PH205 Solution Set 9

1. (a) The forces concerned in this problem are (In rotating frame) the centrifugal force and the force along the CM-center-of-tube line which originates from the spring. Since the number of forces concerned is only two, they should lie parallel to each other, otherwise there can be no equilibrium at all. Furthermore, since the spring force is proportional to the stretched distance and the centrifugal force is proportional to the stretched distance (Plus or minus a, of cource) and the rotation frequency is very high comparing to $\sqrt{k/m}$, the CM position should be on the spring to reduce the magnitude of centrifugal force. Thus, the equilibrium point should be at $\theta = 0$, and

$$kr = m(r-a)\Omega^2 \Rightarrow r = \frac{\Omega^2}{\Omega^2-\omega_0^2}a$$

(comment. Notice that r-a is positive. Some students get two solutions for radial positions. However, it is easy to show that two positions are basically the same.)

(b) If the angle variable is fixed, (this necessarily means the introduction of other fictitious force in angular direction. Thus, considering the angular direction without the consideration of the fictitious force is meaningless.) the effective potentia can be written as,

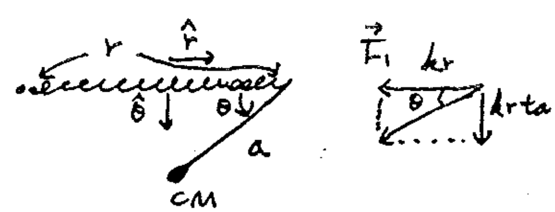
$$V^{eff} = \frac{1}{2}kr^2 - \frac{1}{2}m\Omega^2(r-a)^2 = \frac{1}{2}m\{(\omega_0^2-\Omega^2)r^2 + 2a\Omega^2 r + a^2\Omega^2\}$$

Clearly, the spring constant $\left(= \frac{\partial^2 V^{eff}}{\partial r^2}, \text{here.}\right)$ is negative, which means the motion is unstable.

(c) In general case, there are forces from various origins. As a consequence of both spring and the constraint force which is responsible for the constancy of "a", (assuming there is a rod between CM and the center of the tube is a convenient way of thinking) there is a force,

$$\vec{F}_1 = -kr\,\hat{r} + kr\tan\theta\,\hat{\theta}$$

where the coordinates are shown right. r component of this force is determined by the spring and component is determined by the requirement that the true force on CM lies along the line connecting CM and center of the tube. The centrifugal force is given by



$$\vec{F}_{cen} = m(r - a\cos\theta)\Omega^2\,\hat{r} + ma\sin\theta\,\Omega^2\,\hat{\theta} = m(\vec{r}+\vec{a})\Omega^2$$

and the Coriolis force is given by

$$\vec{F}_{cor.} = 2m\vec{\Omega}\times(\dot{r}\,\hat{r} + a\dot{\theta}\,\hat{\theta}) = 2m\Omega a\dot{\theta}\,\hat{r} - 2m\Omega\dot{r}\,\hat{\theta}$$

where one should be cautious about the sign convention. Thus, the total equation is given by (for small oscillation.)

$$m\ddot{r} = -kr + m(r - a\cos\theta)\Omega^2 + 2m\Omega a\dot{\theta}$$

$$m\ddot{\theta} = kr\tan\theta + ma\sin\theta\,\Omega^2 - 2m\Omega\dot{r}$$

Since we are considering small oscillation, we set $r = \frac{\Omega^2}{\Omega^2-\omega_0^2}a + r_1$, $r_1 \ll a$

and assume θ is small. Then the equations of motion become

$$\ddot{r}_1 = (\Omega^2-\omega_0^2)\, r_1 + 2\Omega a\dot{\theta}$$
$$a\ddot{\theta} = \left(\omega_0^2 \frac{\Omega^2}{\Omega^2-\omega_0^2} + \Omega^2\right) a\theta - 2\Omega \dot{r}_1$$

Since $\frac{\omega_0^2}{\Omega^2}$ is very small, we consider only up to leading order in $\frac{\omega_0^2}{\Omega^2}$. Then,

$$\ddot{r}_1 = (\Omega^2-\omega_0^2)\, r_1 + 2\Omega a\dot{\theta}$$
$$a\ddot{\theta} = (\Omega^2+\omega_0^2)\, a\theta - 2\Omega \dot{r}_1$$

We set $r_1 = Ae^{i\omega t}$ and $a\theta = Be^{i\omega t}$, then see if there exist real roots for ω. After the substitution, above equations yield,

$$\begin{pmatrix} \omega^2+(\Omega^2-\omega_0^2) & 2i\Omega\omega \\ -2i\Omega\omega & \omega^2+(\Omega^2+\omega_0^2) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

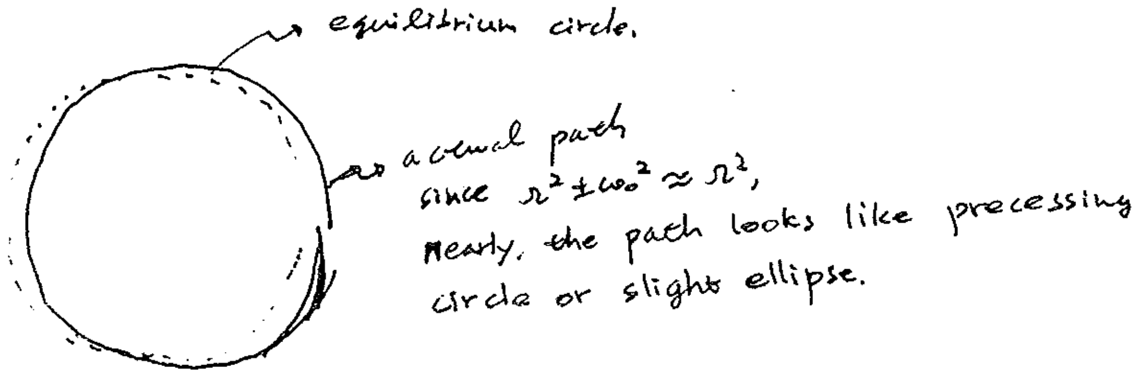
Since the determinant of the above matrix should vanish, we have,

$$(\omega^2+(\Omega^2-\omega_0^2))(\omega^2+(\Omega^2+\omega_0^2)) - 4\Omega^2\omega^2 = (\omega^2-(\Omega^2-\omega_0^2))(\omega^2-(\Omega^2+\omega_0^2)) = 0$$

which shows that ~~two~~ 4 solutions are,

$$\omega = \pm\sqrt{\Omega^2 \pm \omega_0^2}$$

Since these solutions are all real, we can assert that the resulting motion is stable. One thing we should notice here is that without the off-diagonal terms (Coriolis force) there can not be any stable motion. The approximate sketches are shown below.

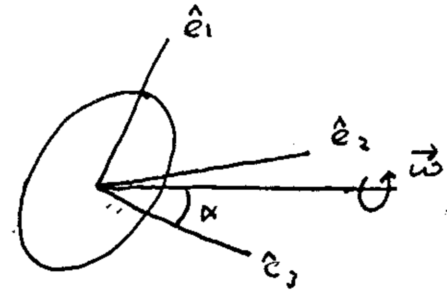


comment. One can use Lagrangian method here. After some approximations, we can show that the Lagrangian can be written as,

$$L = \frac{m}{2}\left\{ \dot{r}_1^2 + a^2\dot{\theta}^2 + 2a\dot{r}_1\theta\Omega - 2ar_1\Omega\dot{\theta} + (\Omega^2-\omega_0^2)\, r_1^2 + ar_0\Omega^2\theta^2 \right\}$$

Try to derive this result!

2. (a) From the symmetry of the disk, it is clear that the two mutually perpendicular symmetry axes should be on the plane of the disk and one additional axis should lie along the line perpendicular to the disk, as shown right. Using perpendicular axis theorem, the principal moments of inertia are



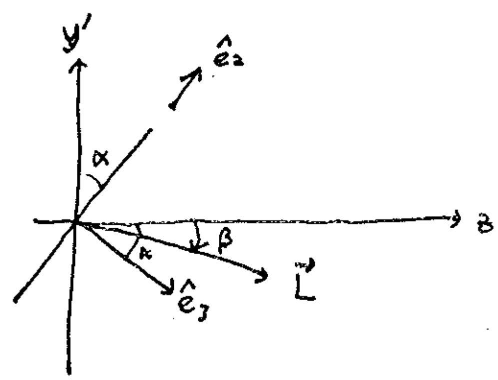
$$I_3 = \frac{1}{2}MR^2, \quad I_1 = I_2 = \frac{1}{4}MR^2$$

Notice that these axes are rotating with the disk. Since $\vec{\omega}$ can be decomposed into

$$\vec{\omega} = \omega(\cos\alpha\,\hat{e}_3 + \sin\alpha\,\hat{e}_2)$$

the angular momentum of the disk is given by

$$\vec{L} = \begin{pmatrix} \frac{1}{4}MR^2 & 0 & 0 \\ 0 & \frac{1}{4}MR^2 & 0 \\ 0 & 0 & \frac{1}{2}MR^2 \end{pmatrix}\begin{pmatrix} 0 \\ \omega\sin\alpha \\ \omega\cos\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{4}MR^2\omega\sin\alpha \\ \frac{1}{2}MR^2\omega\cos\alpha \end{pmatrix}$$



Decomposing $\vec{L}$ into the coordinates shown right which is also rotating, we have

$$\tan\beta = -L_{y'}/L_{z'}$$
$$= \{(\tfrac{1}{2}MR^2\omega\cos\alpha)\sin\alpha - (\tfrac{1}{4}MR^2\omega\sin\alpha)\cos\alpha\}/\{(\tfrac{1}{2}MR^2\omega\cos\alpha)\cos\alpha + (\tfrac{1}{4}MR^2\omega\sin\alpha)\sin\alpha\}$$
$$= \tan\alpha/(2+\tan^2\alpha)$$

(b) Now we consider the fixed frame x-y-z which coincides with x'-y'-z' at t=0.

One easy way of understanding the motions of the pistons is to imagine the disk is fixed and the pistons are rotating in an opposite sense. (suggested by H. Chan) Bearing this fact on mind, it is obvious that,

$$z_1 = y_1\tan\alpha$$
$$y_1 = r\cos\omega t$$

Thus,

$$z_1 = r\cos\omega t\tan\alpha$$

where we refer to the above figure. Consequently,

$$z_2 = r\cos(\omega t + 90^\circ)\tan\alpha = -r\sin\omega t\tan\alpha$$
$$z_3 = r\cos(\omega t + 180^\circ)\tan\alpha = -r\cos\omega t\tan\alpha$$
$$z_4 = r\cos(\omega t + 270^\circ)\tan\alpha = r\sin\omega t\tan\alpha$$

We see that these motions are simple harmonic motion. Above results immediately give

$$\vec{L}_p = \sum_i \vec{r}_i\times\vec{p}_i = m\sum_i \vec{r}_i\times\hat{z}\,\dot{z}_i = 2mr^2\omega\tan\alpha(\cos\omega t\,\hat{y} - \sin\omega t\,\hat{x}) = 2mr^2\omega\tan\alpha\,\hat{y}' \quad (1)$$

The angular momentum of the disk in the direction perpendicular to the shaft is,

$$\vec{L}' = (\tfrac{1}{4}MR^2\omega\sin\alpha)\cos\alpha\,\hat{y}' + (\tfrac{1}{2}MR^2\omega\cos\alpha)(-\sin\alpha\,\hat{y}') = -\tfrac{1}{4}MR^2\omega\sin\alpha\cos\alpha\,\hat{y}' \quad (2)$$

From (1) and (2), we find that if

$$2mr^2\omega\tan\alpha = \frac{1}{4}MR^2\omega\sin\alpha\cos\alpha \Rightarrow m = \frac{MR^2\cos^2\alpha}{8r^2}$$

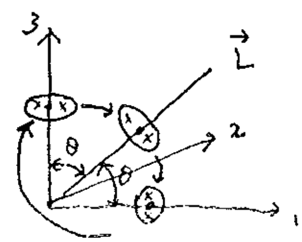
then, the engine will be balanced.

3. (a) After the initial impulse, the angular momentum should be conserved since there is no external torque. The initial angular momentum is

$$\vec{L} = \overleftrightarrow{I}\cdot\vec{\omega} = \frac{1}{4}ma^2(2\omega_3\hat{3} + \omega_1\hat{1})$$

where we used the inertia tensor

$$\overleftrightarrow{I} = \frac{1}{4}ma^2\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}$$



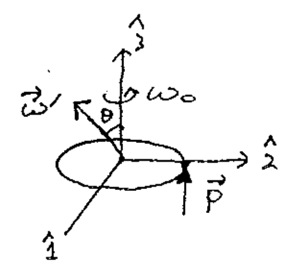
The actual motion would be precession about L as shown right. Thus, the angle between axis 3 and L is less 45 [illegible] same face of the coin is always exposed to an observer. Thus,

$$\frac{2\omega_3}{\omega_1} > 1$$

That means the minimum ratio is 1/2.

(b) After the impulse, the angular momentum becomes

$$\vec{L} = \frac{1}{2}ma^2\omega_0\hat{3} + ap\,\hat{1}$$



due to the angular momentum transfer ($ap\hat{1}$) by meteorite, where we used the coordinate system shown to the right. The above result can be written as

$$\vec{L} = \frac{1}{2}ma^2\omega_0\hat{3} + \frac{1}{4}ma^2\left(\frac{ap}{\frac{1}{4}ma^2}\right)\hat{1} = \overleftrightarrow{I}\cdot\left(\omega_0\hat{3} + \frac{4P}{ma}\hat{1}\right)$$

which shows that the instantaneous angular velocity vector is

$$\vec{\omega}' = \omega_0\hat{3} + \frac{4P}{ma}\hat{1}$$

Consequently the angle between this vector and axis 3 is,

$$\tan\theta = \frac{4P}{ma\omega_0}$$

(c) From component 1 of Euler's equation, we have

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = -C\omega_1 \Rightarrow \dot{\omega}_1 = -\frac{C}{I_1}\omega_1$$

which can be easily integrated to yield

$$\omega_1 = \omega_{1i}\exp\left(-\frac{C}{I_1}t\right)$$; exponential decreacing!

Component 2 and component 3 of Euler's equation are

$$I_2\dot{\omega}_2 + (I_1 - I_2)\omega_1\omega_3 = -C\omega_2$$
$$I_2\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 = -C\omega_3$$

ω_2 (1) + ω_3 (2) gives,

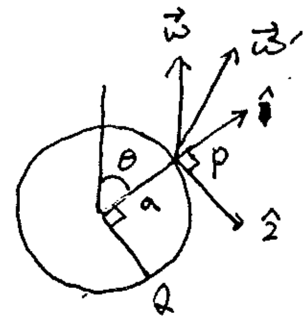
$$\frac{I_2}{2}\frac{d}{dt}(\omega_2^2 + \omega_3^2) = -C(\omega_2^2 + \omega_3^2)$$

which can be integrated to yield

$$\omega_2^2 + \omega_3^2 = (\omega_{2,init}^2 + \omega_{3,init}^2)\,e^{-\frac{2}{I_2}Ct}$$

Thus,

$$\tan\theta = \frac{\sqrt{\omega_2^2 + \omega_3^2}}{\omega_1} = \tan\theta_0\cdot e^{-\left(\frac{1}{I_2} - \frac{1}{I_1}\right)Ct} = \tan\theta_0\exp\left(-\frac{C(I_1 - I_2)}{I_1 I_2}t\right)$$



$\hat{1} \times \hat{2} = \hat{3}$

4. (a) Since there is no impulse-type torque about point P, angular momentum about P should be continuous. Before the fixture, the angular momentum is

$$\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega} = \frac{2}{5} m a^2 \vec{\omega} \qquad (1)$$

where we used the fact that the moment of inertia of sphere is $\frac{2}{5} m a^2$ ~~where~~ and $\vec{\omega}$ is the initial angular momentum. As explained in p.203 of Goldstein, the parallel axis theorem states that the inertia tensor becomes

$$\overleftrightarrow{I}_{new} = \overleftrightarrow{I}_{CM} + M(R^2 \overleftrightarrow{1} - \vec{R}\vec{R})$$

after the coordiante translation along $\vec{R}$. Here $\overleftrightarrow{1}$ denotes identity matrix. If we use 1-2-3 coordinate system as shown above, the inertia tensor for this coordinate system is given by,

$$\overleftrightarrow{I}' = \frac{2}{5} m a^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + m a^2 \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \hat{1}\hat{1} \right) = \frac{m a^2}{5} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

according to the theorem. Thus, final angular momentum can be written as,

$$\vec{L}' = \overleftrightarrow{I}' \cdot \vec{\omega}' = \frac{m a^2}{5} (2\omega_1' \, \hat{1} + 7\omega_2' \, \hat{2} + 7\omega_3' \, \hat{3})$$

Due to the continuity of the angular momentum, (1) and (2) should be the same. If we use the decomposition,

$$\vec{\omega} = \omega \cos\theta \, \hat{1} - \omega \sin\theta \, \hat{2}$$

in 1-2-3 coordinates, we get,

$$\omega_1' = \omega \cos\theta, \quad \omega_2' = -\frac{2}{7} \omega \sin\theta$$

The direction is denoted in the above figure.

(comment. Although the calculation involved is a bit complicated, ~~it~~ solving this problem using "CM motion + rotation about CM" concept is more pedagogical.)

(b) The coordinate of the point Q in 1-2-3 coordiante is

$$(-a, a, 0)$$

Thus, its speed is given by

$$\vec{v}_Q = \begin{vmatrix} \hat{1} & \hat{2} & \hat{3} \\ \omega\cos\theta & -\frac{2}{7}\omega\sin\theta & 0 \\ -a & a & 0 \end{vmatrix} = \omega a \left(\cos\theta - \frac{2}{7}\sin\theta\right)\hat{3} \Rightarrow |\vec{v}_Q| = \omega a \left|\frac{2}{7}\sin\theta - \cos\theta\right|$$

If $\cos\theta > \frac{2}{7}\sin\theta$, the direction is into the page and otherwise, it's out of the page.

(c) The velocity of the CM after the fixture is given by

$$\vec{v}_{CM} = \vec{\omega} \times (-a\hat{1}) = -\frac{2}{7} a\omega \sin\theta \, \hat{3}$$

Thus, the required impulse is

$$\vec{J} = m\Delta\vec{v}_{cm} = -\frac{2}{7} ma\omega\sin\theta\,\hat{j}$$

since the initial velocity of CM is 0. The direction is out of page, which is obvious.

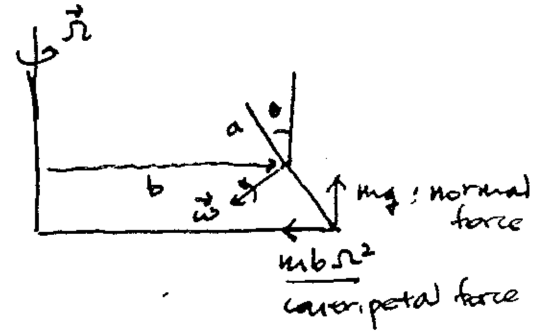
(d) The first component (component 1) of $\vec{\omega}'$ represents the spin of the sphere about axis 1, which requires no external force. The component 2 of $\vec{\omega}'$ represents the rotation of the sphere about axis 2. It needs centripetal force,

$$|\vec{F}| = ma(\omega_2')^2 = \frac{4}{49} ma\omega^2 \sin^2\theta$$

The direction of the force is directly from CM toward P

5. (a) The contact point should momentarily stop. Thus, we should have,

$$(b + a\sin\theta)\Omega = a\omega$$



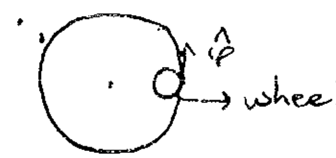
(b) If we measure torque from CM, the normal force and the centripetal force which is responsible for circular motion can contribute to it. Referring to the right figure, and considering the fact that the wheel is in equilibrium vertically, we have

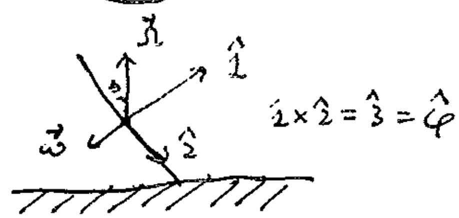
$$\vec{N} = \vec{r} \times \vec{F} = (mab\Omega^2\cos\theta - mga\sin\theta)\hat{\varphi}$$

(c) We observe in lab frame. In this frame, $\vec{\Omega}$ remains constant while $\vec{\omega}$ is rotating with an angular velocity $\vec{\Omega}$. Thus,

$$\frac{d}{dt}\vec{\omega}_t = \frac{d}{dt}(\vec{\Omega} + \vec{\omega}) = \frac{d}{dt}\vec{\omega} = \vec{\Omega} \times \vec{\omega} = -\Omega\omega\cos\theta\,\hat{\varphi}$$

where $\hat{\varphi}$ = angular motion direction in CM motion as shown right. We consider the moment when the axis of wheel looks like the figure right. Then $\vec{\omega}_t$ can be decomposed into





$$\vec{\omega}_t = (\Omega\sin\theta - \omega)\,\hat{1} - \Omega\cos\theta\,\hat{2}$$

and

$$\dot{\vec{\omega}}_t = -\Omega\omega\cos\theta\,\hat{3}$$

It can be easily verified that 1 and 2 components of Euler's equation vanishes without any further effect. Component 3 gives, $(I_1 = ma^2,\ I_2 = I_3 = \frac{1}{2}I_1)$

$$I_3\dot{\omega}_{t_3} + (I_2 - I_1)\,\omega_{t_1}\omega_{t_2} = N_3$$

$$\frac{1}{2}ma^2\{-\Omega\omega\cos\theta - (\Omega\sin\theta - \omega)(-\Omega\cos\theta)\} = mab\Omega^2\cos\theta - mga\sin\theta$$

where we used the result of (b) for N_3. Using the result of (a) to delete ω, we get

$$\Omega^2 = \frac{2g\tan\theta}{4b + a\sin\theta}$$

(d) In this case we consider the same 1-2-3 coordinate system, but we also assume that there is no $\vec{\omega}$ rotation (in coordinates). After the same decomposition as (c), we get

$$\vec{L} = \overset{\leftrightarrow}{I}\cdot\vec{\omega}_t = \frac{1}{2}ma^2\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} \Omega\sin\theta - \omega \\ -\Omega\cos\theta \\ 0 \end{pmatrix}$$

$$= \frac{1}{2}ma^2\,(+2(\Omega\sin\theta - \omega)\,\hat{1} - \Omega\cos\theta\,\hat{2})$$

Thus,

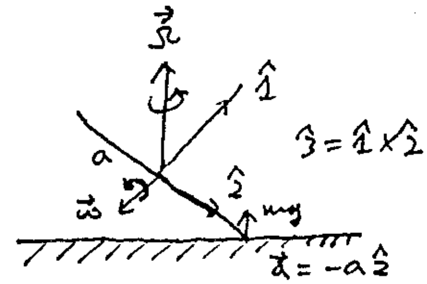
$$\vec{\Omega} \times \vec{L} = \frac{1}{2}ma^2\begin{vmatrix} \hat{1} & \hat{2} & \hat{3} \\ \Omega\sin\theta & -\Omega\cos\theta & 0 \\ 2(\Omega\sin\theta - \omega) & -\Omega\cos\theta & 0 \end{vmatrix} = \frac{1}{2}ma^2\,(+2(\Omega\sin\theta - \omega)\Omega\cos\theta - \Omega^2\sin\theta\cos\theta)\,\hat{3}$$

Using the torque expression derived in (b) and ω in (a), we get

$$\vec{N} = \vec{\Omega} \times \vec{L} \rightarrow \frac{ma^2}{2}\left(2\Omega^2\sin\theta\cos\theta - \frac{2\Omega}{a}\cos\theta\,(b + a\sin\theta)\Omega - \Omega^2\cos\theta\sin\theta\right) = mab\Omega^2\cos\theta - mga\sin\theta$$

$$\therefore\ \Omega^2 = \frac{2g\tan\theta}{4b + a\sin\theta}$$

6. (a) Referring to the right figure, the rolling without slipping condition is



$$\Omega a \sin\theta = a\omega$$

where ω is the spinning frequency. Thus, the total angular velocity can be written as,

$$\vec{\omega}_t = -\Omega\cos\theta\,\hat{2} + (\Omega\sin\theta - \omega)\,\hat{1} = -\Omega\cos\theta\,\hat{2}$$

As expected, the total rotation is the rotation about the instantaneous axis which corresponds to $-\hat{2}$. In lab frame this axis is rotating with angular velocity $\vec{\Omega}$. Thus, the time derivative of the angular momentum

$$\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega}_t = -\frac{1}{4} m a^2 \Omega \cos\theta\,\hat{2}$$

is given by

$$\frac{d\vec{L}}{dt} = -\frac{1}{4} m a^2 \Omega\cos\theta \frac{d}{dt}\hat{2} = -\frac{1}{4} m a^2 \Omega\cos\theta\, \vec{\Omega} \times \hat{2} = -\frac{1}{4} m a^2 \Omega^2 \cos\theta\sin\theta\,\hat{3}$$

Since the only external torque in this case is the torque due to the normal force, we have

$$\frac{d\vec{L}}{dt} = \vec{\lambda} \times (mg\,\hat{z}) = -mga\sin\theta\,\hat{3}$$

which gives

$$\Omega^2 = \frac{4g}{a\cos\theta}$$

(b) As the contact point moves with the angular velocity $\vec{\Omega}$, the figure moves forward by $\Omega\,dt$ during the time dt. Whereas, due to the rolling, the figure should roll back by $\omega\,dt$ during the same time. Thus, the precession frequency of the figure is.

$$\frac{\Omega\,dt - \omega\,dt}{dt} = \Omega(1 - \sin\theta)$$

Thus, during one revolution time T which is defined by $\Omega T = 2\pi$, the face moves

$$\omega T(1 - \sin\theta) = 2\pi(1 - \sin\theta)$$