Abraham, Minkowski and "Hidden" Mechanical Momentum

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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1 Problem

Discuss the relation of "hidden" mechanical momentum to the so-called Abraham-Minkowski debate[1] as to the significance of the expressions $\mathbf{E}\times\mathbf{H}/4\pi c$ and $\mathbf{D}\times\mathbf{B}/4\pi c$ for "electromagnetic" momentum (in Gaussian units), where c is the speed of light in vacuum.

2 Solution

In 1903 Max Abraham noted [3] that the Poynting vector [4], which describes the flow of energy in the electromagnetic field,

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H}, \tag{1}$$

when divided by c^2 has the additional significance of being the density of momentum stored in the electromagnetic field,[2]

$$\mathbf{p}_{\rm EM}^{(\rm A)} = \frac{\mathbf{E}\times\mathbf{H}}{4\pi c} \qquad \text{(Abraham)}. \tag{2}$$

The corresponding total Abraham momentum is,

$$\mathbf{P}_{\rm EM}^{(\rm A)} = \int \frac{\mathbf{E}\times\mathbf{H}}{4\pi c}\, d\text{Vol}. \qquad \text{(Abraham)}. \tag{3}$$

In general, $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ and $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$, where $\mathbf{P}$ and $\mathbf{M}$ are the densities of electric and magnetic polarization, respectively, while Abraham (and Minkowski) considered only linear media, where $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$ when extending their arguments to include a stress tensor.

All linear media except vacuum have both ϵ and μ different from unity. However, most media have $\mu \ll \epsilon$, and most discussion [2] of the Abraham and Minkowski momenta assume that $\mu = 1$. In this idealized case, the Abraham momenta are the same as the momenta,

$$\mathbf{p}_{\rm EM} = \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\,, \qquad \mathbf{P}_{\rm EM} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d\text{Vol}. \tag{4}$$

[1] This debate has been characterized by Ginzburg as a "perpetual problem" [1]. For a lengthy bibliography on this topic, see [2].

[2] J.J. Thomson wrote the electromagnetic momentum as $\mathbf{D}\times\mathbf{H}/4\pi c$ in 1891 [5] and again in 1904 [6]. This form was also used Poincaré in 1900 [7], following Lorentz' convention [8] that the force on electric charge q be written $q(\mathbf{D} + \mathbf{v}/c\times\mathbf{H})$ and that the Poynting vector is $(c/4\pi)\,\mathbf{D}\times\mathbf{H}$. For discussion of these forms, see, for example, [9].

As such, most discussion of the Abraham momentum actually concerns the momentum (4).

In 1908 Hermann Minkowski gave an alternative derivation [10] that the electromagnetic-momentum density is,[3,4]

$$\mathbf{p}_{\mathrm{EM}}^{(\mathrm{M})} = \frac{\mathbf{D} \times \mathbf{B}}{4\pi c} \qquad \text{(Minkowksi)}, \tag{5}$$

and the debate over the merits of these two expressions continues to this day. Minkowski died before adding to the debate, while Abraham published several times on it [15, 16, 17]. For recent reviews, see, for examples [18, 19, 20].

A general consensus has emerged that in dielectric media (with $\mu = 1$) the Abraham momentum (2) is indeed the momentum stored in the electromagnetic field,[5,6] while the Minkowski momentum (5) includes the momentum of matter that interacts with the electromagnetic fields.[7] This suggests that the quantity,

$$\mathbf{P}_{\mathrm{hidden}}^{(\mathrm{A-M})} \stackrel{?}{=} \int \left(\mathbf{p}_{\mathrm{EM}}^{(\mathrm{M})} - \mathbf{p}_{\mathrm{EM}}^{(\mathrm{A})} \right) d\mathrm{Vol} = \int \frac{\mathbf{D} \times \mathbf{B} - \mathbf{E} \times \mathbf{H}}{4\pi c}\, d\mathrm{Vol} = \int \frac{\mathbf{P} \times \mathbf{B} + \mathbf{E} \times \mathbf{M}}{c}\, d\mathrm{Vol}, \tag{6}$$

might have the significance of mechanical momentum "hidden" within the system.[8]

Note that in systems where all materials have permittivity ϵ_0 and permeability μ_0, $\mathbf{p}_{\mathrm{EM}}^{(\mathrm{A})} = \mathbf{p}_{\mathrm{EM}}^{(\mathrm{M})}$, so that according to eq. (6), such systems would contain no "hidden" momentum. However, the concept of "hidden" momentum was developed to characterize unusual features of exactly such systems.

2.1 Shockley's Version of "Hidden" Mechanical Momentum

The term "hidden" mechanical momentum is more commonly associated with a different context, first noted by Shockley [25], in which a system whose center of mass/energy is at rest but for which the electromagnetic field momentum, $\mathbf{P}_{\mathrm{EM}}$, is nonzero. The total

[3]Heaviside gave the form (5) in 1891, p. 108 of [11], and a derivation (1902) essentially that of Minkowski on pp. 146-147 of [12].

[4]See also, for example, sec. 2.1 of [14].

[5]However, this author considers that the nonmechanical, electromagnetic momentum is given by eq. (4), $\mathbf{P}_{\mathrm{EM}} = \int \mathbf{E} \times \mathbf{B}\, d\mathrm{Vol}/4\pi c$.

[6]When dealing with waves of angular frequency ω in a dispersive medium with index $n(\omega)$ it is useful to introduce the quantity $n_g = c/v_g = c\, dk/d\omega = d(\omega n)/d\omega = n + \omega\, dn/d\omega$, which is sometimes called the group-velocity index. This velocity is positive in a passive medium, but can be negative in a gain medium [21]. The emerging consensus [13, 14, 18, 19, 20] is that the Abraham momentum density (for media with $\mu = 1$) corresponds to the momentum of a photon of angular frequency ω in a dielectric medium of group-velocity index n_g being $\hbar\omega/cn_g$, and is sometimes called the kinetic momentum density [13]. The Minkowski momentum density (in a dielectric) corresponds to the momentum $n^2\hbar\omega/cn_g$ of a photon of angular frequency ω, and is sometimes called the pseudomomentum or the quasimomentum. The momentum of a photon most often used in quantum theory is $\hbar\mathbf{k} = n\hbar\omega\,\hat{\mathbf{k}}/c$, which is often called the canonical momentum. In a nondispersive medium with $n > 0$ the Minkowski momentum is the same as the canonical momentum. For discussion of negative-index materials, see [22].

[7]A similar issue arises in acoustics, where one sometimes speaks of the "pseudomomentum" of sound waves, which is analogous to the Minkowski momentum in electrodynamics. See [23] and references therein.

[8]This conjecture was endorsed in [24].

momentum of such a system must be zero [26], so there must be an equal an opposite "hidden" mechanical momentum,[9]

$$\mathbf{P}_{\text{hidden,mech}} = -\mathbf{P}_{\text{EM}}. \tag{7}$$

The Abraham-Minkowski debate over the meaning of $\mathbf{P}_{\text{EM}}$ indicates that the meaning of "hidden momentum" is also ambiguous if it is only defined by eq. (7). A more general definition of "hidden momentum" for any subsystem of a possibly larger system is given in [28],

$$\mathbf{P}_{\text{hidden}} \equiv \mathbf{P} - M\mathbf{v}_{\text{cm}} - \oint_{\text{boundary}} (\mathbf{x} - \mathbf{x}_{\text{cm}})\,(\mathbf{p} - \rho\mathbf{v}_b) \cdot d\mathbf{Area} = -\int \frac{f^0}{c}(\mathbf{x} - \mathbf{x}_{\text{cm}})\, d\text{Vol}, \tag{8}$$

where $\mathbf{P}$ is the total momentum of the subsystem, $M = U/c^2$ is its total "mass", U is its total energy, $\mathbf{x}_{\text{cm}}$ is its center of mass/energy, $\mathbf{v}_{\text{cm}} = d\mathbf{x}_{\text{cm}}/dt$, $\mathbf{p}$ is its momentum density, $\rho = u/c^2$ is its "mass" density, u is its energy density, $\mathbf{v}_b$ is the velocity (field) of its boundary, and,

$$f^\mu = \frac{\partial T^{\mu\nu}}{\partial x_\nu}\,, \tag{9}$$

is the 4-force density exerted by the subsystem on the rest of the system, with $T^{\mu\nu}$ being the stress-energy-momentum 4-tensor of the subsystem.

The definition (8) indicates that the value of the "hidden" momentum depends on the subsystem under consideration. In the classic examples considered by Shockley and others, the entire system was partitioned into two subsystems that occupied that same volume, the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$, and the "mechanical" components of the system; the (relative) permittivity ϵ and the (relative) permeability μ were both unity.

For an isolated, closed system with total stress-energy-momentum tensor $T^{\mu\nu}$, the 4-divergence of the latter is zero, $\partial T^{\mu\nu}/\partial x^\nu = 0$. If the system contains two subsystems A and B which occupy the same volume, then $f^\mu_A = \partial T^{\mu\nu}_A/\partial x^\nu = -\partial T^{\mu\nu}_B/\partial x^\nu = -f^\mu_B$, where f^μ_B is the 4-force density exerted by subsystem A on B. Hence, according to the last form of the definition (8), subsystems A and B have equal and opposite "hidden" momenta. In particular, if the entire system is partitioned into "electromagnetic" and "mechanical" subsystems, we have that,

$$\mathbf{P}_{\text{hidden,EM}} = -\mathbf{P}_{\text{hidden,mech}}. \tag{10}$$

For the electromagnetic subsystem the macroscopic electromagnetic energy-momentum-stress tensor (secs. 32-33 of [29], sec. 12.10B of [30]) is, in a linear medium,[10]

$$\mathsf{T}^{\mu\nu}_{EM} = \left(\begin{array}{c|c} u_{EM} & c\,\mathbf{p}_{EM} \\ \hline c\,\mathbf{p}_{EM} & -T^{ij}_{EM} \end{array} \right), \tag{11}$$

[9]Classical systems with nonzero "hidden" mechanical momentum have moving parts, as noted in [27]. If magnetic charges existed, a system of static electric and magnetic charges would have no "hidden" mechanical momentum, and its total field momentum must also be zero.

[10]For a nonlinear medium, Minkowski's stress tensor [10] is not symmetric, whereas Abraham's [15] is.

where u_{EM} is the electromagnetic field energy density, $\mathbf{p}_{EM}$ is the electromagnetic momentum density, and T^{ij}_{EM} is the 3-dimensional (symmetric) electromagnetic stress tensor. If the tensor (11) is independent of time (as, for example, in the rest frame of a medium with static charge and steady current distributions), then the quantity f^0 in eq. (8) is,

$$f^0 = \frac{\partial T^\nu}{\partial x_\nu} = c\boldsymbol{\nabla} \cdot \mathbf{p}_{\rm EM}, \tag{12}$$

for the electromagnetic subsystem, and hence,

$$\mathbf{P}_{\rm hidden,EM} = -\int \frac{f^0}{c}(\mathbf{x} - \mathbf{x}_{\rm cm})\, d{\rm Vol} = -\int \mathbf{x}(\boldsymbol{\nabla} \cdot \mathbf{p}_{\rm EM})\, d{\rm Vol} + \mathbf{x}_{\rm cm} \int \boldsymbol{\nabla} \cdot \mathbf{p}_{\rm EM}\, d{\rm Vol}. \tag{13}$$

The last integral in eq. (13) transforms into a surface integral at infinity that is negligible for a system with bounded charge and current distributions. The term $-\int \mathbf{x}(\boldsymbol{\nabla} \cdot \mathbf{p}_{\rm EM})\, d{\rm Vol}$ can be integrated by parts, with the resulting surface integral at infinity also being negligible, such that,

$$\mathbf{P}_{\rm hidden,EM} = \int \mathbf{p}_{\rm EM}\, d{\rm Vol} = \mathbf{P}_{\rm EM}. \tag{14}$$

Then, together with eq. (10) we have that,

$$\mathbf{P}_{\rm hidden,EM} = \mathbf{P}_{\rm EM} = -\mathbf{P}_{\rm hidden,mech}. \tag{15}$$

For a "static" case, the "visible" mechanical momentum is zero in the rest frame of the medium, and any mechanical momentum is "hidden". that is,

$$\mathbf{P}_{\rm hidden,EM} = \mathbf{P}_{\rm EM} = -\mathbf{P}_{\rm hidden,mech} = -\mathbf{P}_{\rm mech}, \tag{16}$$

and the total momentum of the system is zero,

$$\mathbf{P}_{\rm total} = \mathbf{P}_{\rm EM} + \mathbf{P}_{\rm mech} = 0. \tag{17}$$

Thus, the definition (8) is consistent with concept of "hidden" momentum as discussed by Shockley and others as explaining how/why the total momentum of an electromechanical system "at rest" is zero.

The result (17) holds for any (valid) form of the electromagnetic field momentum density $\mathbf{p}_{\rm EM}$ and the associated stress-energy-momentum tensor $\mathsf{T}^{\mu\nu}_{EM}$, so the present considerations of "hidden" momentum cannot resolve the Abraham-Minkowski debate. That is, if one accepts either the Abraham or the Minkowski form of the stress-energy-momentum tensor, the definition (8) leads one to a computation of the "hidden" mechanical momentum that is consistent with eqs. (16)-(17).[11] One the other hand, one expects that mechanical momentum, "hidden" or not, is uniquely specifiable for a given system, so that two different values for the "hidden" mechanical momentum cannot both be correct. If, by some argument other than that presented here, the value of the "hidden" mechanical momentum in a medium with electric and magnetic polarization could be determined, this could provide a resolution of the Abraham-Minkowski debate, as least for "static" examples.

In any case, the definition (8) is not consistent with the conjecture (6), as further illustrated in the examples below.

[11] This conclusion appears to differ from that in [36].

2.2 Romer's Example

Following Romer [31],[12] we consider a spherical shell of radius a with free surface-charge density proportional to $\cos\theta$ (with respect to the z-axis), such that the free charge distribution has electric dipole moment $\mathbf{p}_{\rm free}$ and the electric field $\mathbf{E}_{\rm free}$ has the form,

$$\mathbf{E}_{\rm free} = \begin{cases} -\frac{\mathbf{p}_{\rm free}}{a^3} & (r < a), \\ \frac{3(\mathbf{p}_{\rm free}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}_{\rm free}}{r^3} & (r > a), \end{cases} \tag{18}$$

for which the tangential component of $\mathbf{E}_{\rm free}$ is continuous across $r = a$.[13] The system also includes an electrically neutral spherical shell of radius b with free surface currents proportional to $\sin\theta'$ (with respect to the z'-axis), such that the free current distribution has magnetic dipole moment $\mathbf{m}_{\rm free}$, and the magnetic field $\mathbf{B}_{\rm free}$ has the form,

$$\mathbf{B}_{\rm free} = \begin{cases} \frac{2\mathbf{m}_{\rm free}}{b^3} & (r < b), \\ \frac{3(\mathbf{m}_{\rm free}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_{\rm free}}{r^3} & (r > b), \end{cases} \tag{19}$$

for which the normal component of $\mathbf{B}$ is continuous across $r = b$. The system is in vacuum.

We consider the case that $a > b$.

The usual argument in vacuum is that the electromagnetic-field momentum $\mathbf{P}_{\rm EM}$ can be computed as,

$$\begin{aligned} \mathbf{P}_{\rm EM} &= \mathbf{P}_{\rm EM}^{(\rm A)} = \mathbf{P}_{\rm EM}^{(\rm M)} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} \\ &= \int_{r<b} \frac{-\mathbf{p}_{\rm free}\times 2\mathbf{m}_{\rm free}}{4\pi a^3 b^3 c}\, d{\rm Vol} + \int_{b<r<a} \frac{-\mathbf{p}_{\rm free}\times[3(\mathbf{m}_{\rm free}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_{\rm free}]}{4\pi a^3 r^3 c}\, d{\rm Vol} \\ &\quad + \int_{r>a} \frac{[3(\mathbf{p}_{\rm free}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}_{\rm free}]\times[3(\mathbf{m}_{\rm free}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_{\rm free}]}{4\pi r^6 c}\, d{\rm Vol} \\ &= -\frac{2\mathbf{p}_{\rm free}\times\mathbf{m}_{\rm free}}{3a^3 c} - \frac{\mathbf{p}_{\rm free}\times\mathbf{m}_{\rm free}}{a^3 c}\ln\frac{a}{b} + \frac{\mathbf{p}_{\rm free}\times\mathbf{m}_{\rm free}}{a^3 c}\ln\frac{a}{b} - \frac{2\mathbf{p}_{\rm free}\times\mathbf{m}_{\rm free}}{3a^3 c} + \frac{\mathbf{p}_{\rm free}\times\mathbf{m}_{\rm free}}{3a^3 c} \\ &= \frac{\mathbf{m}_{\rm free}\times\mathbf{p}_{\rm free}}{a^3 c}\,. \end{aligned} \tag{20}$$

If $a < b$ the result is $\mathbf{P}_{\rm EM} = \mathbf{m}_{\rm free}\times\mathbf{p}_{\rm free}/b^3 c$.

This system is at rest and must have zero total momentum [26], which leads to eq. (7). Hence, we infer that the system also contains "hidden" mechanical momentum,

$$\mathbf{P}_{\rm hidden,mech} = -\mathbf{P}_{\rm EM} = \frac{\mathbf{p}_{\rm free}\times\mathbf{m}_{\rm free}}{a^3 c}\,, \tag{21}$$

when $a > b$. However, $\mathbf{P}_{\rm EM}^{(\rm A)} - \mathbf{P}_{\rm EM}^{(\rm M)} = 0$ in this case.

[12] A precursor to Romer's example, with $a = b$ and a uniform surface-charge density, was discussed in [32], where it is attributed to J.J. Thomson around 1904. This example has zero field momentum but the field angular momentum is $\mathbf{L} = 2Q\mathbf{M}/2ac$, where Q is the total electric charge and $\mathbf{M}$ is the magnetic moment of the sphere.

[13] Another example of this type has been considered in [33].

Rather than supposing the fields to be due to free charges and currents, we can consider the cases that there exist uniform electric polarization density $\mathbf{P} = 3\mathbf{p}_{\rm free}/4\pi a^3$ for $r < a$ (electret), and/or uniform magnetic polarization density $\mathbf{M} = 3\mathbf{m}_{\rm free}/4\pi b^3$ for $r < b$ (permanent magnet). Then, the fields $\mathbf{E}$ and $\mathbf{B}$ are identical to those of eqs. (18)-(19), so we suppose that the field-only momentum is still given by eq. (20).

So, in addition to Romer's original examples with charge and current densities, we now have three more variants. We consider these only for the case that $a > b$.

2.2.1 Electric Polarization Density and Electric Current Density

Here, $\rho = 0$, $\mathbf{P} \neq 0$, $\mathbf{J} \neq 0$, $\mathbf{M} = 0$, and $\mathbf{B} = \mathbf{H}$.

$$\mathbf{P}_{\rm EM}^{\rm (A)} = \int \frac{\mathbf{E}\times\mathbf{H}}{4\pi c}\, d{\rm Vol} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} = \mathbf{P}_{\rm EM}, \tag{22}$$

$$\begin{aligned}
\mathbf{P}_{\rm EM}^{\rm (M)} &= \mathbf{P}_{\rm EM} + \int_{r<a} \frac{\mathbf{P}\times\mathbf{B}}{c}\, d{\rm Vol} \\
&= \frac{\mathbf{m}_{\rm free}\times\mathbf{p}_{\rm free}}{a^3 c} + \frac{1}{c}\int_{r<b} \frac{3\mathbf{p}_{\rm free}}{4\pi a^3} \times \frac{2\mathbf{m}_{\rm free}}{b^3}\, d{\rm Vol} + \frac{1}{c}\int_{b<r<a} \frac{3\mathbf{p}_{\rm free}}{4\pi a^3} \times \frac{3(\mathbf{m}_{\rm free}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_{\rm free}}{r^3}\, d{\rm Vol} \\
&= -\frac{\mathbf{m}_{\rm free}\times\mathbf{p}_{\rm free}}{a^3 c} = -\mathbf{P}_{\rm EM}.
\end{aligned} \tag{23}$$

$$\mathbf{P}_{\rm EM}^{\rm (A)} - \mathbf{P}_{\rm EM}^{\rm (M)} = 2\mathbf{P}_{\rm EM} = -2\mathbf{P}_{\rm hidden,mech}. \tag{24}$$

2.2.2 Charge Density and Magnetization Density

Here, $\rho \neq 0$, $\mathbf{P} = 0$, $\mathbf{J} = 0$, $\mathbf{M} \neq 0$, and $\mathbf{D} = \mathbf{E}$.

$$\mathbf{P}_{\rm EM}^{\rm (A)} = \mathbf{P}_{\rm EM} - \int_{r<b} \frac{\mathbf{E}\times\mathbf{M}}{c}\, d{\rm Vol} = \frac{\mathbf{m}_{\rm free}\times\mathbf{p}_{\rm free}}{a^3 c} - \frac{1}{c}\int_{r<b} \frac{-\mathbf{p}_{\rm free}}{a^3} \times \frac{3\mathbf{m}_{\rm free}}{4\pi b^3}\, d{\rm Vol} = 0, \tag{25}$$

$$\mathbf{P}_{\rm EM}^{\rm (M)} = \int \frac{\mathbf{D}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} = \mathbf{P}_{\rm EM}, \tag{26}$$

$$\mathbf{P}_{\rm EM}^{\rm (A)} - \mathbf{P}_{\rm EM}^{\rm (M)} = -\mathbf{P}_{\rm EM} = \mathbf{P}_{\rm hidden,mech}. \tag{27}$$

2.2.3 Electric Polarization Density and Magnetization Density

Here, $\rho = 0$, $\mathbf{P} \neq 0$, $\mathbf{J} = 0$, $\mathbf{M} \neq 0$.

$$\mathbf{P}_{\rm EM}^{\rm (A)} = \mathbf{P}_{\rm EM} - \int_{r<b} \frac{\mathbf{E}\times\mathbf{M}}{c}\, d{\rm Vol} = \frac{\mathbf{m}_{\rm free}\times\mathbf{p}_{\rm free}}{a^3 c} - \frac{1}{c}\int_{r<b} \frac{-\mathbf{p}_{\rm free}}{a^3} \times \frac{3\mathbf{m}_{\rm free}}{4\pi b^3}\, d{\rm Vol} = 0, \tag{28}$$

$$\begin{aligned}
\mathbf{P}_{\rm EM}^{\rm (M)} &= \mathbf{P}_{\rm EM} + \int_{r<a} \frac{\mathbf{P}\times\mathbf{B}}{c}\, d{\rm Vol} \\
&= \frac{\mathbf{m}\times\mathbf{p}}{a^3 c} + \frac{1}{c}\int_{r<b} \frac{3\mathbf{p}_{\rm free}}{4\pi a^3} \times \frac{2\mathbf{m}_{\rm free}}{b^3}\, d{\rm Vol} + \frac{1}{c}\int_{b<r<a} \frac{3\mathbf{p}_{\rm free}}{4\pi a^3} \times \frac{3(\mathbf{m}_{\rm free}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_{\rm free}}{r^3}\, d{\rm Vol} \\
&= -\frac{\mathbf{m}_{\rm free}\times\mathbf{p}_{\rm free}}{a^3 c} = -\mathbf{P}_{\rm EM}.
\end{aligned} \tag{29}$$

$$\mathbf{P}_{\rm EM}^{\rm (A)} - \mathbf{P}_{\rm EM}^{\rm (M)} = \mathbf{P}_{\rm EM} = -\mathbf{P}_{\rm hidden,mech}. \tag{30}$$

2.2.4 Romer's Example with Linear Media *(October, 2017)*

Since Abraham and Minkowski considered only linear media, where $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$, it may not be so surprising that their expressions for field momentum are not particularly relevant to examples with permanent electric and magnetic dipoles. So, we now consider Romer's example where the sphere of radius a has constant (relative) permittivity $\epsilon \neq 1$ and the sphere of radius b has constant (relative) permeability $\mu \neq 1$.

For there to be nonzero field momentum, the electric and magnetic fields must both be nonzero, which would not be the case if either the free surface-charge density $\sigma_{\rm free}$ or the free surface-current density $\mathbf{K}_{\rm free}$ were zero. Hence, we suppose that these densities have the forms assumed at the beginning of this section, and that the corresponding dipole moments are $\mathbf{p}_{\rm free}$ and $\mathbf{m}_{\rm free}$. Then, bound charges and currents are induced, with the same spatial dependences as for the free charges and currents, which lead to dipole moments $\mathbf{p}_{\rm bound}$ and $\mathbf{m}_{\rm bound}$, and total moments $\mathbf{p}_{\rm total}$ and $\mathbf{m}_{\rm total}$, that are parallel to the free moments, respectively. The (total) electric and magnetic fields have the same form as eqs. (18)-(19), with $\mathbf{p} \to \mathbf{p}_{\rm total}$ and $\mathbf{m} \to \mathbf{m}_{\rm total}$,

$$\mathbf{E} = \begin{cases} -\frac{\mathbf{p}_{\rm total}}{a^3} & (r < a), \\ \frac{3(\mathbf{p}_{\rm total}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}_{\rm total}}{r^3} & (r > a), \end{cases} \qquad \mathbf{B} = \begin{cases} \frac{2\mathbf{m}_{\rm total}}{b^3} & (r < b), \\ \frac{3(\mathbf{m}_{\rm total}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_{\rm total}}{r^3} & (r > b). \end{cases} \tag{31}$$

To determine the field $\mathbf{D} = \epsilon\mathbf{E}$, we note that the free and total surface-charge densities at $r = a$ have the forms,

$$\sigma_{\rm free} = \frac{3}{4\pi a^3}\mathbf{p}_{\rm free}\cdot\hat{\mathbf{r}}, \qquad \sigma_{\rm total} = \frac{3}{4\pi a^3}\mathbf{p}_{\rm total}\cdot\hat{\mathbf{r}}. \tag{32}$$

Then, the boundary condition for $\mathbf{D}$ at $r = a$ is,

$$\begin{aligned} 4\pi\sigma_{\rm free} &= \frac{3}{a^3}\mathbf{p}_{\rm free}\cdot\hat{\mathbf{r}} = D_r(r = a^+) - D_r(r = a^-) = E_r(r = a^+) - \epsilon E_r(r = a^-) \\ &= \frac{2\mathbf{p}_{\rm total}\cdot\hat{\mathbf{r}}}{a^3} + \frac{\epsilon\mathbf{p}_{\rm total}\cdot\hat{\mathbf{r}}}{a^3}, \end{aligned} \tag{33}$$

from which we infer that,

$$\mathbf{p}_{\rm total} = \frac{3}{2+\epsilon}\mathbf{p}_{\rm free}, \qquad \mathbf{p}_{\rm bound} = \mathbf{p}_{\rm total} - \mathbf{p}_{\rm free} = \frac{\epsilon - 1}{2+\epsilon}\mathbf{p}_{\rm free}. \tag{34}$$

To determine the field $\mathbf{H} = \mathbf{B}/\mu$, we note that the free, bound and total surface-charge densities at $r = b$ all have the forms,

$$\mathbf{K} = \frac{3c}{4\pi b^3}\mathbf{m}\times\hat{\mathbf{r}}. \tag{35}$$

Then, the boundary condition for $\mathbf{H}$ at $r = b$ is,

$$\begin{aligned} \frac{4\pi}{c}\mathbf{K}_{\rm free} &= \frac{3}{b^3}\mathbf{m}_{\rm free}\times\hat{\mathbf{r}} = -\mathbf{H}(r = b^+)\times\hat{\mathbf{r}} + \mathbf{H}(r = b^-)\times\hat{\mathbf{r}} \\ &= -\mathbf{B}(r = b^+)\times\hat{\mathbf{r}} + \frac{\mathbf{B}(r = b^-)\times\hat{\mathbf{r}}}{\mu} \\ &= \frac{\mathbf{m}_{\rm total}\times\hat{\mathbf{r}}}{b^3} + \frac{2\mathbf{m}_{\rm total}\times\hat{\mathbf{r}}}{\mu b^3}, \end{aligned} \tag{36}$$

from which we infer that,

$$\mathbf{m}_{\rm total} = \frac{3\mu}{2+\mu}\mathbf{m}_{\rm free}, \qquad \mathbf{m}_{\rm bound} = \mathbf{m}_{\rm total} - \mathbf{m}_{\rm free} = 2\frac{\mu-1}{2+\mu}\mathbf{m}_{\rm free}. \tag{37}$$

We could also determine the field $\mathbf{H} = \mathbf{B}/\mu$, by supposing that the magnetic fields are due to fictitious magnetic charges, whose free, bound and total surface-charge densities at $r = b$ all have the forms,

$$\tilde\sigma = \frac{3}{4\pi b^3}\mathbf{m}\cdot\hat{\mathbf{r}}. \tag{38}$$

The $\mathbf{B}$ field is determined by the total fictitious charge, while the $\mathbf{H}$ field is determined by the bound fictitious charge (or alternatively by the free currents) Then, the boundary condition for $\mathbf{H}$ at $r = b$ is,

$$\begin{aligned} 4\pi\tilde\sigma_{\rm bound} &= \frac{3}{b^3}\mathbf{m}_{\rm bound}\cdot\hat{\mathbf{r}} = H_r(r=b^+) - H_r(r=b^-) = B_r(r=b^+) - \frac{B_r(r=b^-)}{\mu} \\ &= \frac{2\mathbf{m}_{\rm total}\cdot\hat{\mathbf{r}}}{b^3} - \frac{2\mathbf{m}_{\rm total}\cdot\hat{\mathbf{r}}}{\mu b^3}, \end{aligned} \tag{39}$$

from which we infer that,

$$\mathbf{m}_{\rm bound} = \frac{2(\mu-1)}{3\mu}\mathbf{m}_{\rm total}, \qquad \mathbf{m}_{\rm free} = \mathbf{m}_{\rm total} - \mathbf{m}_{\rm bound} = \frac{2+\mu}{3\mu}\mathbf{m}_{\rm total},$$
$$\mathbf{m}_{\rm total} = \frac{3\mu}{2+\mu}\mathbf{m}_{\rm free}, \qquad \mathbf{m}_{\rm bound} = 2\frac{\mu-1}{2+\mu}\mathbf{m}_{\rm free}. \tag{40}$$

With these expressions for the total moments in terms of the free moments, the electromagnetic fields can now be written as,

$$\mathbf{D} = \begin{cases} -\frac{3\epsilon}{2+\epsilon}\frac{\mathbf{p}_{\rm free}}{a^3}, \\ \frac{3}{2+\epsilon}\frac{3(\mathbf{p}_{\rm free}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}_{\rm free}}{r^3}, \end{cases} \qquad \mathbf{E} = \begin{cases} -\frac{3}{2+\epsilon}\frac{\mathbf{p}_{\rm free}}{a^3} & (r<a), \\ \frac{3}{2+\epsilon}\frac{3(\mathbf{p}_{\rm free}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}_{\rm free}}{r^3} & (r>a), \end{cases} \tag{41}$$

$$\mathbf{H} = \begin{cases} \frac{3}{2+\mu}\frac{2\mathbf{m}_{\rm free}}{b^3}, \\ \frac{3\mu}{2+\mu}\frac{3(\mathbf{m}_{\rm free}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_{\rm free}}{r^3}, \end{cases} \qquad \mathbf{B} = \begin{cases} \frac{3\mu}{2+\mu}\frac{2\mathbf{m}_{\rm free}}{b^3} & (r<b), \\ \frac{3\mu}{2+\mu}\frac{3(\mathbf{m}_{\rm free}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_{\rm free}}{r^3} & (r>b). \end{cases} \tag{42}$$

For $a > b$, the various field momenta are, recalling from eq. (20) that the momentum density $\mathbf{p}_{\rm EM}$ sums to zero in the region $b < r < a$, and that for $r > a$ the momentum $\mathbf{P}_{\rm EM}$ now is $3/(2+\epsilon)3\mu/(2+\mu)$ times that in eq. (20), *i.e.*, $-\mathbf{p}_{\rm free}/(2+\epsilon)a^3c \times 3\mu\mathbf{m}_{\rm free}/(2+\mu)$,

$$\begin{aligned} \mathbf{P}_{\rm EM} &= \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\,d{\rm Vol} = \int_{r<b} \frac{-3\mathbf{p}_{\rm free}\times 6\mu\mathbf{m}_{\rm free}}{4\pi(2+\epsilon)(2+\mu)a^3b^3c}\,d{\rm Vol} - \frac{\mathbf{p}_{\rm free}}{(2+\epsilon)a^3c}\times\frac{3\mu\mathbf{m}_{\rm free}}{2+\mu} \\ &= -\frac{2\mathbf{p}_{\rm free}}{(2+\epsilon)a^3c}\times\frac{3\mu\mathbf{m}_{\rm free}}{2+\mu} - \frac{\mathbf{p}_{\rm free}}{(2+\epsilon)a^3c}\times\frac{3\mu\mathbf{m}_{\rm free}}{2+\mu} = -\frac{3\mathbf{p}_{\rm free}}{(2+\epsilon)a^3c}\times\frac{3\mu\mathbf{m}_{\rm free}}{2+\mu} \\ &= \frac{\mathbf{E}(r<a)\times\mathbf{m}_{\rm total}}{c}, \end{aligned} \tag{43}$$

$$\mathbf{P}_{\rm EM}^{\rm (A)} = \int \frac{\mathbf{E}\times\mathbf{H}}{4\pi c}\,d{\rm Vol} = \int_{r<b} \frac{-3\mathbf{p}_{\rm free}\times 6\mathbf{m}_{\rm free}}{4\pi(2+\epsilon)(2+\mu)a^3b^3c}\,d{\rm Vol} - \frac{\mathbf{p}_{\rm free}}{(2+\epsilon)a^3c}\times\frac{3\mu\mathbf{m}_{\rm free}}{2+\mu}$$

$$= -\frac{2\mathbf{p}_{\rm free}}{(2+\epsilon)a^3 c} \times \frac{3\mathbf{m}_{\rm free}}{2+\mu} - \frac{\mathbf{p}_{\rm free}}{(2+\epsilon)a^3 c} \times \frac{3\mu\mathbf{m}_{\rm free}}{2+\mu} = -\frac{3\mathbf{p}_{\rm free}}{(2+\epsilon)a^3 c} \times \mathbf{m}_{\rm free}, \tag{44}$$

$$\mathbf{P}_{\rm EM}^{(\rm M)} = \int \frac{\mathbf{D}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} = \int_{r<b} \frac{-3\epsilon\mathbf{p}_{\rm free} \times 6\mu\mathbf{m}_{\rm free}}{4\pi(2+\epsilon)(2+\mu)a^3 b^3 c}\, d{\rm Vol} - \frac{\mathbf{p}_{\rm free}}{(2+\epsilon)a^3 c} \times \frac{3\mu\mathbf{m}_{\rm free}}{2+\mu}$$

$$= -\frac{2\epsilon\mathbf{p}_{\rm free}}{(2+\epsilon)a^3 c} \times \frac{3\mu\mathbf{m}_{\rm free}}{2+\mu} - \frac{\mathbf{p}_{\rm free}}{(2+\epsilon)a^3 c} \times \frac{3\mu\mathbf{m}_{\rm free}}{2+\mu} = -\frac{(2\epsilon+1)\mathbf{p}_{\rm free}}{(2+\epsilon)a^3 c} \times \frac{3\mu\mathbf{m}_{\rm free}}{2+\mu}, \tag{45}$$

$$\mathbf{P}_{\rm EM}^{(\rm A)} - \mathbf{P}_{\rm EM}^{(\rm M)} = (\epsilon\mu - 1)\frac{3\mathbf{p}_{\rm free}}{(2+\epsilon)a^3 c} \times \frac{2\mathbf{m}_{\rm free}}{2+\mu}\,. \tag{46}$$

For completeness, note that,

$$\mathbf{P}_{\rm EM}^{(\rm DH)} \equiv \int \frac{\mathbf{D}\times\mathbf{H}}{4\pi c}\, d{\rm Vol} = \int_{r<b} \frac{-3\epsilon\mathbf{p}_{\rm free} \times 6\mathbf{m}_{\rm free}}{4\pi(2+\epsilon)(2+\mu)a^3 b^3 c}\, d{\rm Vol} - \frac{\mathbf{p}_{\rm free}}{(2+\epsilon)a^3 c} \times \frac{3\mu\mathbf{m}_{\rm free}}{2+\mu}$$

$$= -\frac{2\epsilon\mathbf{p}_{\rm free}}{(2+\epsilon)a^3 c} \times \frac{3\mathbf{m}_{\rm free}}{2+\mu} - \frac{\mathbf{p}_{\rm free}}{(2+\epsilon)a^3 c} \times \frac{3\mu\mathbf{m}_{\rm free}}{2+\mu} = -\frac{(6\epsilon+3\mu)\mathbf{p}_{\rm free}}{(2+\epsilon)a^3 c} \times \frac{\mathbf{m}_{\rm free}}{2+\mu} \tag{47}$$

Hidden Momentum

Romer's example with linear media contains "hidden" mechanical momentum associated both with the free and bound current densities.

As discussed, for example, in [34, 35], the "hidden" momentum associated with a magnetic moment $\mathbf{m}$ due to electrical currents is given by $\mathbf{m}\times\mathbf{E}/c$ if the electric field is uniform over the currents. In the present example, the electric field on the both the free and bound electrical currents (at $r \le b$) is $\mathbf{E} = -3\mathbf{p}_{\rm free}/(2+\epsilon)a^3$, so the corresponding "hidden" mechanical momenta are,

$$\mathbf{P}_{\rm hidden,free} = \int \frac{\mathbf{m}_{\rm free}\times\mathbf{E}}{c}\, d{\rm Vol} = \mathbf{m}_{\rm free} \times \frac{-3\mathbf{p}_{\rm free}}{(2+\epsilon)a^3 c}\,, \tag{48}$$

$$\mathbf{P}_{\rm hidden,bound} = \int \frac{\mathbf{m}_{\rm bound}\times\mathbf{E}}{c}\, d{\rm Vol} = \mathbf{m}_{\rm bound} \times \frac{-3\mathbf{p}_{\rm free}}{(2+\epsilon)a^3 c}$$

$$= \frac{2(\mu-1)\mathbf{m}_{\rm free}}{\mu+2}\mathbf{m}_{\rm free} \times \frac{-3\mathbf{p}_{\rm free}}{(2+\epsilon)a^3 c}\,. \tag{49}$$

The total "hidden" mechanical momentum is,

$$\mathbf{P}_{\rm hidden,total} = \mathbf{P}_{\rm hidden,bound} + \mathbf{P}_{\rm hidden,bound} = \frac{3\mu\mathbf{m}_{\rm free}}{2+\mu} \times \frac{-3\mathbf{p}_{\rm free}}{(2+\epsilon)a^3 c}$$

$$= \frac{\mathbf{m}_{\rm total}\times\mathbf{E}(r<b)}{c} = -\mathbf{P}_{\rm EM} = -\int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol}. \tag{50}$$

The "hidden" mechanical momentum is equal and opposite to the field momentum based only on $\mathbf{E}$ and $\mathbf{B}$.

In this static example, the only mechanical momentum is the "hidden" momentum (50), and the total momentum is zero (in the frame in which the two spheres are at rest).

Note that bound "hidden" momentum (49) can exist for $\epsilon = 1$, but not for $\mu = 1$, since bound "hidden" momentum is associated with bound electric currents (which exist in linear media only for $\mu \ne 1$).

Neither the Abraham momentum (44) nor the Minkowski momentum (45) are equal and opposite to the (hidden) mechanical momentum. Hence, both of these momentum are a combination of field momentum (based on $\mathbf{E}\times\mathbf{B}$) and mechanical momentum. As such, they do not have a crisp physical interpretation, as least for examples that contain "hidden" mechanical momentum.

Note that the Abraham momentum can be written as,

$$
\begin{aligned}
\mathbf{P}_{\rm EM}^{(\rm A)} &= \int \frac{\mathbf{E}\times\mathbf{H}}{4\pi c}\, d{\rm Vol} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} + \int \frac{\mathbf{E}\times\mathbf{M}}{c}\, d{\rm Vol} = \mathbf{P}_{\rm EM} - \mathbf{P}_{\rm hidden,bound} \\
&= -\mathbf{P}_{\rm hidden,free} - 2\mathbf{P}_{\rm hidden,bound}. \qquad (51)
\end{aligned}
$$

For the present example with an idealized, linear dielectric medium, with $\epsilon \neq 1$ but $\mu = 1$, $\mathbf{P}_{\rm hidden,bound} = 0$, and $\mathbf{P}_{\rm EM}^{(\rm A)} = \mathbf{P}_{\rm EM} = -\mathbf{P}_{\rm hidden,free}$.

In the case of electromagnetic waves, rather than static fields, in and around media, it can be that there is no "hidden" mechanical momentum, and the Abraham and/or Minkowski momentum has more "overt" physical significance.

All the various results of this section confirm that eq. (6) is not, in general, a suitable expression for the hidden mechanical momentum (21) of the system.

Still, it may be useful to consider another example.

2.3 Hnizdo's Example: Uniformly Magnetized Toroid

V. Hnizdo notes that a toroid with uniform azimuthal magnetization $\mathbf{M} = M\,\hat{\boldsymbol{\phi}}$ (in cylindrical coordinates (ϱ, ϕ, z)) has $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} = 0$ everywhere and $\mathbf{B} = 4\pi\mathbf{M}$ inside the toroid.[14] If this toroid is subject to a radial electric field $\mathbf{E}$, then $\mathbf{E}\times\mathbf{B}$ is nonzero inside the toroid and parallel to z-axis while $\mathbf{E}\times\mathbf{H}$ is zero everywhere. If we accept that this example contains "hidden" mechanical momentum, then the Abraham momentum cannot be the "correct" one (in the sense of being equal and opposite to the "hidden" mechanical momentum such that the total momentum of the system, which is "at rest", is zero).

A body with uniform magnetization is a collection of identical Ampèrian magnetic dipoles.[15] As noted in [38], an Ampèrian magnetic dipole $\mathbf{m}$ in an external electric field $\mathbf{E}$ is a system "at rest" with nonzero field momentum $\mathbf{E}\times\mathbf{m}/c$ (computed from $\int \mathbf{E}\times\mathbf{B}\, d{\rm Vol}/4\pi c$), and so this system must contain "hidden" mechanical momentum in the direction of $\mathbf{m}\times\mathbf{E}$. We infer that a collection such as Hnizdo's example of identical magnetic dipoles, all in an external electric field such that $\mathbf{m}\times\mathbf{E}$ is always in the same direction, also contains nonzero "hidden" mechanical momentum.

Hence, it appears that neither the Abraham nor the Minkowski momenta are the "correct" field momenta in the sense of being equal and opposite to the "hidden" mechanical momentum of such systems "at rest" that possess this.

[14] For uniform magnetization $\mathbf{M}$, the volume density $\rho_m = -\boldsymbol{\nabla}\cdot\mathbf{M}$ of "fictitious" magnetic charges is zero. For a toroid with azimuthal magnetization, the surface density $\sigma_m = \mathbf{M}\cdot\hat{\mathbf{n}}$ of "fictitious" magnetic charges is also zero (where $\hat{\mathbf{n}}$ is normal to the surface). Then, there are no sources for the $\mathbf{H}$-field, which is hence zero.

[15] Experimental evidence that magnetization is due to Ampèrian magnetic dipoles is reviewed in [37].

It seems to this author that only the form $\int \mathbf{E}\times\mathbf{B}\, d\mathrm{Vol}/4\pi c$ should be called the electromagnetic field momentum, and that both the Abraham and Minkowski momenta represent combinations of electromagnetic field momentum with mechanical momentum associated with electric and/or magnetic polarization.

2.3.1 The Toroid is Also an Electret

We record two examples of radial electric fields as discussed above. First, we consider the toroid also to be an electret with uniform radial electric polarization $\mathbf{P} = P\,\hat{\boldsymbol{\varrho}}$.

We suppose the toroid has a rectangular cross section $a < \varrho < b$, $|z| < l$ and that the toroid is long, $l \gg b$. The surface density of bound electric charge is then,

$$\sigma_e(\varrho = a) = -P, \qquad \sigma_e(\varrho = b) = P. \tag{52}$$

The electric field inside the long toroid, at locations not close to its ends, is approximately radial, and follows from Gauss' law as,

$$E_r(a < \varrho < b) = -4\pi P \frac{a}{\varrho}\,, \tag{53}$$

and hence the interior $\mathbf{D}$-field is,

$$D_r(a < \varrho < b) = E_r + 4\pi P_r = 4\pi P \left(1 - \frac{a}{\varrho}\right)\,. \tag{54}$$

2.3.2 The Toroid is a Cylindrical Capacitor

Alternatively, the surfaces $\varrho = a$ and b could be conductors, forming a cylindrical capacitor, and the medium of the toroid could be a (linear) dielectric with (relative) permittivity ϵ such that $\mathbf{D} = \epsilon\mathbf{E}$. These conductors can support densities of free charge given by eq. (52) such that the electric field inside the toroid is again given by eq. (53). In this case, the $\mathbf{D}$-field inside the toroid is given by,

$$D_r(a < \varrho < b) = \epsilon E_r = -4\pi P \frac{a}{\varrho}\,. \tag{55}$$

Forces and Momenta If the Magnetization Goes to Zero

This variant permits a noteworthy phenomenon if the electrically charged conductors are physically isolated from the magnetized toroid, and that magnetization goes to zero at some time.

When the magnetization is $\mathbf{M} = M\,\hat{\boldsymbol{\phi}}$, the (field only) electromagnetic momentum per unit length in z associated with the (long) system is,

$$\mathbf{P}_{\mathrm{EM}} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d\mathrm{Area} \approx \int_a^b \frac{-(4\pi Pa/\varrho)\,\hat{\mathbf{r}}\times 4\pi M\,\hat{\boldsymbol{\phi}}}{4\pi c}\, 2\pi\varrho\, d\varrho = -\frac{8\pi^2 MPa(b-a)}{c}\,\hat{\mathbf{z}}, \tag{56}$$

where $\mp P$ is the surface density of electric charge on the conductors (at $\varrho = a^-$ and b^+). At this time, there is an equal and opposite "hidden" mechanical momentum per unit length,

$$\mathbf{P}_{\mathrm{hidden,mech}} = -\mathbf{P}_{\mathrm{EM}} = \int \frac{\mathbf{B}\times\mathbf{E}}{4\pi c}\, d\mathrm{Area} = \int \frac{\mathbf{M}\times\mathbf{E}}{c}\, d\mathrm{Area}, \tag{57}$$

such that the total momentum of the system is zero. Equation (57) is consistent with the earlier comment that the "hidden" mechanical momentum of a magnetic dipole $\mathbf{m}$ in an electric field $\mathbf{E}$ is $\mathbf{m} \times \mathbf{E}/c$ [38], such that the volume density of "hidden" mechanical momentum is $\mathbf{M} \times \mathbf{E}/c$ inside the magnetization density $\mathbf{M}$.

If the magnetization drops to zero at some later time, a transient electric field is induced when the $\mathbf{B}$-field is changing, which electric field is conveniently deduced from the changing vector potential,

$$\mathbf{E}_{\rm ind} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\,. \tag{58}$$

We ignore the small additional magnetic field and vector potential associated with the transient currents, and approximate the vector potential as the quasistatic value related to the instantaneous magnetization $\mathbf{M}(t)$, still supposed to be spatially uniform as it drops to zero. This vector potential has only a z-component, related by,

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \qquad B_\phi = -\frac{\partial A_z}{\partial \varrho} = \begin{cases} 0 & (\varrho < a), \\ 4\pi M & (a < \varrho < b), \\ 0 & (\varrho > b). \end{cases} \tag{59}$$

We take the vector potential to be zero at $\varrho = \infty$, such that,

$$A_z = 4\pi M \begin{cases} b - a & (\varrho < a), \\ b - \varrho & (a < \varrho < b), \\ 0 & (\varrho > b). \end{cases} \tag{60}$$

When the magnetization drops at rate $\dot{M} < 0$, the induced electric field is,

$$E_{{\rm ind},z} = -\frac{1}{c}\frac{\partial A_z}{\partial t} = -\frac{4\pi \dot{M}}{c} \begin{cases} b - a & (\varrho < a), \\ b - \varrho & (a < \varrho < b), \\ 0 & (\varrho > b). \end{cases} \tag{61}$$

There is no induced electric field at the outer conductor ($\varrho = b$), while the field on the inner conductor ($\varrho = a$) is in the $+z$ direction when $\dot{M} < 0$. If the inner and outer conductors are free to move separately, only the inner conductor will move, and in the $-z$ direction as its surface charge density $-P$ is negative. The force per unit length on the inner conductor is,

$$F_z = 2\pi a(-P) E_{{\rm ind},z}(\varrho = a^-) = \frac{8\pi^2 \dot{M} P a(b - a)}{c}\,, \tag{62}$$

so the (negatively charged) inner conductor takes on mechanical momentum per unit length,

$$P_{{\rm inner\ conductor},\,z} = \int F_z\, dt = -\frac{8\pi^2 M P a(b - a)}{c}\,, \tag{63}$$

when the magnetization drops from M to zero. If the conductors are free to move, the inner conductor has final velocity in the $-z$ direction, while the outer conductor remains at rest.

Meanwhile, both the initial electromagnetic field momentum and the "hidden" mechanical momentum have dropped to zero, so if only the inner conductor has final momentum, there would be a violation of momentum conservation.

It remains to consider forces on the magnetized toroid while the magnetization changes.

The volume-force density $\mathbf{f}_M$ on the (Ampèrian) magnetization $\mathbf{M}$ is given, for example, in eq. (18) of [37],

$$\mathbf{f}_M = (\mathbf{M}\cdot\boldsymbol{\nabla})\mathbf{B} + \mathbf{M}\times\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + c\mathbf{M}\times(\boldsymbol{\nabla}\times\mathbf{M}). \tag{64}$$

This force density acts to change the mechanical momentum density of the toroid, which consists of the "overt" momentum density $\mathbf{p}_{\rm overt} = \rho_{\rm mass}\mathbf{v}$ as well as the density $\mathbf{p}_{\rm hidden,mech} = \mathbf{M}\times\mathbf{E}/c$ of "hidden" mechanical momentum.[16] In the present example, the first and last terms in eq. (63) vanish, so the mechanical momentum of the toroid varies according to,

$$\mathbf{f}_M = \mathbf{M}\times\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{d}{dt}\left(\mathbf{p}_{\rm toroid,overt} + \frac{\mathbf{M}\times\mathbf{E}}{c}\right), \qquad \frac{d\mathbf{p}_{\rm toroid,overt}}{dt} = -\frac{\partial \mathbf{M}}{\partial t}\times\frac{\mathbf{E}}{c}\,. \tag{65}$$

As the magnetization $\mathbf{M}$ drops to zero, the overt mechanical momentum of the toroid changes, until finally the overt mechanical momentum per unit length of the toroid is,

$$\begin{aligned}\mathbf{P}_{\rm toroid,overt} &= \int d{\rm Area}\int\frac{d\mathbf{p}_{\rm overt}}{dt}\,dt = \int \mathbf{M}_{\rm initial}\times\frac{\mathbf{E}}{c}\,d{\rm Area} = \int_a^b M\,\hat{\boldsymbol{\phi}}\times -\frac{4\pi Pa\,\hat{\mathbf{r}}}{c\varrho}\,2\pi\varrho\,d\varrho \\ &= \frac{8\pi^2 MPa(b-a)}{c}\,\hat{\mathbf{z}} = -\mathbf{P}_{\rm inner\ conductor}\,.\end{aligned} \tag{66}$$

Hence, the final, total momentum, $\mathbf{P}_{\rm inner\ conductor} + \mathbf{P}_{\rm toroid,overt}$, of the system is zero, as expected.

The force density (64) is radial in the present example, so its volume integral vanishes, with the implication that the mechanical momentum of the toroid remains constant as the magnetization drops to zero. If the toroid is free to move, its final velocity is in the $+z$ direction. Hence, the appearance of the final, "overt" mechanical momentum of the toroid can be regarded as evidence of the initial, "hidden" mechanical momentum. This suggests that a laboratory demonstration of the present example would be useful in convincing skeptics of the existence of "hidden" mechanical momentum.

So, we consider some numbers for a possible demonstration experiment. We take $a \approx b \approx 1$ cm. A practical voltage across the 1-cm cylindrical capacitor might be around 1000 V = 3.3 statvolt. This voltage is also given by $V = \int_a^b E\,d\varrho \approx 4\pi P\ln 2 \approx 3P$, so the surface charge density is $P \approx 1$ statCoulomb/cm^2. The magnetic field inside a strong permanent magnet is about $B \approx 1$ T = 10,000 G = $4\pi M$, so $M \approx 1000$ in Gaussian units. Then, the final, overt momentum would be $\approx 8\pi^2 MP/c \approx 10^{-7}$ g-cm/s, and for a toroid with mass of a few grams, its final velocity would be $\approx 10^{-7}$ cm/s, too small to be observable in a simple demonstration.

[16] This argument was made implicitly by Shockley [25], and explictly on p. 53 of [39].

Forces and Momenta If the Electric Field Goes to Zero

If the electric field of the cylindrical capacitor drops to zero but the magnetization of the toroid remains constant, then according to eq. (65) there is no change in the "overt" mechanical momentum of the toroid, which therefore remains at rest as its "hidden" mechanical momentum drops to zero.

Meanwhile, the charges on the conductors of the cylindrical capacitor experience no axial electric field as the radial electric field drops to zero, so the conductors remain at rest.

In the final state, with zero electric field and nonzero **B** and **M** inside the toroid, there is no mechanical momentum anywhere, "hidden" or "overt", and the electromagnetic field momentum is also zero.[17]

2.3.3 Comments

The Minkowski momenta, $\int \mathbf{D} \times \mathbf{B}\, d\mathrm{Vol}/4\pi c$, for these two cases have the opposite signs, yet the microscopic electromagnetic field momenta of the magnetic dipoles is the same in both cases. This reinforces the conclusion of sec. 2.2 that the Minkowski momentum is not the "correct" one to be equal and opposite to the "hidden" mechanical momentum (which is the same in both cases).

2.3.4 Appendix: Toroid with Radial Magnetization and Azimuthal Polarization

For possible amusement we consider a toroid with geometry as in sec. 2.3.1 but with radial magnetization, $\mathbf{M} = M\,\hat{\boldsymbol{\varrho}}$, and azimuthal polarization, $\mathbf{P} = P\,\hat{\boldsymbol{\phi}}$. In this case there is no free or bound electric charge densities, either in the bulk or on the surface of the toroid. Hence, there are no sources of the electric field and $\mathbf{E} = 0$ everywhere. Inside the toroid the displacement field in nonzero, $\mathbf{D}_{\mathrm{interior}} = \mathbf{E} + 4\pi\mathbf{P} = 4\pi P\,\hat{\boldsymbol{\phi}}$.

There is no bulk density of "fictitious" magnetic charge associated with the radial magnetization, but there are "fictitious" surface magnetic charge densities given by,

$$\sigma_m(\varrho = a) = -M, \qquad \sigma_m(\varrho = b) = M. \tag{67}$$

The **H**-field inside the long toroid, at locations not close to its ends, is approximately radial, and follows from Gauss' law (here $\boldsymbol{\nabla} \cdot \mathbf{H} = 4\pi\rho_m$) as,

$$H_r(a < \varrho < b) = -4\pi M \frac{a}{\varrho}\,, \tag{68}$$

and hence the interior **B**-field is,

$$B_r(a < \varrho < b) = H_r + 4\pi M_r = 4\pi M \left(1 - \frac{a}{\varrho}\right)\,. \tag{69}$$

[17] An example of this type was considered by J.J. Thomson on p. 348 of [6]; see also [40]. For examples with "hidden" mechanical momentum in systems with an electric dipole in a magnetic fields due to current loops, all "at rest", such that various equal and opposite "overt" mechanical momenta arise as the electromagnetic fields are brought to zero in various ways, see [41], especially secs. IV and V.

This case is the dual of that described in sec. 2.3.1, with the duality relations $\mathbf{M} \leftrightarrow \mathbf{P}$, $\mathbf{E} \leftrightarrow \mathbf{H}$ and $\mathbf{D} \leftrightarrow \mathbf{B}$.

If we accept that "hidden" mechanical momentum is due to the "external" electric field $\mathbf{E}$ on the Ampèrian currents associated with the magnetization $\mathbf{M}$, then there in no "hidden" mechanical momentum in the example of this Appendix. This is consistent with the "field only" momentum $\int \mathbf{E} \times \mathbf{B}\, d\mathrm{vol}/4\pi c$ being zero. Of course, the Abraham momentum is also zero in the case, but the Minkowski momentum is nonzero and in the $-z$ direction.

2.4 Hidden Momentum and a Wave in a Linear Medium

This section was inspired by [24] (October, 2017).

A feature of the definition (8) is that the electromagnetic momentum (4) of a (source-free) electromagnetic wave in vacuum has no "hidden" momentum.

For example, a plane electromagnetic wave,

$$\mathbf{E} = E_0 \cos(kz - \omega t)\, \hat{\mathbf{x}}, \qquad \mathbf{B} = E_0 \cos(kz - \omega t)\, \hat{\mathbf{y}}, \tag{70}$$

propagates with velocity $\mathbf{v} = (\omega/k)\, \hat{\mathbf{z}} = c\, \hat{\mathbf{z}}$, and has energy, effective mass, and field momentum densities,

$$u_{\mathrm{EM}} = \frac{E^2 + B^2}{8\pi} = \frac{E_0^2 \cos^2(kz - \omega t)}{4\pi}, \qquad \rho_{\mathrm{eff,EM}} = \frac{u_{\mathrm{EM}}}{c^2}, \tag{71}$$

$$\mathbf{p}_{\mathrm{EM}} = \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} = \frac{E_0^2 \cos^2(kz - \omega t)}{4\pi c}\, \hat{\mathbf{z}} = m_{\mathrm{eff,EM}}\, c\, \hat{\mathbf{z}}, \tag{72}$$

such that the density of (electromagnetic) "hidden" momentum is,

$$\mathbf{p}_{\mathrm{hidden,EM}} = \mathbf{p}_{\mathrm{EM}} - \rho_{\mathrm{eff}} \mathbf{v} = 0. \tag{73}$$

Turning to the case of an electromagnetic wave propagating inside a linear medium that is at rest, we note that the definition (8) of "hidden" momentum applies to an entire system, or to a subsystem thereof.

In most considerations of "hidden" momentum, one emphasizes the "mechanical" subsystem, which is considered distinct from the (macroscopic) "electromagnetic field" subsystem. This distinction is reasonably clear in examples where all material has unit (relative) permittivity and permeability, as has been the case in essentially all past discussion of "hidden" momentum, which also have been restricted to (quasi)static examples where electromagnetic waves were neglected. In such quasistatic examples, the total "hidden" momentum is always zero, but there can be nonzero "hidden" mechanical momentum that is equal and opposite to "hidden" electromagnetic field momentum (see sec. 4.1.4 of [28]).[18]

However, in examples with waves, where the total momentum of the system can be nonzero, it is not clear that the total "hidden" momentum must be zero.

[18] Nonzero total electromagnetic field momentum in a static example is always "hidden" according to definition (8), in that the velocity $\mathbf{v}_{\mathrm{EM}}$ of the center of electromagnetic field energy is zero, and $\mathbf{P}_{\mathrm{hidden,EM}} = \mathbf{P}_{\mathrm{EM}} - U_{\mathrm{EM}} \mathbf{v}_{\mathrm{EM}}/c^2 = \mathbf{P}_{\mathrm{EM}}$.

We now consider a plane electromagnetic wave in a linear medium, with index of refraction $n = \sqrt{\epsilon\mu}$, such that the (phase) velocity of propagation is $\mathbf{v} = (\omega/k)\,\hat{\mathbf{z}} = c\,\hat{\mathbf{z}}/n$,

$$\mathbf{E} = E_0 \cos(kz - \omega t)\,\hat{\mathbf{x}} = \frac{\mathbf{D}}{\epsilon}\,, \qquad \mathbf{B} = nE_0 \cos(kz - \omega t)\,\hat{\mathbf{y}} = \mu\mathbf{H}. \tag{74}$$

If we consider the "electromagnetic field" subsystem to consist only of the (macroscopic) fields $\mathbf{E}$ and $\mathbf{B}$, then the energy, effective mass, and field momentum densities are,

$$u_{\rm EM} = \frac{E^2 + B^2}{8\pi} = \frac{E_0^2(1+n^2)\cos^2(kz-\omega t)}{8\pi}\,, \qquad \varrho_{\rm eff,EM} = \frac{u_{\rm EM}}{c^2}\,, \tag{75}$$

$$\mathbf{p}_{\rm EM} = \frac{\mathbf{E}\times\mathbf{B}}{4\pi c} = \frac{2nE_0^2\cos^2(kz-\omega t)}{8\pi c}\,\hat{\mathbf{z}}, \tag{76}$$

such that the density of "hidden" momentum is,

$$\begin{aligned}\mathbf{p}_{\rm hidden,EM} &= \mathbf{p}_{\rm EM} - \varrho_{\rm eff,EM}\mathbf{v} = \frac{E_0^2\cos^2(kz-\omega t)}{8\pi c}\left(2n - \frac{1+n^2}{n}\right)\hat{\mathbf{z}} \\ &= \frac{n^2-1}{n}\frac{E_0^2\cos^2(kz-\omega t)}{8\pi c}\,\hat{\mathbf{z}} = \frac{\epsilon\mu - 1}{\sqrt{\epsilon\mu}}\frac{E_0^2\cos^2(kz-\omega t)}{8\pi c}\,\hat{\mathbf{z}}.\end{aligned} \tag{77}$$

Since the medium is at rest (in the frame of the present analysis), it has zero "overt" mechanical momentum, but it might have "hidden" mechanical momentum associated with the bound electric-current density,

$$\mathbf{J}_{\rm bound} = \frac{\partial\mathbf{P}}{\partial t} + c\boldsymbol{\nabla}\times\mathbf{M} = \frac{\epsilon-1}{4\pi}\frac{\partial\mathbf{E}}{\partial t} + c\frac{\mu-1}{4\pi\mu}\boldsymbol{\nabla}\times\mathbf{B} = \left(\frac{\epsilon-1}{4\pi} - \frac{\mu-1}{4\pi\mu}\right)\frac{\partial\mathbf{E}}{\partial t}\,, \tag{78}$$

recalling that for a linear medium, $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \epsilon\mathbf{E}$ and $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} = \mathbf{B}/\mu$. The current density (78) appears to be in the x-direction, and with zero time average, so it might seem that there can be no net, time-average mechanical momentum associated with it.

However, the Ampèrian model of magnetization $\mathbf{M}$ is that it is due to "molecular currents" which flow in small (atom-sized) loops in the plane perpendicular to $\mathbf{M}$ (and $\mathbf{B}$). Then, the argument which led to an understanding of "hidden" mechanical momentum of current loops in an electric field (see, for example, [34, 35]) would appear to apply to the present case, such that for waves with wavelength large compared to the radius of an atom (which includes optical waves) there should be a density of bound, "hidden" mechanical momentum given by,

$$\mathbf{p}_{\rm hidden,mechanical} = \frac{\mathbf{M}\times\mathbf{E}}{c} = \frac{\mu-1}{4\pi\mu}\frac{\mathbf{B}\times\mathbf{E}}{c} = -\frac{\mu-1}{\mu}\mathbf{p}_{\rm EM}. \tag{79}$$

This suggests that the total density of "hidden" momentum in the system is,

$$\mathbf{p}_{\rm hidden,total} = \mathbf{p}_{\rm hidden,EM} + \mathbf{p}_{\rm hidden,mechanical} = \frac{E_0^2\cos^2(kz-\omega t)}{8\pi c}\left(\frac{\epsilon\mu-1}{\sqrt{\epsilon\mu}} - 2\frac{\mu-1}{\mu}\right)\hat{\mathbf{z}}. \tag{80}$$

One premise of the above argument is that the electric field $\mathbf{E}$ be uniform over the loop, which implies that in the case of a wave field, the wavelength λ is large compared to the sides h and w of the loop.

In, addition, for the instantaneous "hidden" mechanical momentum to be $\mathbf{m} \times \mathbf{E}(t)/c$, it must be the period of the circulation of charges in the current loop be less than the period of the external wave field. That is, the angular velocity of the circulating charges must be larger than the angular velocity of the wave field.

We examine this requirement for a diamagnetic medium in the following subsection.

2.4.1 Diamagnetism

Diamagnetism is associated with magnetic momentum to due orbital motion of atomic electrons that is induced by the external electromagnetic field.

For an order-of-magnitude estimate, we take the effective radius of the magnetic moment to be the Bohr radius $r_B = \lambda\!\!\!^{-}_C/\alpha$, where $\alpha = e^2/\hbar c$ is the fine-structure constant and e is the charge of an electron.

Then, the requirement that the external field be uniform over the magnetic moment is that the wavelength of the external field be large compared to the Bohr radius, which is well satisfied by optical fields.

The angular frequency of the induced orbital motion of electrons is less than $v_{\rm max}/r_B$, where,

$$v_{\rm max} \approx a_{\rm max} t_{\rm wave} \approx \frac{eE_0}{m}\frac{1}{\omega}, \tag{81}$$

where E_0 is the peak electric field of the wave. For $\mathbf{m} \times \mathbf{E}(t)/c$ to describe the "hidden" mechanical momentum associated with diamagnetism, we must also have,

$$\omega_{\rm orbital} \approx \frac{v_{\rm max}}{r_B} \approx \frac{eE_0}{m\omega r_B} = \alpha\frac{eE_0}{m\omega\lambda\!\!\!^{-}_C} \gg \omega, \qquad E_0 \gg \frac{m\omega^2\lambda\!\!\!^{-}_C}{\alpha e} = \frac{\hbar^2\omega^2}{e^3} = \frac{E_{\rm crit}}{\alpha}\left(\frac{\lambda\!\!\!^{-}_C}{\lambda\!\!\!^{-}}\right)^2, \tag{82}$$

where $E_{\rm crit} = m^2c^3/e\hbar = 1.6 \times 10^{16}$ V/cm is the so-called QED critical field strength (above which a static electric field would spontaneously produce electron-positron pairs [42]). For optical waves, this requirement is that $E_0 \gg 10^4$ V/cm, which is a reasonably strong field, but which is readily achieved in laser beams.

2.4.2 Abraham and Minkowski Momenta

Assuming that the requirement (82 is satisfied, we can also consider the Abraham and Minkowski momenta, although it is not immediately clear to which subsystem they apply. The Abraham momentum density in the present example is,

$$\begin{aligned}\mathbf{p}_{\rm EM}^{\rm (A)} &= \frac{\mathbf{E}\times\mathbf{H}}{4\pi c} = \sqrt{\frac{\epsilon}{\mu}}\frac{E_0^2\cos^2(kz-\omega t)}{4\pi c}\,\hat{\mathbf{z}} \\ &= \frac{\mathbf{E}\times\mathbf{B}}{4\pi c} - \frac{\mathbf{E}\times\mathbf{M}}{c} = \mathbf{p}_{\rm EM} + \mathbf{p}_{\rm hidden,mechanical} = \mathbf{p}_{\rm total}.\end{aligned} \tag{83}$$

Thus, it appears that the Abraham momentum is not a subsystem momentum, but the total momentum of the system (in the rest frame of the medium).

The Minkowski momentum density is,

$$\mathbf{p}_{\rm EM}^{(\rm M)} = \frac{\mathbf{D}\times\mathbf{B}}{4\pi c} = \epsilon\mathbf{p}_{\rm EM} = \epsilon\sqrt{\epsilon\mu}\frac{E_0^2\cos^2(kz-\omega t)}{4\pi c}\,\hat{\mathbf{z}} \tag{84}$$

The arguments that lead to the forms of the Abraham and Minkowski momentum both associate with them a field-energy density,

$$u_{\rm A,M} = \frac{\epsilon E^2+\mu H^2}{8\pi} = \epsilon\frac{E_0^2\cos^2(kz-\omega t)}{4\pi}\,, \qquad \varrho_{\rm eff,A,M} = \frac{u_{\rm A,M}}{c^2}\,, \tag{85}$$

such that one might consider densities of "hidden" momentum associated with the Abraham and Minkowski momenta to be, using eq. (83) for the Abraham momentum,

$$\mathbf{p}_{\rm hidden}^{(\rm A)} = \mathbf{p}_{\rm EM}^{(\rm A)} - \varrho_{\rm eff,A,M}\mathbf{v} = 0, \tag{86}$$

$$\mathbf{p}_{\rm hidden}^{(\rm M)} = \mathbf{p}_{\rm EM}^{(\rm M)} - \varrho_{\rm eff,A,M}\mathbf{v} = \epsilon(\sqrt{\epsilon\mu}-1)\frac{E_0^2\cos^2(kz-\omega t)}{8\pi c}\,\hat{\mathbf{z}}. \tag{87}$$

The results (83) and (86) for the Abraham momentum of a plane wave in a linear medium are suggestive, but do not hold in general, as shown in sec. 2.2.4 for the case of a linear medium, at rest, together with free electric charges and currents.

2.4.3 Canonical and Kinetic Momentum

The above discussion of "hidden" momentum has been based on the notion that it might be possible to partition a system into "electromagnetic" and "mechanical" subsystems. While this concept seems well motivated from a "classical" perspective, its is doubtful in the quantum picture. In the latter, the interaction between electromagnetic waves and matter is associated with quanta that are neither purely "electromagnetic" nor "mechanical," but which are rather "quasiparticles" of the electromechanical interaction.

Within a "classical" context, one can seek a description that will not be in great contradiction with the quantum view. For this, we might start by considering a wave packet (photon) inside the linear medium, which we take to be nondispersive for now, such that the phase and group velocities are the same. This wave packet propagates with velocity $\mathbf{v} = \omega k = c\,\hat{\mathbf{k}}/n < c$, where $\mathbf{k}$ is the wave vector of the packet, and it seems reasonable to associate an effective mass $m_{\rm eff} = U_{\rm packet}/c^2$ with the wave packet. Then, the overt/kinetic momentum of the wave packet is,

$$\mathbf{P}_{\rm kinetic} = m_{\rm eff}\mathbf{v} = \frac{U_{\rm packet}}{cn}\,\hat{\mathbf{k}} = \int \frac{u_{\rm packet}}{c\sqrt{\epsilon\mu}}\,d{\rm Vol}\,\hat{\mathbf{k}}. \tag{88}$$

Regarding the wave packet as an electromechanical effect, is it plausible to identify its energy density as,

$$u_{\rm packet} = u_{\rm A,M} = \frac{\epsilon E^2+\mu H^2}{8\pi} \approx \frac{\epsilon E^2}{4\pi}\,, \tag{89}$$

rather than $u_{\rm EM} = (E^2+B^2)/8\pi$, as would be appropriate if the wave packet were purely "electromagnetic". Then, the kinetic momentum is,

$$\mathbf{P}_{\rm kinetic} \approx \int \mathbf{p}_{\rm EM}^{(\rm A)}\,d{\rm Vol}, \tag{90}$$

recalling eq. (83). That is, the kinetic momentum of the wave packet is its total Abraham momentum.[19]

In contrast, when one works in a Hamiltonian/Lagrangian formalism, one naturally considers canonical momenta (in addition to kinetic momenta $m\mathbf{v}$). Shortly after the original papers of Abraham [3, 15, 16, 17] and Minkowski [10], this was done by various authors [45, 46, 47], who identified the canonical stress-energy-momentum tensor (and hence the canonical momentum) of an electromechanical system as the Minkowski tensor (and momentum).[20]

A Appendix: If Magnetic Charges Also Existed

This Appendix was suggested by P. Saldanha (October, 2017).

If magnetic charges (magnetic monopoles) existed in addition to electric charges the story is more complex.

In particular, one would then consider magnetic charge and current densities ρ_m and $\mathbf{J}_m$ in addition to electric charge and current densities, ρ_e and $\mathbf{J}_e$ respectively (which were called ρ and $\mathbf{J}$ in the main text).

As discussed in [9], Maxwell's equations for the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ would then be,

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\rho_e, \quad \boldsymbol{\nabla}\cdot\mathbf{B} = 4\pi\rho_m, \quad -c\boldsymbol{\nabla}\times\mathbf{E} = \frac{\partial}{\partial t}\mathbf{B} + 4\pi\mathbf{J}_m, \quad c\boldsymbol{\nabla}\times\mathbf{B} = \frac{\partial\mathbf{E}}{\partial t} + 4\pi\mathbf{J}_e. \tag{91}$$

There could then be magnetic dipoles $\mathbf{m}_m$ formed from magnetic charges and electric dipoles $\mathbf{p}_m$ formed from magnetic currents (in addition to electric dipoles $\mathbf{p}_e$ formed from electric charges and magnetic dipoles $\mathbf{m}_e$ formed from electric currents, previously denoted without the subscript e).

Further, we should consider polarization densities $\mathbf{P}_m$ due to electric dipoles formed by magnetic currents and $\mathbf{M}_m$ due to magnetic dipoles formed by magnetic charges (in addition to polarization densities $\mathbf{P}_e$ due to electric dipoles formed by electric charges and $\mathbf{M}_e$ due to magnetic dipoles formed by electric currents, as considered previously).

When we spoke of "free" and "bound" charges and currents, we would then include effects associated with magnetic charges and currents. The "free" charge and current densities will be denoted as $\tilde{\rho}_e$, $\tilde{\mathbf{J}}_e$, $\tilde{\rho}_m$ and $\tilde{\mathbf{J}}_m$, and are related to the total charge and current densities (without a ˜) by,

$$\rho_e = \tilde{\rho}_e - \boldsymbol{\nabla}\cdot\mathbf{P}_e, \qquad \mathbf{J}_e = \tilde{\mathbf{J}}_e + \frac{\partial\mathbf{P}_e}{\partial t} + c\boldsymbol{\nabla}\times\mathbf{M}_e, \tag{92}$$

$$\rho_m = \tilde{\rho}_m - \boldsymbol{\nabla}\cdot\mathbf{M}_m, \qquad \mathbf{J}_m = \tilde{\mathbf{J}}_m + \frac{\partial\mathbf{M}_m}{\partial t} - c\boldsymbol{\nabla}\times\mathbf{P}_m. \tag{93}$$

[19]Arguments of this type can be traced to [43] (if not earlier), while the term "kinetic momentum" was perhaps first introduced in footnote 3 of [44].

[20]This insight was somewhat forgotten until rediscovered in the late 1940's [48, 49, 50], and then rediscovered in the new millennium [18, 52, 53].

It is customary in macroscopic electrodynamics to use versions of Maxwell's equations in which only "free" charge and current densities appear. For this we introduce the fields,[21]

$$\mathbf{D}_e = \mathbf{E} + 4\pi\mathbf{P}_e, \qquad \mathbf{H}_e = \mathbf{B} - 4\pi\mathbf{M}_e, \qquad \mathbf{D}_m = \mathbf{E} - 4\pi\mathbf{P}_m, \qquad \mathbf{H}_m = \mathbf{B} + 4\pi\mathbf{M}_m, \tag{94}$$

such that $\mathbf{D}_e$ and $\mathbf{H}_m$, and also $\mathbf{H}_e$ and $\mathbf{D}_m$, have similar forms, and,

$$\boldsymbol{\nabla}\cdot\mathbf{D}_e = 4\pi\tilde{\rho}_e, \qquad \boldsymbol{\nabla}\cdot\mathbf{H}_m = 4\pi\tilde{\rho}_m, \tag{95}$$

$$-\boldsymbol{\nabla}\times\mathbf{D}_m = \frac{\partial\mathbf{H}_m}{\partial t} + 4\pi\tilde{\mathbf{J}}_m, \qquad \boldsymbol{\nabla}\times\mathbf{H}_e = \frac{\partial\mathbf{D}_e}{\partial t} + 4\pi\tilde{\mathbf{J}}_e, \tag{96}$$

where in the absence of magnetic charges $\mathbf{D}_e$ and $\mathbf{H}_e$ are the familiar fields $\mathbf{D}$ and $\mathbf{H}$.[22]

When the arguments of Abraham and Minkowski are extended to include magnetic charges and currents, one finds [9] that the Abraham and Minkowski momentum densities are,

$$\mathbf{p}_{\mathrm{EM}}^{(\mathrm{A})} = \frac{\mathbf{S}}{c^2} = \frac{\mathbf{D_m}\times\mathbf{H}_e}{4\pi c}, \qquad \mathbf{p}_{\mathrm{EM}}^{(\mathrm{M})} = \frac{\mathbf{D_e}\times\mathbf{H_m}}{4\pi c}, \tag{97}$$

which revert to the familiar forms (2) and (4) if magnetic charges and currents do not exist.

A.1 Romer's Example with Permanent Polarizations

We now reconsider Romer's example [31], sec. 2.2 above, including the (hypothetical) magnetic charges as well as electric charges. Among the many possible variants, we only discuss the case of no free charges or currents, as in sec. 2.2.3 above, and for simplicity we consider only a single sphere of radius a that supports both electric and magnetic polarization.

We now have four (permanent) polarization densities to consider, $\mathbf{P}_e$, $\mathbf{P}_m$, $\mathbf{M}_e$ and $\mathbf{M}_m$.

We desire that the fields $\mathbf{E}$ due to both $\mathbf{P}_e$ and $\mathbf{P}_m$ be those of eq. (18) outside the sphere of electric polarization, and the fields $\mathbf{B}$ due to both $\mathbf{M}_e$ and $\mathbf{M}_m$ be those of eq. (19) outside the sphere of magnetic polarization. As inferred from the top of p. 6, this implies that the polarization densities associated with electric charges and currents are, for $r < a$,

$$\mathbf{P}_e = \frac{3\mathbf{p}_e}{4\pi a^3}, \qquad \mathbf{P}_m = \frac{3\mathbf{p}_m}{4\pi a^3}, \qquad \mathbf{M}_e = \frac{3\mathbf{m}_e}{4\pi a^3}, \qquad \mathbf{M}_m = \frac{3\mathbf{m}_m}{4\pi a^3}. \tag{98}$$

However, the $\mathbf{E}$ and $\mathbf{B}$ fields inside the polarized sphere are different for the cases of electric charges/currents and magnetic charges/currents. For polarization due to electric charges and currents the fields are those given in eq. (18)-(19),

$$\mathbf{E}_e = \begin{cases} -\frac{\mathbf{p}_e}{a^3} = -\frac{4\pi\mathbf{P}_e}{3} & (r<a), \\ \frac{3(\mathbf{p}_e\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}_e}{r^3} & (r>a), \end{cases} \qquad \mathbf{B}_e = \begin{cases} \frac{2\mathbf{m}_e}{a^3} = \frac{8\pi\mathbf{M}_e}{3} & (r<a), \\ \frac{3(\mathbf{m}_e\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}_e}{r^3} & (r>a), \end{cases} \qquad \text{(electric charges).} \tag{99}$$

[21]See Appendix D.2 of [9] for a justification of eq. (94) via the concept of electromagnetic duality.

[22]The relation $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$ seems to have been first introduced by W. Thomson in 1871, eq. (r), p. 401 of [56], and appears in sec. 399 of Maxwell's *Treatise* [57].

The case of magnetic dipole density $\mathbf{M}_m$ due to magnetic charges is analogous to the case of electric dipole density $\mathbf{P}_e$, and similarly the case of electric dipole density $\mathbf{P}_m$ due to magnetic currents is analogous to the case of magnetic dipole density $\mathbf{M}_e$. Hence, we have,

$$\mathbf{E}_m = \begin{cases} \frac{2\mathbf{p}_m}{a^3} = \frac{8\pi\mathbf{P}_m}{3} & (r < a), \\ \frac{3(\mathbf{p}_m\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}_m}{r^3} & (r > a), \end{cases} \qquad \mathbf{B}_m = \begin{cases} -\frac{\mathbf{m}_m}{a^3} = -\frac{4\pi\mathbf{M}_m}{3} & (r < a), \\ \frac{3(\mathbf{m}_m\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}_m}{r^3} & (r > a), \end{cases} \qquad \text{(magnetic charges)} \tag{100}$$

We now consider four cases, each with one type of electric polarization, and one type of magnetic polarization.

In all four cases, the field momentum (4) outside the sphere is the same, and given by last two terms of the next to last line of eq. (20),

$$\mathbf{P}_{\text{EM}}(r > a) = \int_{r>a} \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d\text{Vol} = \frac{\mathbf{m}\times\mathbf{p}}{3a^3 c}\,. \tag{101}$$

And, in all cases the total momentum is zero, since the systems are all "at rest," such that the ("hidden") mechanical momentum is equal and opposite to the electromagnetic momentum,

$$\mathbf{P}_{\text{mech}} = \mathbf{P}_{\text{hidden,mech}} = -\mathbf{P}_{\text{EM}}. \tag{102}$$

A.1.1 $\mathbf{P}_e \neq 0,\ \mathbf{P}_m = 0,\ \mathbf{M}_e \neq 0,\ \mathbf{M}_m = 0$

This case was discussed in sec. 2.2.3 above, so we just quote the results,

$$\mathbf{P}_{\text{EM}} = -\mathbf{P}_{\text{hidden,mech}} = \frac{\mathbf{m}_e\times\mathbf{p}_e}{a^3 c} = -\frac{\mathbf{m}_e\times\mathbf{E}_e}{c}\,, \tag{103}$$

$$\mathbf{P}_{\text{EM}}^{(\text{A})} = 0, \tag{104}$$

$$\mathbf{P}_{\text{EM}}^{(\text{M})} = -\mathbf{P}_{\text{EM}} = \mathbf{P}_{\text{hidden,mech}}. \tag{105}$$

A.1.2 $\mathbf{P}_e = 0,\ \mathbf{P}_m \neq 0,\ \mathbf{M}_e = 0,\ \mathbf{M}_m \neq 0$

$$\begin{aligned} \mathbf{P}_{\text{EM}} &= \frac{\mathbf{m}_m\times\mathbf{p}_m}{3a^3 c} + \int_{r<a} \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d\text{Vol} = \frac{\mathbf{m}_m\times\mathbf{p}_m}{3a^3 c} + \frac{2\mathbf{p}_m}{a^3}\times\frac{-\mathbf{m}_m}{a^3}\frac{a^3}{3c} = \frac{\mathbf{m}_m\times\mathbf{p}_m}{a^3 c} \\ &= \frac{\mathbf{p}_m\times\mathbf{B}_m}{c} = -\mathbf{P}_{\text{hidden,mech}}, \end{aligned} \tag{106}$$

$$\begin{aligned} \mathbf{P}_{\text{EM}}^{(\text{A})} &= \int \frac{\mathbf{D_m}\times\mathbf{H}_e}{4\pi c}\, d\text{Vol} = \int \frac{(\mathbf{E}-4\pi\mathbf{P}_m)\times\mathbf{B}}{4\pi c}\, d\text{Vol} = \mathbf{P}_{\text{EM}} - \int_{r<a} \frac{\mathbf{P}_m\times\mathbf{B}}{c}\, d\text{Vol} \\ &= \frac{\mathbf{p}_m\times\mathbf{B}_m}{c} - \frac{\mathbf{p}_m\times\mathbf{B}_m}{c} = 0, \end{aligned} \tag{107}$$

$$\begin{aligned} \mathbf{P}_{\text{EM}}^{(\text{M})} &= \int \frac{\mathbf{D_e}\times\mathbf{H}_m}{4\pi c}\, d\text{Vol} = \int \frac{\mathbf{E}\times(\mathbf{B}+4\pi\mathbf{M}_m)}{4\pi c}\, d\text{Vol} = \mathbf{P}_{\text{EM}} + \int_{r<a} \frac{\mathbf{E}\times\mathbf{M}_m}{c}\, d\text{Vol} \\ &= \frac{\mathbf{p}_m\times\mathbf{B}_m}{c} - \frac{2\mathbf{p}_m\times\mathbf{B}_m}{c} = -\frac{\mathbf{p}_m\times\mathbf{B}_m}{c} = -\mathbf{P}_{\text{EM}} = \mathbf{P}_{\text{hidden,mech}}. \end{aligned} \tag{108}$$

A.1.3 $\mathbf{P}_e = 0,\ \mathbf{P}_m \neq 0,\ \mathbf{M}_e \neq 0,\ \mathbf{M}_m = 0$

$$\begin{aligned}
\mathbf{P}_{\rm EM} &= \frac{\mathbf{m}_e \times \mathbf{p}_m}{3a^3c} + \int_{r<a} \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} = \frac{\mathbf{m}_e \times \mathbf{p_m}}{3a^3c} + \frac{2\mathbf{p}_m}{a^3} \times \frac{2\mathbf{m}_e}{a^3}\,\frac{a^3}{3c} = \frac{\mathbf{p}_m \times \mathbf{m}_e}{a^3c} \\
&= \frac{\mathbf{p}_m \times \mathbf{B}_e(r<a)}{4c} - \frac{\mathbf{m}_e \times \mathbf{E}_m(r<a)}{4c} = \int \left(\frac{\mathbf{P}_m \times \mathbf{B}_e}{4c} - \frac{\mathbf{M}_e \times \mathbf{E}_m}{4c}\right) d{\rm Vol} \\
&= -\mathbf{P}_{\rm hidden,mech}, \qquad (109)
\end{aligned}$$

Also,[23]

$$\begin{aligned}
\mathbf{P}_{\rm EM}^{\rm (A)} &= \int \frac{\mathbf{D_m}\times\mathbf{H}_e}{4\pi c}\, d{\rm Vol} = \int \frac{(\mathbf{E} - 4\pi\mathbf{P}_m)\times(\mathbf{B} - 4\pi\mathbf{M}_e)}{4\pi c}\, d{\rm Vol} \\
&= \int \frac{\mathbf{E}\times\mathbf{B}}{c}\, d{\rm Vol} - \int_{r<a} \frac{\mathbf{P}_m\times\mathbf{B}}{c}\, d{\rm Vol} - \int_{r<a} \frac{\mathbf{E}\times\mathbf{M}_e}{c}\, d{\rm Vol} + 4\pi \int_{r<a} \frac{\mathbf{P}_m\times\mathbf{M}_e}{c}\, d{\rm Vol} \\
&= \frac{\mathbf{p}_m\times\mathbf{m}_e}{a^3c} - \frac{2\mathbf{p}_m\times\mathbf{m}_e}{a^3c} - \frac{2\mathbf{p}_m\times\mathbf{m}_e}{a^3c} + \frac{3\mathbf{p}_m\times\mathbf{m}_e}{a^3c} = 0, \qquad (113) \\
\mathbf{P}_{\rm EM}^{\rm (M)} &= \int \frac{\mathbf{D_e}\times\mathbf{H}_m}{4\pi c}\, d{\rm Vol} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} = \mathbf{P}_{\rm EM} = -\mathbf{P}_{\rm hidden,mech}. \qquad (114)
\end{aligned}$$

A.1.4 $\mathbf{P}_e \neq 0,\ \mathbf{P}_m = 0,\ \mathbf{M}_e = 0,\ \mathbf{M}_m \neq 0$

$$\begin{aligned}
\mathbf{P}_{\rm EM} &= \frac{\mathbf{m}_m \times \mathbf{p}_e}{3a^3c} + \int_{r<a} \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} = \frac{\mathbf{m}_m \times \mathbf{p}_e}{3a^3c} + \frac{-\mathbf{p}_e}{a^3} \times \frac{-\mathbf{m}_m}{a^3}\,\frac{a^3}{3c} = 0, \qquad (115) \\
\mathbf{P}_{\rm EM}^{\rm (A)} &= \int \frac{\mathbf{D_m}\times\mathbf{H}_e}{4\pi c}\, d{\rm Vol} = \int \frac{(\mathbf{E} - 4\pi\mathbf{P}_m)\times\mathbf{B}}{4\pi c}\, d{\rm Vol} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} \\
&= \mathbf{P}_{\rm EM} = 0, \qquad (116) \\
\mathbf{P}_{\rm EM}^{\rm (M)} &= \int \frac{\mathbf{D_e}\times\mathbf{H}_m}{4\pi c}\, d{\rm Vol} = \int \frac{(\mathbf{E} + 4\pi\mathbf{P}_e)\times\mathbf{B}}{4\pi c}\, d{\rm Vol} = \mathbf{P}_{\rm EM} + \int_{r<a} \frac{\mathbf{P}_e\times\mathbf{B}}{c}\, d{\rm Vol} \\
&= \frac{\mathbf{p}_e \times \mathbf{B}_m}{c} \qquad (\text{while } \mathbf{P}_{\rm hidden,mech} = 0). \qquad (117)
\end{aligned}$$

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[23] *(June 22, 2020)* This case has subtleties discussed in Appendix B.2 of [35], so it may be useful to record some details supposing the sphere of uniform electric magnetization $\mathbf{M}_e$ has radius $a > b$, where b is the radius of the sphere of uniform magnetic polarization $\mathbf{P}_m$. Then,

$$\mathbf{P}_{\rm EM} = \int_{r>a} \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} + \int_{b<r<a} \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} + \int_{r<b} \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol}. \qquad (110)$$

As in the computation of eq. (20), the integral over $b < r < a$ vanishes, while the integral for $r > a$ has the form of eq. (101). Hence,

$$\mathbf{P}_{\rm EM} = \frac{\mathbf{m}_e \times \mathbf{p_m}}{3a^3c} + \left(\frac{2\mathbf{p}_m}{b^3} \times \frac{2\mathbf{m}_e}{a^3}\right) \frac{4\pi b^3}{3(4\pi c)} = \frac{\mathbf{p}_m \times \mathbf{m}_e}{a^3c} = -\mathbf{P}_{\rm hidden,mech}, \qquad (111)$$

$$\begin{aligned}
&\int \left(\frac{\mathbf{P}_m \times \mathbf{B}_e}{4c} - \frac{\mathbf{M}_e \times \mathbf{E}_m}{4c}\right) d{\rm Vol} = \int_{r<b} \left(\frac{\mathbf{P}_m \times \mathbf{B}_e}{4c} - \frac{\mathbf{M}_e \times \mathbf{E}_m}{4c}\right) d{\rm Vol} - \int_{b<r<a} \frac{\mathbf{M}_e \times \mathbf{E}_m}{4c}\, d{\rm Vol} \\
&= \int_{r<b} \left(\frac{3\mathbf{p}_m}{4\pi b^3} \times \frac{2\mathbf{m}_e}{a^3} - \frac{3\mathbf{m}_e}{4\pi a^3} \times \frac{2\mathbf{p}_m}{b^3}\right) \frac{d{\rm Vol}}{4c} - \int_{b<r<a} \frac{3\mathbf{m}_e}{4\pi a^3} \times \frac{3(\mathbf{p}_m\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p}_m}{r^3}\, \frac{d{\rm Vol}}{4c} = \frac{\mathbf{p}_m \times \mathbf{m}_e}{a^3c}. \qquad (112)
\end{aligned}$$

Thus, the forms for $\mathbf{P}_{\rm EM}$ found in eq. (109) do not depend on the spheres of polarization $\mathbf{P}_m$ and $\mathbf{M}_e$ having precisely the same radii.

A.1.5 Comments

The cases reported in secs. A1.1, A1.2 and A1.3 all have charges (electric and/or magnetic) in motion, and all have "hidden" mechanical momentum $\mathbf{m}\times\mathbf{p}/a^3c$, while the case in sec. A.1.4 has no charges in motion and no "hidden" mechanical momentum.

The "hidden" mechanical momenta associated with dipoles due to charges in motion can be summarized as,

$$\mathbf{P}_{\rm hidden,mech}(\mathbf{m}_e) = \frac{\mathbf{m}_e \times \mathbf{E}}{c}\,, \qquad \mathbf{P}_{\rm hidden,mech}(\mathbf{p}_m) = -\frac{\mathbf{p}_m \times \mathbf{B}}{c}\,, \tag{118}$$

for electric and magnetic fields from both electric and magnetic charges/currents.[24] Further, these forms hold for $\mathbf{E}$ and $\mathbf{B}$ due to either electric charges/currents or magnetic charges/currents (or both).[25]

This is an example of the duality transformation between cases with only electric charges/currents and those with only magnetic charges/currents, as discussed, for example, in [27],

$$\rho_e \to \rho_m, \quad \mathbf{J}_e \to \mathbf{J}_m, \quad \Rightarrow \quad \mathbf{E}_e \to \mathbf{B}_m, \quad \mathbf{B}_e \to -\mathbf{E}_m, \quad \mathbf{p}_e \to \mathbf{m}_m, \quad \mathbf{m}_e \to -\mathbf{p}_m. \tag{119}$$

The reverse transformation is,[26]

$$\rho_m \to -\rho_e, \quad \mathbf{J}_m \to -\mathbf{J}_e, \quad \Rightarrow \quad \mathbf{E}_m \to \mathbf{B}_e, \quad \mathbf{B}_m \to -\mathbf{E}_e, \quad \mathbf{p}_m \to \mathbf{m}_e, \quad \mathbf{m}_m \to -\mathbf{p}_e. \tag{120}$$

Thus, the duality transformation of the "hidden" mechanical momentum of dipoles is,

$$\frac{\mathbf{m}_e \times \mathbf{E}}{c} \leftrightarrow -\frac{\mathbf{p}_m \times \mathbf{B}}{c}\,. \tag{121}$$

While it is intriguing that the Abraham momentum is zero in all four variants of Romer's example for media at rest with permanent dipole moments, recall from sec. 2.2 above that the Abraham momentum is nonzero in the variants with "free" surface electric currents.

A.2 Romer's Example with Linear Media (June 2020)

This section extends sec. 2.2.4 above to include the possibility of linear media with magnetic charges and currents, which would be characterized by relative permittivity ϵ_m and relative permeability μ_m. We now denote ϵ_e as the relative permittivity, and μ_e as the relative permeability, of linear media with electric charges and currents.

We discuss only that case when the radii of the two spheres are the same, a, and that the relative permittivity and permeabilities are both unity for $r > a$.

[24]For an example in which the "hidden" mechanical momentum is $-\mathbf{p}_m \times \mathbf{B}/c$ for a loop of magnetic current, see Appendix B of [35].

[25]The case presented in sec. A.1.3, in which electric and magnetic currents coexist within the same volume, is an exception, with "hidden" mechanical momentum $\mathbf{m}_e \times \mathbf{E}_m/4c - \mathbf{p}_m \times \mathbf{B}_e/4c$ (although this does equal $\mathbf{m}_e \times \mathbf{p}_m/a^3c$). This unusual case is considered in greater detail in Appendix B.2.2 of [35].

[26]The identity transformation consists of a sequence of four transformations, $e \to m$, $m \to e$, $e \to m$, $m \to e$ (rather than only two, $e \to m$, $m \to e$).

In eqs. (41)-(42) of sec. 2.2.4, we found the fields associated with electric charges and currents, which can be written in the notation of the present section as,

$$\mathbf{D}_e = \begin{cases} -\frac{3\epsilon_e}{2+\epsilon_e}\frac{\mathbf{p}_{e,\mathrm{free}}}{a^3} & (r<a), \\ \frac{3}{2+\epsilon_e}\frac{3(\mathbf{p}_{e,\mathrm{free}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}_{e,\mathrm{free}}}{r^3} & (r>a), \end{cases} \qquad \mathbf{E}_e = \begin{cases} -\frac{3}{2+\epsilon_e}\frac{\mathbf{p}_{e,\mathrm{free}}}{a^3} & (r<a), \\ \frac{3}{2+\epsilon_e}\frac{3(\mathbf{p}_{e,\mathrm{free}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}_{e,\mathrm{free}}}{r^3} & (r>a), \end{cases} \tag{122}$$

$$\mathbf{H}_e = \begin{cases} \frac{3}{2+\mu_e}\frac{2\mathbf{m}_{e,\mathrm{free}}}{a^3} & (r<a), \\ \frac{3\mu_e}{2+\mu_e}\frac{3(\mathbf{m}_{e,\mathrm{free}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}_{e,\mathrm{free}}}{r^3} & (r>a), \end{cases} \qquad \mathbf{B}_e = \begin{cases} \frac{3\mu_e}{2+\mu_e}\frac{2\mathbf{m}_{e,\mathrm{free}}}{a^3} & (r<a), \\ \frac{3\mu_e}{2+\mu_e}\frac{3(\mathbf{m}_{e,\mathrm{free}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}_{e,\mathrm{free}}}{r^3} & (r>a). \end{cases} \tag{123}$$

We can obtain the expressions associated with magnetic charges and currents from eqs. (122)-(123) via the duality transformations (119) above and eq. (114) of [9],

$$\mathbf{D}_e \to \mathbf{H}_m, \qquad \mathbf{H}_e \to -\mathbf{D}_m, \qquad \mathbf{D}_m \to \mathbf{H}_e, \qquad \mathbf{H}_m \to -\mathbf{D}_e. \tag{124}$$

We also note the duality relations for the relative permittivities and permeabilities,

$$\mathbf{D}_e = \epsilon_e \mathbf{E}, \qquad \mathbf{H}_e = \frac{\mathbf{B}}{\mu_e}, \qquad \mathbf{D}_m \to \epsilon_m \mathbf{E}, \qquad \mathbf{H}_m = \frac{\mathbf{B}}{\mu_m}, \tag{125}$$

$$\epsilon_e \to \frac{1}{\mu_m}, \qquad \mu_e \to \frac{1}{\epsilon_m}, \qquad \epsilon_m \to \frac{1}{\mu_e}, \qquad \mu_m \to \frac{1}{\epsilon_e}. \tag{126}$$

Then,

$$\mathbf{H}_m = \begin{cases} -\frac{3}{2\mu_m+1}\frac{\mathbf{m}_{m,\mathrm{free}}}{a^3}, \\ \frac{3\mu_m}{2\mu_m+1}\frac{3(\mathbf{m}_{m,\mathrm{free}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}_{m,\mathrm{free}}}{r^3}, \end{cases} \qquad \mathbf{B}_m = \begin{cases} -\frac{3\mu_m}{2\mu_m+1}\frac{\mathbf{m}_{m,\mathrm{free}}}{a^3} & (r<a), \\ \frac{3\mu_m}{2\mu_m+1}\frac{3(\mathbf{m}_{m,\mathrm{free}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}_{m,\mathrm{free}}}{r^3} & (r>a), \end{cases} \tag{127}$$

$$\mathbf{D}_m = \begin{cases} \frac{3\epsilon_m}{2\epsilon_m+1}\frac{2\mathbf{p}_{m,\mathrm{free}}}{a^3}, \\ \frac{3}{2\epsilon_m+1}\frac{3(\mathbf{p}_{m,\mathrm{free}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}_{m,\mathrm{free}}}{r^3}, \end{cases} \qquad \mathbf{E}_m = \begin{cases} \frac{3}{2\epsilon_m+1}\frac{2\mathbf{p}_{m,\mathrm{free}}}{a^3} & (r<a), \\ \frac{3}{2\epsilon_m+1}\frac{3(\mathbf{p}_{m,\mathrm{free}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}_{m,\mathrm{free}}}{r^3} & (r>a). \end{cases} \tag{128}$$

It is useful to recall from eq. (20) that,

$$\int_{r>a} \frac{3(\mathbf{p}_{\mathrm{free}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}_{\mathrm{free}}}{r^3} \times \frac{3(\mathbf{m}_{\mathrm{free}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}_{\mathrm{free}}}{r^3}\frac{d\mathrm{Vol}}{4\pi c} = \frac{\mathbf{m}_{\mathrm{free}}\times\mathbf{p}_{\mathrm{free}}}{3a^3 c}. \tag{129}$$

A.2.1 $\mathbf{p}_e \neq 0$, $\mathbf{p}_m = 0$, $\mathbf{m}_e \neq 0$, $\mathbf{m}_m = 0$

This case was treated in sec. 2.2.4 above.

$$\mathbf{P}_{\mathrm{EM}} = -\frac{3\mathbf{p}_{e,\mathrm{free}}}{(2+\epsilon_e)a^3 c} \times \frac{3\mu_e \mathbf{m}_{e,\mathrm{free}}}{2+\mu_e}, \tag{130}$$

$$\mathbf{P}_{\mathrm{EM}}^{(\mathrm{A})} = \frac{3\mathbf{m}_{m,\mathrm{free}}}{(2\mu_m+1)a^3 c} \times \mathbf{p}_{m,\mathrm{free}}, \tag{131}$$

$$\mathbf{P}_{\mathrm{EM}}^{(\mathrm{M})} = -\frac{(2\epsilon_e+1)\mathbf{p}_{e,\mathrm{free}}}{(2+\epsilon_e)a^3 c} \times \frac{3\mu_e \mathbf{m}_{e,\mathrm{free}}}{2+\mu_e}. \tag{132}$$

A.2.2 $\mathbf{p}_e = 0,\ \mathbf{p}_m \neq 0,\ \mathbf{m}_e = 0,\ \mathbf{m}_m \neq 0$

This case is the dual of that in sec. A.2.1.

$$\mathbf{P}_{\rm EM} = \frac{3\mu_m \mathbf{m}_{m,\rm free}}{(2\mu_m + 1)a^3 c} \times \frac{3\mathbf{p}_{m,\rm free}}{2\epsilon_m + 1}\,, \tag{133}$$

$$\mathbf{P}_{\rm EM}^{\rm (A)} = -\frac{3\mathbf{p}_{e,\rm free}}{(2+\epsilon_e)a^3 c} \times \mathbf{m}_{e,\rm free}, \tag{134}$$

$$\mathbf{P}_{\rm EM}^{\rm (M)} = -\frac{(2+\mu_m)\mathbf{m}_{m,\rm free}}{(2\mu_m + 1)a^3 c} \times \frac{3\mathbf{p}_{m,\rm free}}{2\epsilon_\mu + 1}\,. \tag{135}$$

A.2.3 $\mathbf{p}_e = 0,\ \mathbf{p}_m \neq 0,\ \mathbf{m}_e \neq 0,\ \mathbf{m}_m = 0$

$$\begin{aligned}
\mathbf{P}_{\rm EM} &= \int \frac{\mathbf{E}_m \times \mathbf{B}_e}{4\pi c}\, d{\rm Vol} = \int_{r<a} \frac{6}{2\epsilon_m + 1} \frac{\mathbf{p}_{m,\rm free}}{a^3} \times \frac{6\mu_e}{2+\mu_e} \frac{\mathbf{m}_{e,\rm free}}{a^3} \frac{d{\rm Vol}}{4\pi c} \\
&+ \int_{r>a} \frac{3}{2\epsilon_m + 1} \frac{3(\mathbf{p}_{m,\rm free}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}_{m,\rm free}}{r^3} \times \frac{3\mu_e}{2+\mu_e} \frac{3(\mathbf{m}_{e,\rm free}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_{e,\rm free}}{r^3} \frac{d{\rm Vol}}{4\pi c} \\
&= \frac{3}{2\epsilon_m + 1} \frac{3\mu_e}{2+\mu_e} \left(\frac{4\mathbf{p}_{m,\rm free} \times \mathbf{m}_{e,\rm free}}{3a^3 c} + \frac{\mathbf{m}_{e,\rm free} \times \mathbf{p}_{m,\rm free}}{3a^3 c} \right) \\
&= \frac{3\mu_e}{(2\epsilon_m + 1)(2+\mu_e)} \frac{\mathbf{p}_{m,\rm free} \times \mathbf{m}_{e,\rm free}}{a^3 c}\,,
\end{aligned} \tag{136}$$

$$\begin{aligned}
\mathbf{P}_{\rm EM}^{\rm (A)} &= \int \frac{\mathbf{D}_m \times \mathbf{H}_e}{4\pi c}\, d{\rm Vol} = \int_{r<a} \frac{\epsilon_m \mathbf{E}_m \times \mathbf{B}_e/\mu_e}{4\pi c}\, d{\rm Vol} + \int_{r>a} \frac{\mathbf{E}_m \times \mathbf{B}_e}{4\pi c}\, d{\rm Vol} \\
&= \frac{\epsilon_m}{\mu_e} \int_{r<a} \frac{6}{2\epsilon_m + 1} \frac{\mathbf{p}_{m,\rm free}}{a^3} \times \frac{6\mu_e}{2+\mu_e} \frac{\mathbf{m}_{e,\rm free}}{a^3} \frac{d{\rm Vol}}{4\pi c} \\
&+ \int_{r>a} \frac{3}{2\epsilon_m + 1} \frac{3(\mathbf{p}_{m,\rm free}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}_{m,\rm free}}{r^3} \times \frac{3\mu_e}{2+\mu_e} \frac{3(\mathbf{m}_{e,\rm free}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_{e,\rm free}}{r^3} \frac{d{\rm Vol}}{4\pi c} \\
&= \frac{3}{2\epsilon_m + 1} \frac{3\mu_e}{2+\mu_e} \left(\frac{4\epsilon_m}{\mu_e} \frac{\mathbf{p}_{m,\rm free} \times \mathbf{m}_{e,\rm free}}{3a^3 c} + \frac{\mathbf{m}_{e,\rm free} \times \mathbf{p}_{m,\rm free}}{3a^3 c} \right) \\
&= \frac{3(4\epsilon_m - \mu_e)}{(2\epsilon_m + 1)(2+\mu_e)} \frac{\mathbf{p}_{m,\rm free} \times \mathbf{m}_{e,\rm free}}{a^3 c}\,,
\end{aligned} \tag{137}$$

$$\mathbf{P}_{\rm EM}^{\rm (M)} = \int \frac{\mathbf{D}_e \times \mathbf{H}_m}{4\pi c}\, d{\rm Vol} = \int \frac{\mathbf{E}_m \times \mathbf{B}_e}{4\pi c}\, d{\rm Vol} = \mathbf{P}_{\rm EM}. \tag{138}$$

A.2.4 $\mathbf{p}_e \neq 0,\ \mathbf{p}_m = 0,\ \mathbf{m}_e = 0,\ \mathbf{m}_m \neq 0$

This case, like that of sec. A.1.4 above, has no moving charges, so $\mathbf{P}_{\rm EM} = 0$. We can confirm this using,

$$\begin{aligned}
\mathbf{P}_{\rm EM} &= \int \frac{\mathbf{E}_e \times \mathbf{B}_m}{4\pi c}\, d{\rm Vol} = \int_{r<a} -\frac{3}{2+\epsilon_e} \frac{\mathbf{p}_{e,\rm free}}{a^3} \times -\frac{3\mu_m}{2\mu_m + 1} \frac{\mathbf{m}_{m,\rm free}}{a^3} \frac{d{\rm Vol}}{4\pi c} \\
&+ \int_{r>a} \frac{3}{2+\epsilon_e} \frac{3(\mathbf{p}_{e,\rm free}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}_{e,\rm free}}{r^3} \times \frac{3\mu_m}{2\mu_m + 1} \frac{3(\mathbf{m}_{m,\rm free}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_{m,\rm free}}{r^3} \frac{d{\rm Vol}}{4\pi c} \\
&= \frac{3}{2+\epsilon_e} \frac{3\mu_m}{2\mu_m + 1} \left(\frac{\mathbf{p}_{e,\rm free} \times \mathbf{m}_{m,\rm free}}{3a^3 c} + \frac{\mathbf{m}_{m,\rm free} \times \mathbf{p}_{e,\rm free}}{3a^3 c} \right) = 0.
\end{aligned} \tag{139}$$

Also,

$$\mathbf{P}_{\rm EM}^{\rm (A)} = \int \frac{\mathbf{D}_m \times \mathbf{H}_e}{4\pi c}\, d{\rm Vol} = \int \frac{\mathbf{E}_e \times \mathbf{B}_m}{4\pi c}\, d{\rm Vol} = 0. \qquad (140)$$

$$\begin{aligned}
\mathbf{P}_{\rm EM}^{\rm (M)} = \int \frac{\mathbf{D}_e \times \mathbf{H}_m}{4\pi c}\, d{\rm Vol} &= \int_{r<a} \frac{\epsilon_e \mathbf{E}_e \times \mathbf{B}_m/\mu_m}{4\pi c}\, d{\rm Vol} + \int_{r>a} \frac{\mathbf{E}_e \times \mathbf{B}_m}{4\pi c}\, d{\rm Vol} \\
&= \frac{\epsilon_e}{\mu_m} \int_{r<a} -\frac{3}{2+\epsilon_e} \frac{\mathbf{p}_{e,\rm free}}{a^3} \times -\frac{3\mu_m}{2\mu_m+1} \frac{\mathbf{m}_{m,\rm free}}{a^3} \frac{d{\rm Vol}}{4\pi c} \\
&+ \int_{r>a} \frac{3}{2+\epsilon_e} \frac{3(\mathbf{p}_{e,\rm free}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}_{e,\rm free}}{r^3} \times \frac{3\mu_m}{2\mu_m+1} \frac{3(\mathbf{m}_{m,\rm free}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_{m,\rm free}}{r^3} \frac{d{\rm Vol}}{4\pi c} \\
&= \frac{3}{2+\epsilon_e} \frac{3\mu_m}{2\mu_m+1} \left(\frac{\epsilon_e}{\mu_m} \frac{\mathbf{p}_{e,\rm free} \times \mathbf{m}_{m,\rm free}}{3a^3 c} + \frac{\mathbf{m}_{m,\rm free} \times \mathbf{p}_{e,\rm free}}{3a^3 c} \right) \\
&= \frac{3(\epsilon_e - \mu_m)}{(2+\epsilon_e)(2\mu_m+1)} \frac{\mathbf{p}_{e,\rm free} \times \mathbf{m}_{m,\rm free}}{a^3 c}\,. \qquad (141)
\end{aligned}$$

B Appendix: Why Does $\mathbf{p}_m = -\int \mathbf{r} \times \mathbf{J}_m\ d{\rm Vol}/2c$?

The duality transformation (119) contains the prescription that the dual of a magnetic-dipole moment due to electric currents is the negative of an electric-dipole moment due to magnetic currents,

$$\mathbf{m}_e \to -\mathbf{p}_m, \qquad (142)$$

which minus sign is perhaps counterintuitive. Now,

$$\mathbf{m}_e = \int \frac{\mathbf{r} \times \mathbf{J}_e}{2c}\, d{\rm Vol} \to \int \frac{\mathbf{r} \times \mathbf{J}_m}{2c}\, d{\rm Vol}, \qquad (143)$$

so eqs. (142)-(143) imply that,

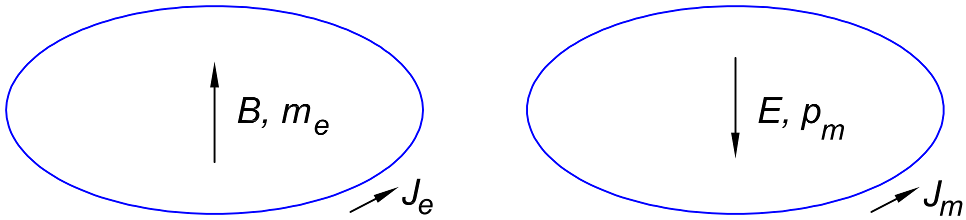
$$\mathbf{p}_m = -\int \frac{\mathbf{r} \times \mathbf{J}_m}{2c}\, d{\rm Vol}, \qquad (144)$$

which also is perhaps surprising.

We recall that for a magnetic dipole $\mathbf{m}_e$ associated with an electric-current density $\mathbf{J}_e$ that flows in a loop, say in a static situation, the Maxwell equation $\boldsymbol{\nabla} \times \mathbf{B} = 4\pi \mathbf{J}_e/c$ implies that,

$$\frac{4\pi}{c} \oint_{\rm loop} \mathbf{J}_e \cdot d\mathbf{l} = \oint_{\rm loop} \boldsymbol{\nabla} \times \mathbf{B} \cdot d\mathbf{l} = \int_{\rm loop} \mathbf{B} \cdot d\mathbf{Area}. \qquad (145)$$

That is, the direction of the magnetic field $\mathbf{B}$ at the center of the loop is related to the direction of $\mathbf{J}_e$ by the righthand rule, as sketched in the left figure below.



This is consistent with the usual relation,

$$\mathbf{m}_e = \int \frac{\mathbf{r} \times \mathbf{J}_e}{2c}\, d\text{Vol}. \tag{146}$$

In the case of a loop of magnetic current density $\mathbf{J}_m$, again in a static situation, the Maxwell equation (91), $\boldsymbol{\nabla} \times \mathbf{E} = -4\pi\mathbf{J}_m/c$, implies that,

$$\frac{4\pi}{c}\oint_{\text{loop}} \mathbf{J}_m \cdot d\mathbf{l} = -\oint_{\text{loop}} \boldsymbol{\nabla} \times \mathbf{E} \cdot d\mathbf{l} = \int_{\text{loop}} \mathbf{E} \cdot d\mathbf{Area}. \tag{147}$$

That is, the direction of the magnetic field $\mathbf{E}$ at the center of the loop is related to the direction of $\mathbf{J}_m$ by the lefthand rule, as sketched in the right figure above. This is consistent with the relation (144), which is in turn consistent with the duality relation (142), $\mathbf{m}_e \to -\mathbf{p}_m$.

Thus, the difference in sign between the relations (144) and (146) is due to the difference in signs of the terms in the current densities in the Maxwell equations,

$$\frac{c}{4\pi}\boldsymbol{\nabla} \times \mathbf{E} = -\left(\frac{1}{4\pi}\frac{\partial \mathbf{B}}{\partial t} + \mathbf{J}_m\right), \qquad \frac{c}{4\pi}\boldsymbol{\nabla} \times \mathbf{B} = \frac{1}{4\pi}\frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}_e. \tag{148}$$

The equation for $\boldsymbol{\nabla} \times \mathbf{E}$ with magnetic currents was first discussed by Heaviside in 1885 [58]. He argued (p. 448 of [58]) that just as in the equation for $\boldsymbol{\nabla} \times \mathbf{B}$ where the current density $\mathbf{J}_e$ and the "displacement-current density" $(1/4\pi)\,\partial\mathbf{E}/\partial t$ have the same sign, the current density $\mathbf{J}_m$ and the "magnetic-displacement-current density" $(1/4\pi)\,\partial\mathbf{B}/\partial t$ should have the same sign in the equation for $\boldsymbol{\nabla} \times \mathbf{E}$.[27]

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