 2\frac{g}{\ell}\left(1 + \frac{\ell}{2R}\right)\omega^2 + \frac{g^2}{\ell^2}\frac{\ell}{R} = 0$$

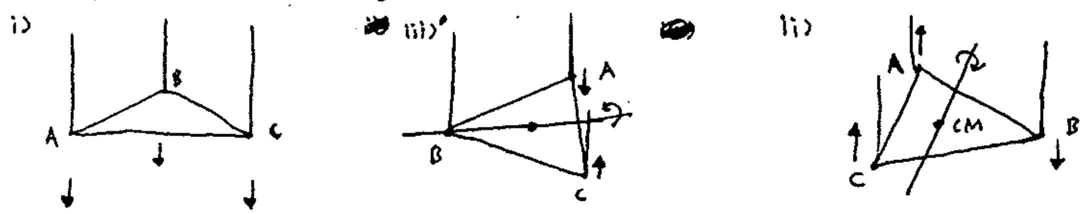
$$\omega^2 = \frac{g}{l}\left(1+\frac{l}{2R} \pm \sqrt{1+\frac{l^2}{4R^2}}\right)$$

Thus, if R = 1 / 2, we have

$$\omega^2 = \frac{g}{l}(2 \pm \sqrt{2})$$

comment. There is a typographical mistake in a problem set. One important fact to note is the one frequency is larger than $\sqrt{\frac{g}{l}}$ and another is less than that. Try to understand the simple physical picture behind this fact.

8. First, we guess the three modes. Clearly the first mode is the translation of the triangle without any rotation about its CM. Another mode is the rotation about CM as shown below. Additionally, we can choose rotation about the line connecting CM and other end of triangle as another mode. (Or, rotation orthogonal to the second case.)

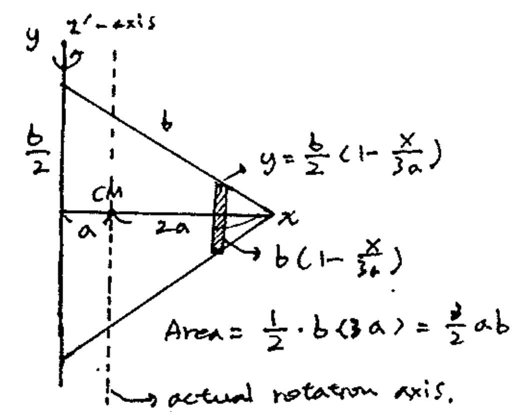


In first case, the displacement from the equilibrium position (equilibrium including gravity), x, causes restoring force 3kx. Thus, the equation of motion is,

$$m\ddot{x} = -3kx$$

Thus, the frequency is,

$$\omega_1 = \sqrt{\frac{3k}{m}}$$



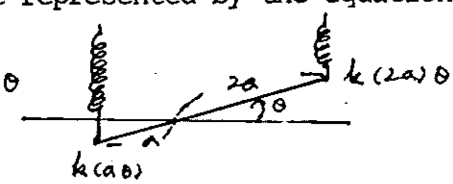
Now we calculate the moment of inertia of the triangle shown right. Clearly, the moment of inertia about the axis shown is,

$$I' = \int_0^{3a} x^2 b\left(1-\frac{x}{3a}\right)dx \cdot m/\text{Area}$$
$$= \frac{3}{2}ma^2$$

Using parallel axis theorem, the moment of inertia about the axis passing through the CM is given by,

$$ma^2 + I_{cm} = \frac{3}{2}ma^2 \Rightarrow I_{cm} = \frac{1}{2}ma^2.$$

In the second mode, the rotation shown right can be represented by the equation of motion,

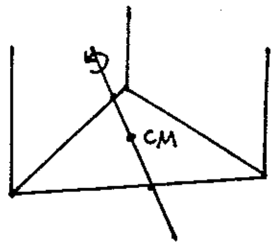


$$\frac{1}{2}ma^2\ddot{\theta} = -k(2a)\theta(2a) - a\theta k\cdot a - a\theta\cdot k\cdot a \Rightarrow \ddot{\theta} = -\frac{12k}{m}\theta$$

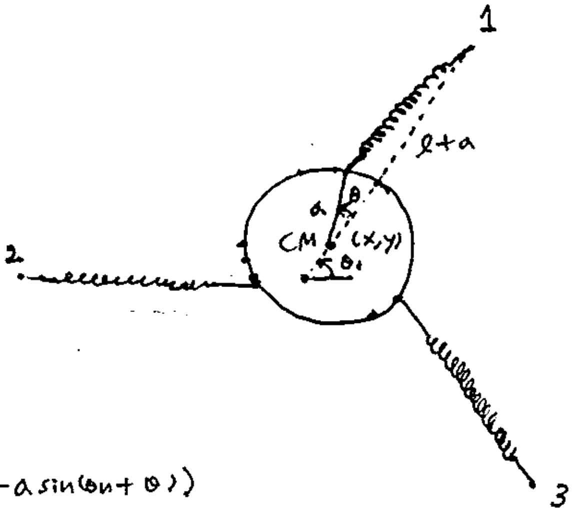
Thus, the frequency is,

$$\omega_2 = 2\sqrt{\frac{3k}{m}}$$

By the similar fashion, another frequency is calculated to give the same result. This result permit us to define arbitrary mode having arbitrary line about which rotation occurs. One of them, for example, would look like,



9. The kinetic energy of the disk is the sum of the translational energy of CM and the rotational energy about CM. Thus,



$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}\left(\frac{1}{2}ma^2\right)\dot{\theta}^2$$

To calculate potential energy stored in each spring, we should know the stretched length of the springs. The stretched length vector of spring n is given by, referring to the right figure,

$$\vec{l}_n = \left((l+a)\cos\theta_n - x - a\cos(\theta_n+\theta),\ (l+a)\sin\theta_n - y - a\sin(\theta_n+\theta)\right)$$

Thus, its magnitude square is,

$$l_n^2 = l^2 + x^2 + y^2 + 2a(l+a)(1-\cos\theta) - 2xl\cos\theta_0 - 2yl\sin\theta_0 - (2ax\cos\theta_0 + 2ay\sin\theta_0)(1-\cos\theta)$$
$$- 2a(x\sin\theta_0 - y\cos\theta_0)\sin\theta$$

Thus,

$$(l_n - l_0)^2 = l_0^2 - 2l_0 l_n + l_n^2$$
$$\approx l_0^2 + x^2 + y^2 + l^2 + a(l+a)\theta^2 - 2l(x\cos\theta_0 + y\sin\theta_0) - 2a\theta(x\sin\theta_0 - y\cos\theta_0)$$
$$-2l_0 l\Big(1 + \frac{x^2}{2l^2} + y^2/2l^2 + a(l+a)\theta^2/2l^2 - (x\cos\theta_0 + y\sin\theta_0)/l - a\theta(x\sin\theta_0 - y\cos\theta_0)/l^2$$
$$- \frac{1}{2}\frac{x^2}{l^2}\cos^2\theta_0 - \frac{1}{2}\frac{y^2}{l^2}\sin^2\theta_0 - \frac{1}{2}\frac{xy}{l^2}\sin\theta_0\cos\theta_0\Big)$$

where we used the Taylor expansion, $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$, $\cos\theta = 1 - \frac{\theta^2}{2} + \cdots$, and $\sin\theta = \theta + \cdots$ and we retained terms only up to the second order in amplitude. From the geometry of this problem, we have,

$$\sum_n \cos^2\theta_n = \sum_n \sin^2\theta_n = \frac{3}{2}, \quad \sum_n \cos\theta_n = \sum_n \sin\theta_n = \sum_n \sin\theta_n\cos\theta_n = 0.$$

Thus, the whole Lagrangian can be written as, (parts vanish!)

$$L = T + \sum_n \frac{1}{2}k(l_n - l_0)^2$$
$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{4}ma^2\dot{\theta}^2 + \frac{3}{2}k\frac{2l-l_0}{2l}(x^2+y^2) + \frac{3k}{2}\frac{a(l+a)(l-l_0)}{l}\theta^2 + \frac{3}{2}k(l-l_0)^2$$

, which gives three decoupled equations of motion via Euler-Lagrange equation.

$$\ddot{x} + \frac{3k}{2m}\frac{2l-l_0}{l}x = 0, \quad \ddot{y} + \frac{3k}{2m}\frac{2l-l_0}{l}y = 0$$
$$\ddot{\theta} + \frac{6k}{m}\frac{(a+l)(l-l_0)}{al}\theta = 0$$

Thus, the three frequencies of this problem are

$$\omega_1 = \omega_2 = \sqrt{\frac{3k}{2m}\frac{2l-l_0}{l}}, \quad \omega_3 = \sqrt{\frac{6k}{m}\frac{(l-l_0)(a+l)}{al}}$$

comment. The calculational method shown here looks a little bit cumbersome. As long as the amount of the calculations is concerned, guessing the modes is the best. Only to give you many possible ways of solving problems, I chose this rather general method.

10. (a) The Lagrangian of this system is,

$$L = \frac{1}{2}m_A(\dot{x}_1^2 + \dot{x}_3^2) + \frac{1}{2}m_B\dot{x}_2^2 - \frac{1}{2}k((x_1 - x_2)^2 + (x_3 - x_2)^2)$$

Thus, the equations of motions are,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0 \Rightarrow m_A\ddot{x}_1 + kx_1 - kx_2 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0 \Rightarrow m_B\ddot{x}_2 - kx_1 + 2kx_2 - kx_3 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_3}\right) - \frac{\partial L}{\partial x_3} = 0 \Rightarrow m_A\ddot{x}_3 - kx_2 + kx_3 = 0$$

We assume the oscillatory solutions, $\ddot{x}_1 = -\omega^2 x_1, \ddot{x}_2 = -\omega^2 x_2$, and $\ddot{x}_3 = -\omega^2 x_3$. Then, the condition for non-trivial solution is to set the determinant of the matrix of the above equation to be zero. That determinant is,

$$\begin{vmatrix} -m_A\omega^2 + k & -k & 0 \\ -k & -m_B\omega^2 + 2k & -k \\ 0 & -k & -m_A\omega^2 + k \end{vmatrix} = 0$$

From the determinant of the form,

$$\begin{vmatrix} a & -d & 0 \\ -d & b & -d \\ 0 & -d & a \end{vmatrix} = aba - d^2a - d^2a = a(ab - 2d^2) = 0 \Rightarrow a = 0 \text{ or } ab - 2d^2 = 0.$$

we have an equation for ω^2.

$$-m_A\omega^2 + k = 0, \quad (-m_A\omega^2 + k)(-m_B\omega^2 + 2k) - 2k^2 = m_A m_B\omega^4 - (2m_A + m_B)k\omega^2 = 0$$

Thus, we have three solutions,

$$\omega^2 = \frac{k}{m_A}, \quad \omega^2 = 0, \quad \omega^2 = \frac{2m_A + m_B}{m_A m_B}k$$

(b) In this case, we should add $-\frac{k}{2}x_2^2$ term to the Lagrangian of problem (a). Then the exactly the same procedure gives the determinant,

$$\begin{vmatrix} -m_A\omega^2 + k & -k & 0 \\ -k & -m_B\omega^2 + 3k & -k \\ 0 & -k & -m_A\omega^2 + k \end{vmatrix} = 0$$

which reduces to,

$$-m_A\omega^2 + k = 0, \quad (-m_A\omega^2 + k)(-m_B\omega^2 + 3k) - 2k^2 = m_A m_B\omega^4 - (3m_A + m_B)k\omega^2 + k^2 = 0$$

Thus, solving the above equations, we have three frequencies,

$$\omega^2 = \frac{k}{m_A}, \quad \omega^2 = k\,\frac{3m_A + m_B \pm \sqrt{9m_A^2 + 2m_A m_B + m_B^2}}{2m_A m_B}$$