Where Does the Power Become AC in an AC Power Source?

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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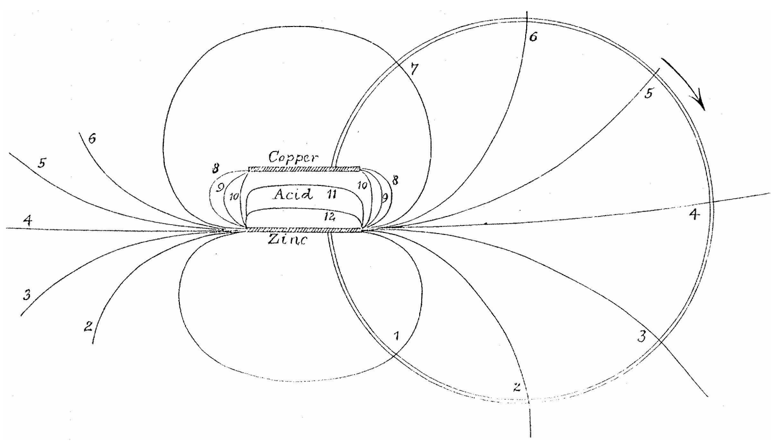
1 Problem

According to Faraday and Maxwell, the electromagnetic field stores energy and momentum. The flow of energy associated with electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ was quantified by Poynting [1] in terms of the vector,

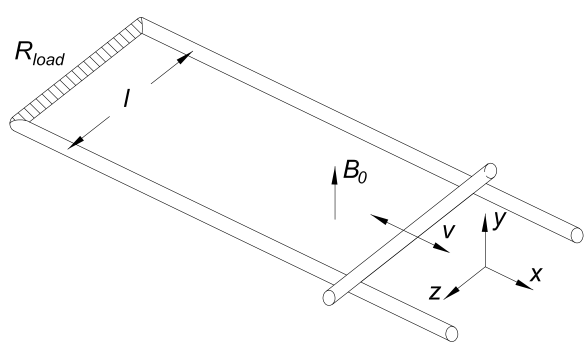
$$\mathbf{S} = \mathbf{E} \times \mathbf{H}, \tag{1}$$

in SI units, whose magnitude is the energy crossing unit area perpendicular to $\mathbf{S}$ per unit time.

Poynting discussed the flow of energy from a battery to a resistive loop of wire. As shown in his figure below, the power does not flow down the wire of the loop, but rather it flows through the air/vacuum along lines of the vector $\mathbf{S}$ and enters the wire at right angles to its surface.[1]



Discuss the flow of power in an electromechanical power supply, as sketched below.



[1]A more detailed discussion of power flow in a DC current loop is given in [2].

A metal bar slides in the x direction along a U-shaped wire with oscillator velocity $\mathbf{v} = v(t)\,\hat{\mathbf{x}} = v_0 \cos\omega t\,\hat{\mathbf{x}}$ through a region of uniform magnetic field,

$$\mathbf{B}_0 = B_0\,\hat{\mathbf{y}}. \tag{2}$$

The metal bar is aligned along the z direction, with distance l between its two points of contact with the U-shaped wire. All materials in this example have magnetic permeability μ_0.

For an application of this type of AC generator in watches, see [3].

2 Solution

It will be difficult to give a complete analytic description of the Poynting vector throughout the circuit, so we will content ourselves with a characterization of the energy flow across the surface of the most relevant circuit elements.

First, we recall the well-known example[2] of a cylindrical resistor of radius a, length l and resistivity ρ. The corresponding resistance is $R = \rho l/\pi a^2$, so that when the resistor carries current I along its axial direction, the power consumed is $P = I^2 R$. An axial electric field $E_z = \rho J = \rho I/\pi a^2$ exists inside the resistor to drive the current, and an azimuthal magnetic field $H_\phi = I/2\pi a$ exists at the surface of the resistor. The Poynting vector at the cylindrical surface of the resistor points inward, $S_r = -E_z H_\phi = -\rho I^2/2\pi^2 a^3$. Since the Poynting vector vanishes on the two circular ends of the resistor, the total power flowing into the resistor via electromagnetic fields is the Poynting vector $|S_r|$ times the surface area $2\pi a l$, namely $P_{\rm in} = I^2 \rho l/\pi a^2 = I^2 R$. Thus, the inward electromagnetic energy flow described by the Poynting vector $\mathbf{S}$ at the surface of the resistor accounts for its $I^2 R$ power consumption.

2.1 EMF and Current in the Circuit

According to Faraday's law, the EMF $\mathcal{E}$ around a circuit is equal to the (negative) time rate of change of the magnetic flux through the circuit. In the present example we find,

$$\mathcal{E}(t) = v(t) l B_0. \tag{3}$$

If the total load resistance in the circuit is R, then a current,

$$I(t) = \mathcal{E}/R = \frac{v(t) l B_0}{R}\,, \tag{4}$$

flows in the circuit, and the (instantaneous) power consumption is,

$$P = I^2 R = \frac{\mathcal{E}^2}{R} = \frac{v^2 l^2 B_0^2}{R}\,. \tag{5}$$

The total resistance R is the sum of the resistances of the four straight line segments of the circuit. The problem is somewhat conceptually simpler if the resistance is negligible for the two long segments on which the bar slides. That is, we approximate,

$$R \approx R_{\rm load} + R_{\rm bar} = \frac{\rho_{\rm load} l}{\pi a_{\rm load}^2} + \frac{\rho l}{\pi a^2}\,, \tag{6}$$

[2] For a typical textbook discussion, see Example 8.1 of [4].

where ρ is the resistivity and a is the radius of a segment (load or sliding bar, both of length l).

The voltage drop across the sliding bar is,

$$\Delta V = I R_{\rm bar} = \frac{R_{\rm bar}}{R} \mathcal{E}\,, \tag{7}$$

which implies that the electric field inside the sliding bar and parallel to its axis (the z-axis) is,

$$\mathbf{E}_{\parallel} = \frac{I R_{\rm bar}}{l}\,\hat{\mathbf{z}}. \tag{8}$$

2.2 Mechanical Power Equals Electrical Power

The current flows in the $+z$-direction inside the sliding bar when its velocity is in the $+x$-direction. Hence, the sliding bar experiences a Lorentz force (in the lab frame),

$$\mathbf{F}(t) = I(t) l\,\hat{\mathbf{z}} \times \mathbf{B}_0 = -I l B_0\,\hat{\mathbf{x}} = -\frac{v(t) l^2 B_0^2}{R}\,\hat{\mathbf{x}}. \tag{9}$$

An external force $\mathbf{F}_{\rm ext} = -\mathbf{F}$ must be applied to keep the bar sliding with velocity $\mathbf{v}(t)$. The (instantaneous) power delivered into the system by this external mechanical force is,

$$P_{\rm ext} = \mathbf{F}_{\rm ext} \cdot \mathbf{v} = \frac{v^2 l^2 B_0^2}{R} = P. \tag{10}$$

That is, a lab-frame analysis indicates that the external mechanical system which drives the sliding bar provides the power (5) that is consumed by $I^2 R$ electrical heating.

We can also give an analysis in the rest frame of the sliding bar (the * frame). For this, we suppose that the magnet which produces the field $\mathbf{B}_0$ is at rest in the lab frame. When this field exerts force $\mathbf{F}$ on the sliding bar (in the lab frame), there is a reaction force $-\mathbf{F}$ on the magnet. To keep the magnet at rest there must be an external force $\mathbf{F}'_{\rm ext} = -(-\mathbf{F}) = -\mathbf{F}_{\rm ext}$ on it. In the rest frame of the sliding bar, the magnet has velocity $-\mathbf{v}$, so the force $\mathbf{F}'^\star = \mathbf{F}'_{\rm ext}$ on the moving magnet must deliver power $P^\star = \mathbf{F}^\star \cdot -\mathbf{v} = \mathbf{F}_{\rm ext} \cdot \mathbf{v} = P$.

2.3 The Motional EMF

The argument of sec. 2.1 did not localize the source of the EMF. It often considered that the EMF is generated in the portion of the sliding bar that is immersed in the external magnetic field. Here, we give an analysis that supports this view.

Faraday's law can be expressed in various forms,[3]

$$\mathcal{E} = -\frac{d\Phi_{\mathbf{B}}}{dt} = -\frac{d}{dt}\int_{\rm loop} \mathbf{B} \cdot d\mathbf{Area} = \oint (\mathbf{v} \times \mathbf{B} + \mathbf{E}) \cdot d\mathbf{l} = \mathcal{E}_{\rm motional} + \mathcal{E}_{\rm fixed\ loop}, \tag{11}$$

where,

$$\mathcal{E}_{\rm motional} = \oint_{\rm loop} \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l}, \tag{12}$$

[3]See, for example, sec. 2.1 of [5].

and,

$$\mathcal{E}_{\text{fixed loop}} = -\frac{\partial}{\partial t}\int_{\text{loop at time } t} \mathbf{B}\cdot d\mathbf{Area} = -\int_{\text{loop}} \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{Area} = \oint_{\text{loop}} \mathbf{E}\cdot d\mathbf{l}, \tag{13}$$

using $\boldsymbol{\nabla}\times\mathbf{E} = -\partial\mathbf{B}/\partial t$ and Stoke's theorem.

In the present example, where the external magnetic field $\mathbf{B}_0$ is independent of time, $\mathcal{E}_{\text{fixed loop}} = 0$, and the EMF is entirely motional,

$$\mathcal{E} = \mathcal{E}_{\text{motional}} = vlB_0, , \tag{14}$$

as in eq. (3). This suggests that the sliding bar is the source of the EMF.

If so, the electromagnetic energy generated in this example should flow from the sliding bar. That is, the integral of lab-frame Poynting vector (1) over the surface of the sliding bar should equal the power delivered to the rest of the circuit.

To compute the lab-frame Poynting vector, we need more details as to the electromagnetic fields at the surface of the sliding bar, which are most easily obtained by transformation of an analysis in the rest frame of the sliding bar. So, we digress to discuss the latter frame, and return to the lab frame in secs. 2.6-7.

2.4 Fields in the Rest Frame of the Bar

First, we consider the fields associated with the lab-frame magnetic field $\mathbf{B}_0$ as observed in rest frame of the sliding bar, in which quantities are designated with the superscript *. The speed v of the bar in the lab frame is much less than the speed of light c, so we will ignore effects of order v^2/c^2, and write $\gamma = 1/\sqrt{1-v^2/c^2} \approx 1$.

Then, the external magnetic field in the frame of the sliding bar is,

$$\mathbf{B}_0^\star = \gamma\mathbf{B}_0 \approx \mathbf{B}_0 = B_0\,\hat{\mathbf{y}}. \tag{15}$$

And, associated with the lab-frame magnetic field $\mathbf{B}_0$, there is an electric field,

$$\mathbf{E}_0^\star = \gamma\mathbf{v}\times\mathbf{B}_0 \approx \mathbf{v}\times\mathbf{B}_0 = v\,\hat{\mathbf{x}}\times B_0\,\hat{\mathbf{y}} = vB_0\,\hat{\mathbf{z}}, \tag{16}$$

in the rest frame of the sliding bar.

Next, there is the magnetic field due to the current I in the circuit. To a good approximation this magnetic field is azimuthal, and independent of coordinate z inside the bar of radius a,

$$\mathbf{B}_\phi^\star(r<a) = \frac{\mu_0 I r}{2\pi a^2}\,\hat{\boldsymbol{\phi}}. \tag{17}$$

Electrons drift inside the bar with velocity $\mathbf{v}_d^\star = v_d\,\hat{\mathbf{z}}$ (which is the same as the lab-frame drift velocity $\mathbf{v}_d$). The Lorentz force on a conduction electron of charge e at radius $r<a$ in the rest frame of the bar is,

$$\mathbf{F}^\star = e\mathbf{v}_d^\star\times\mathbf{B}^\star \approx ev_d\,\hat{\mathbf{z}}\times(\mathbf{B}_0+\mathbf{B}_\phi^\star) = -ev_d\left(B_0\,\hat{\mathbf{x}} + \frac{\mu_0 I r}{2\pi a^2}\,\hat{\mathbf{r}}\right). \tag{18}$$

For steady behavior, this tranverse force must be compensated by an equal and opposite force. The second term in eq. (18) is a "radial pinch," and leads to a very slight difference in the density of electrons and lattice ions, as discussed in [6]. We neglect this tiny effect in the following.

The first term of eq. (18) leads to a charge separation along the x-direction, until surface charges on the wire establish the so-called is the Hall electric field [7],

$$\mathbf{E}^\star_{\mathbf{H}} \approx -\mathbf{v}^\star_d \times \mathbf{B}_0 = v_d B_0\, \hat{\mathbf{x}}, \tag{19}$$

which counteracts the Lorentz force on the drifting, conduction electrons.[4]

If the bar has resistivity ρ, radius a and carries current I, then this current is driven by a longitudinal electric field inside the bar, whose value in the rest frame of the bar is,

$$\mathbf{E}^\star_\parallel = \rho \mathbf{J} = \frac{\rho I}{\pi a^2}\, \hat{\mathbf{z}}, \tag{20}$$

which is the same as the lab-frame field (8) when $v \ll c$.

In greater detail, the motional field (16) leads to charge accumulations on the surface of the wire as needed to establish the longitudinal fields that drive the current. Inside the bar, the field $\mathbf{E}^\star_q$ due to this charge accumulation opposes the motional field (16) and alters the longitudinal field to strength (20). Hence,

$$\mathbf{E}^\star_q = \mathbf{E}^\star_\parallel - \mathbf{E}^\star_0 = \left(\frac{\rho I}{\pi a^2} - v B_0\right) \hat{\mathbf{z}}. \tag{21}$$

Of course, the line integral of the longitudinal electric field around the circuit remains $\mathcal{E} = v l B_0$.

2.5 Poynting Flux at the Surface of the Bar in its Rest Frame

The (instantaneous) Poynting vector, $\mathbf{S}^\star = \mathbf{E}^\star \times \mathbf{B}^\star/\mu_0$ just inside the surface of the bar, in its rest frame, depends on the (instantaneous) fields,

$$\mathbf{E}^\star(r = a^-) = \mathbf{E}^\star_\parallel + \mathbf{E}^\star_{\mathrm{H}} \approx \frac{\rho I}{\pi a^2}\, \hat{\mathbf{z}} + v_d B_0\, \hat{\mathbf{x}}, \tag{22}$$

and,

$$\mathbf{B}^\star(r = a^-) = \mathbf{B}^\star_0 + \mathbf{B}^\star_\phi \approx B_0\, \hat{\mathbf{y}} + \frac{\mu_0 I}{2\pi a}\, \hat{\boldsymbol{\phi}}. \tag{23}$$

Thus,

$$\begin{aligned} \mathbf{S}^\star(r = a^-) &= \frac{(\mathbf{E}^\star_\parallel + \mathbf{E}^\star_{\mathrm{H}} \times (\mathbf{B}^\star_0 + \mathbf{B}^\star_\phi)}{\mu_0} \\ &= \frac{\mathbf{E}^\star_\parallel \times \mathbf{B}^\star_0}{\mu_0} + \frac{\mathbf{E}^\star_{\mathrm{H}} \times \mathbf{B}^\star_0}{\mu_0} + \frac{\mathbf{E}^\star_\parallel \times \mathbf{B}^\star_\phi}{\mu_0} + \frac{\mathbf{E}_{\mathrm{H}} \times \mathbf{B}^\star_\phi}{\mu_0} \\ &= -\frac{\rho I B_0}{\mu_0 \pi a^2}\, \hat{\mathbf{x}} - \frac{v_d B_0^2}{\mu_0}\, \hat{\mathbf{y}} - \frac{\rho I^2}{2\pi^2 a^3}\, \hat{\mathbf{r}} - \frac{v_d I B_0 \cos\phi}{2\pi a}\, \hat{\mathbf{z}}, \end{aligned} \tag{24}$$

[4]See also [8, 9]. See [10] for a discussion of the relation between the Hall field (19) and the Lorentz force (9) felt by the lattice ions of the sliding bar.

noting that $\hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{r}} - \sin\phi\,\hat{\boldsymbol{\phi}}$. Only the third term of eq. (24) leads to net (instantaneous) electromagnetic power flow across the surface of the sliding bar,

$$P^{\star}_{\text{leaving bar}} = \oint_{\text{bar}} \mathbf{S}\cdot d\mathbf{Area} = -\frac{\rho l I^2}{\pi a^2} = -I^2 R_{\text{bar}}. \tag{25}$$

That is, (instantaneous) power $I^2 R_{\text{bar}}$ flows into the bar in its rest frame, which power is consumed by Joule heating of the bar.

As seen in sec. 2.2, the power dissipated in the circuit appears to be provided in the lab frame by the mechanical forces on the sliding bar. However, the result of the analysis in the rest frame of the sliding bar (this section) indicates that the sliding bar is a sink, rather than a source, of power. We infer that in the rest frame of the bar, the power appears to be provided by the forces on the magnet (which is in motion in this frame) that produces the field $\mathbf{B}_0^\star$.[5]

2.6 Lab-Frame Electric and Magnetic Fields at the Surface of the Sliding Bar

We new return to the lab frame, where the task now to describe, via the Poynting vector (1), the flow of power (10) from the sliding bar through the electromagnetic field to the resistive load. For this, we need to know the total electric field $\mathbf{E}$ and total magnetic field $\mathbf{H} = \mathbf{B}/\mu_0$ at the surface of the bar to calculate $\mathbf{S} = \mathbf{E}\times\mathbf{H} = \mathbf{E}\times\mathbf{B}/\mu_0$.

Thus far, the only fields identified in the lab frame are the external magnetic field $\mathbf{B}_0$, given by eq. (2), and the axial electric field $\mathbf{E}_\parallel$ inside the sliding bar, given by (8). Other electric and magnetic fields have been identified in the rest frame of the sliding bar, and must be transformed to the lab frame.

Because the speed v of the sliding bar is much less than c, the magnetic field $\mathbf{B}_\phi$ appears in the lab frame as the magnetic field $\mathbf{B}_\phi \approx \mathbf{B}_\phi^\star$. In the lab frame, $\mathbf{B}_\phi$ is associated with the electric field,[6]

$$\mathbf{E}_v \approx -\mathbf{v}\times\mathbf{B}_\phi^\star = -\mathbf{v}\times\frac{\mu_0 I}{2\pi a}\,\hat{\boldsymbol{\phi}} = -v\,\hat{\mathbf{x}}\times\frac{\mu_0 I}{2\pi a}(-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}) = -\frac{\mu_0 v I\cos\phi}{2\pi a}\,\hat{\mathbf{z}}. \tag{26}$$

Similarly, the electric field $\mathbf{E}^\star = \mathbf{E}_\parallel^\star + \mathbf{E}_H^\star$ at the surface of the sliding bar in its rest frame transforms to the lab-frame electric field,

$$\mathbf{E}_\parallel + \mathbf{E}_H \approx \mathbf{E}_\parallel^\star + \mathbf{E}_H^\star = \frac{\rho I}{\pi a^2}\,\hat{\mathbf{z}} - v_d B_0\,\hat{\mathbf{x}}, \tag{27}$$

as well as to the lab-frame magnetic field,

$$\mathbf{B}_{\mathbf{v}} \approx \mathbf{v}/c^2\times(\mathbf{E}_\parallel^\star + \mathbf{E}_H^\star) = -\frac{\rho v I}{\pi a^2 c^2}\,\hat{\mathbf{y}}. \tag{28}$$

The magnetic field $\mathbf{B}_v$ is of order $1/c^2$, so we ignore it in the same approximation as taking γ to be 1.

[5] This is an illustration of the relativity of energy flow, also discussed in [11].

[6] The field $\mathbf{E}_v$ is also equal to $-\partial\mathbf{A}/\partial t$, where $\mathbf{A}$ is the time-dependent vector potential of the moving magnetic field, $\mathbf{B}_\phi$.

The total electric field in the lab frame at the surface of the sliding bar is (neglecting terms of order $1/c^2$),

$$\mathbf{E} = \mathbf{E}_\parallel + \mathbf{E}_H + \mathbf{E}_v \approx -v_d B_0\,\hat{\mathbf{x}} + \left(\frac{\rho I}{\pi a^2} - \frac{\mu_0 v I\cos\phi}{2\pi a}\right)\hat{\mathbf{z}}, \tag{29}$$

and the magnetic field is,

$$\mathbf{B} \approx \mathbf{B}_0 + \mathbf{B}_\phi \approx \left(B_0 - \frac{\rho v I}{\pi a^2 c^2}\right)\hat{\mathbf{y}} + \frac{\mu_0 I}{2\pi a}\,\hat{\boldsymbol{\phi}}. \tag{30}$$

2.7 Poynting Vector at the Surface of the Sliding Bar in the Lab Frame

The (instantaneous) Poynting vector of fields (29) and (30) at the surface of the sliding bar contains four terms,

$$\begin{aligned} \mathbf{S}(r = a^-) &= \frac{\mathbf{E}\times\mathbf{B}}{\mu_0} \qquad (31)\\ &= -\frac{v_d B_0}{\mu_0}\left(B_0 - \frac{\rho v I}{\pi a^2 c^2}\right)\hat{\mathbf{z}} - \left(\frac{\rho I}{\pi a^2} - \frac{\mu_0 v I\cos\phi}{2\pi a}\right)\left(B_0 - \frac{\rho v I}{\pi a^2 c^2}\right)\frac{\hat{\mathbf{x}}}{\mu_0} \\ &\quad -\frac{v_d I B_0}{2\pi a}\cos\phi\,\hat{\mathbf{r}} - \left(\frac{\rho I}{\pi a^2} - \frac{\mu_0 v I\cos\phi}{2\pi a}\right)\frac{I}{2\pi a}\,\hat{\mathbf{r}}, \end{aligned}$$

noting that $\hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{r}} - \sin\phi\,\hat{\boldsymbol{\phi}}$. We are interested in the integral of radial Poynting flux at the surface of the bar, to which only terms even in $\cos\phi$ will contribute,

$$S(r = a^-)_{r,\mathrm{even}} = \frac{vI\cos^2\phi}{2\pi a}\left(B_0 - \frac{\rho v I}{\pi a^2 c^2}\right) - \frac{\rho I^2}{2\pi^2 a^3}, \tag{32}$$

$$P_{\mathrm{leaving\ \ bar}} = \oint_{\mathrm{bar}} \mathbf{S}\cdot d\mathbf{Area} = \frac{vlI}{2}\left(B_0 - \frac{\rho v I}{\pi a^2 c^2}\right) - \frac{\rho l I^2}{\pi a^2} = \frac{\mathcal{E}I}{2} - I^2 R_{\mathrm{bar}}\left(1 + \frac{v^2}{2c^2}\right). \tag{33}$$

Thus far, we have accounted for only one half of the (instantaneous) power, $\mathcal{E}I$, consumed in the circuit as emanating from the sliding bar.

References

[1] J.H. Poynting, *On the Transfer of Energy in the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **175**, 343 (1884),
http://kirkmcd.princeton.edu/examples/EM/poynting_ptrsl_175_343_84.pdf

[2] M. Heald, *Electric fields and charges in elementary circuits*, Am. J. Phys. **52**, 522 (1984),
http://kirkmcd.princeton.edu/examples/EM/heald_ajp_52_522_84.pdf

[3] R. Sigelmann, *Fundamentals of Tuning Fork Timepieces*,
http://kirkmcd.princeton.edu/examples/EM/sigelmann_tuning_fork_timepieces.pdf

[4] D.J. Griffiths, *Introduction to Electrodynamics*, 3rd ed. (Prentice Hall, 1999), http://kirkmcd.princeton.edu/examples/EM/griffiths_em3.pdf

[5] K.T. McDonald, *Is Faraday's Disk Dynamo a Flux-Rule Exception?* (July 27, 2019), http://kirkmcd.princeton.edu/examples/faradaydisk.pdf

[6] K.T. McDonald, *Charge Density in a Current-Carrying Wire* (Dec. 23, 2010), http://kirkmcd.princeton.edu/examples/wire.pdf

[7] E.H. Hall, *On a New Action of the Magnet on Electric Currents*, Am. J. Math. **2**, 287 (1879), http://kirkmcd.princeton.edu/examples/EM/hall_ajm_2_287_79.pdf

[8] A.C. English, *Force on a Wire in a Magnetic Field*, Am. J. Phys. **35**, 326 (1967), http://kirkmcd.princeton.edu/examples/EM/english_ajp_35_326_67.pdf

[9] E.P. Mosca, *Magnetic Forces Doing Work?* Am. J. Phys. **42**, 295 (1974), http://kirkmcd.princeton.edu/examples/EM/mosca_ajp_42_295_74.pdf

[10] W.R. McKinnon, S.P. McAlister and C.M. Hurd, *Origin of the force on a current-carrying wire in a magnetic field*, Am. J. Phys. **49**, 493 (1981), http://kirkmcd.princeton.edu/examples/EM/mckinnon_ajp_49_493_81.pdf

[11] K.T. McDonald, *Relativity of Steady Energy Flow* (Nov. 3, 2007), http://kirkmcd.princeton.edu/examples/1dgas.pdf

[12] K.T. McDonald, *Perfectly Conducting Cylinder in an External, Static Magnetic Field* (Oct. 29, 2011), http://kirkmcd.princeton.edu/examples/perfect_cylinder.pdf