Radiation from Hertzian Dipoles in a Uniaxial Anisotropic Medium

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1 Problem

The propagation of plane electromagnetic waves in a (uniaxial) anisotropic linear dielectric medium is associated with the phenomenon of double refraction.[1] Consider instead the propagation of waves in such a medium where the source is an idealized Hertzian (point) electric dipole of moment $\mathbf{p}\, e^{-i\omega t}$, or a Hertzian magnetic dipole of moment $\mathbf{m}\, e^{-i\omega t}$. It suffices to consider a medium with unit (relative) permeability and (relative) symmetric dielectric tensor ϵ_{ij} that is diagonal with respect to the axes (x, y, z), such that the electric displacement vector $\mathbf{D}$ and electric field vector $\mathbf{E}$ are related by

$$D_i = \epsilon_i E_i, \qquad i = x, y, z, \tag{1}$$

in Gaussian units. In general, the three dielectric constants ϵ_x, ϵ_x and ϵ_z are all different, but restrict your discussion to the case of a so-called uniaxial medium for which $\epsilon_x = \epsilon_y \equiv \epsilon$.

A Hertzian dipole of arbitrary orientation emits waves with velocities that depend of direction, with extremes of $c/\sqrt{\epsilon}$ and $c/\sqrt{\epsilon_z}$, where c is the speed of light in vacuum. Show that for Hertzian dipoles oriented along the z axis, a magnetic dipole emits ordinary transverse radiation whose wavefronts are spherical, while an electric dipole emits extraordinary radiation for which the wavefronts in the far zone are spheroidal but the Poynting vector is radial (so that at large distances from the source the electric field $\mathbf{E}$ is transverse to the radial direction while the displacement field $\mathbf{D}$ has a longitudinal component).[2]

In practice, the media in which antennas reside are isotropic to an excellent approximation, so the results of this problem are mainly of pedagogic interest. An exception is the Earth's ionosphere; see, for example, [4] and references therein.

2 Solution

Following a brief introduction to Maxwell's equations for a uniaxial medium, we first consider plane waves in sec. 2.1, followed by solutions to Maxwell's equations for point sources in secs. 2.2-4. The methods used in secs. 2.1-4 largely depend on the local properties of the medium, and so can be generalized to the case of nonuniform media. In the Appendix we review a method of scaling vacuum solutions to Maxwell's equations to ones for a uniform uniaxial medium.

[1] See, for example, chap. XI of [1] or chap. XIV of [2].

[2] Wave energy from a localized source flows in straight lines in any uniform (homogeneous) medium, even when that medium is anisotropic [3].

Maxwell's equations for the electromagnetic fields in an anisotropic linear medium for which permeability $\mu = 1$ are (in Gaussian units),

$$\boldsymbol{\nabla} \cdot \mathbf{D} = 4\pi\varrho_{\rm free}, \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0, \qquad \boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \boldsymbol{\nabla} \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{J}_{\rm free}, \quad (2)$$

where $\varrho_{\rm free}$ and $\mathbf{J}_{\rm free}$ are the free-charge and -current densities, respectively. The electric fields $\mathbf{D}$ and $\mathbf{E}$ are related by the constituent equations (1).

The second and third Maxwell equations are unaffected by the anisotropy of the medium, so as usual the fields $\mathbf{B}$ and $\mathbf{E}$ can be related to a scalar potential V and a vector potential $\mathbf{A}$ according to,

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \qquad \mathbf{E} = -\boldsymbol{\nabla} V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}. \quad (3)$$

For sources $\varrho_{\rm free}$ and $\mathbf{J}_{\rm free}$ with a pure sinusoidal time dependence $e^{-i\omega t}$ at frequency ω, the displacement field $\mathbf{D}$ outside the source region can be related to the local magnetic field $\mathbf{B}$ via the fourth Maxwell equation according to,

$$\mathbf{D} = \frac{ic}{\omega}\boldsymbol{\nabla} \times \mathbf{B} = \frac{ic}{\omega}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}). \quad (4)$$

Hence, all three fields $\mathbf{B}$, $\mathbf{D}$ and $\mathbf{E}$ can be deduced from the vector potential $\mathbf{A}$ outside the source region.

Once we have the electromagnetic fields, we can characterize the flow of electromagnetic energy by the (time-average) Poynting vector,

$$\langle \mathbf{S} \rangle = \frac{c}{8\pi} Re(\mathbf{E} \times \mathbf{H}^\star) = \frac{c}{8\pi} Re(\mathbf{E} \times \mathbf{B}^\star), \quad (5)$$

where the latter form holds for media with unit permeability. Thus, the fields $\mathbf{E}$ and $\mathbf{H}$ are always transverse to the direction of the flow of energy, whether or not this direction is that from the source to the observer. In an anisotropic medium there can be both ordinary radiation in which the direction of energy flow is along the line from the source to the observer, as well as extraordinary radiation for which these two directions might be different.

2.1 Plane Waves in a Uniaxial Medium

In any small region of the far zone of the Hertzian dipoles, the fields are plane waves to a good approximation. Hence, it is useful to consider the behavior of plane waves in a uniaxial medium before addressing the full complexity of the 3-dimensional fields.[3]

The spacetime dependence of a plane wave of angular frequency ω will be written as,

$$e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}, \quad (6)$$

where the wave vector $\mathbf{k}$ is also written as,

$$\mathbf{k} = \frac{\omega}{c}\mathbf{n}, \quad (7)$$

[3]This section largely follows chap. XI of [1], but the discussion of the velocity of wave energy is adapted from chap. XIV of [2].

where $n = |\mathbf{n}|$. The phase velocity v_p of the plane wave is,

$$v_p = \frac{c}{n}\,. \tag{8}$$

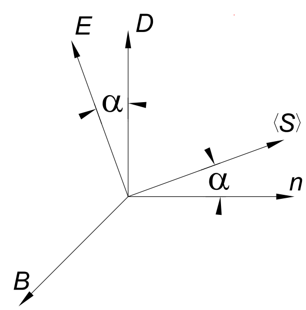
In an isotropic medium with dielectric constant ϵ, we have $\mathbf{n} = n\,\hat{\mathbf{k}}$ where $n = \sqrt{\epsilon}$ is the index of refraction.

For a plane wave (6) far from its sources, Maxwell's equations (2) imply that,

$$\mathbf{n}\cdot\mathbf{D} = 0, \qquad \mathbf{n}\cdot\mathbf{B} = 0, \qquad \mathbf{n}\times\mathbf{E} = \mathbf{B}, \qquad \mathbf{n}\times\mathbf{B} = -\mathbf{D}. \tag{9}$$

Thus, in the far zone the three vectors $\mathbf{n}$, $\mathbf{B}$ and $\mathbf{D}$ are mutually orthogonal. Vectors $\mathbf{B}$ and $\mathbf{E}$ are orthogonal, but in general the electric field $\mathbf{E}$ has a component along the direction $\mathbf{n}$ of the wave vector.

The Poynting vector $\langle\mathbf{S}\rangle = (c/8\pi)Re(\mathbf{E}\times\mathbf{B}^\star)$ is in general not parallel to the vector $\mathbf{n}$. However, the three vectors $\mathbf{B}$, $\mathbf{E}$ and $\langle\mathbf{S}\rangle$ are mutually orthogonal. The two mutually orthogonal triads, $\{\mathbf{n},\mathbf{B},\mathbf{D}\}$ and $\{\mathbf{B},\mathbf{E},\langle\mathbf{S}\rangle\}$, each include the magnetic field $\mathbf{B}$, so the four vectors $\mathbf{D}$, $\mathbf{E}$, $\mathbf{n}$ and $\langle\mathbf{S}\rangle$ all lie in a plane perpendicular to $\mathbf{B}$, as shown in the figure below.



The electric fields $\mathbf{D}$ and $\mathbf{E}$ are related by the 3rd and 4th equations of (9) according to,

$$\mathbf{D} = -\mathbf{n}\times(\mathbf{n}\times\mathbf{E}) = n^2\mathbf{E} - (\mathbf{n}\cdot\mathbf{E})\,\mathbf{n} = n^2[\mathbf{E} - (\hat{\mathbf{n}}\cdot\mathbf{E})\,\hat{\mathbf{n}}] = n^2 E\cos\alpha\,\hat{\mathbf{D}}, \tag{10}$$

where α is the angle between $\mathbf{n}$ and $\mathbf{S}$ as well as that between $\mathbf{D}$ and $\mathbf{E}$. These fields are also related by the constituent eq. (1), which can be combined with the second form of eq. (10) to give three scalar equations, expressible in matrix form as,

$$\begin{pmatrix} n^2 - n_x^2 - \epsilon & -n_x n_y & -n_x n_z \\ -n_x n_y & n^2 - n_y^2 - \epsilon & -n_y n_z \\ -n_x n_z & -n_y n_z & n^2 - n_z^2 - \epsilon_z \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{11}$$

For a solution to exist, the determinant of the matrix must vanish, which after some algebra leads to the factorizable quartic equation,

$$(n^2 - \epsilon)[\epsilon(n_x^2 + n_y^2) + \epsilon_z n_z^2 - \epsilon\epsilon_z] = 0. \tag{12}$$

Thus, there are two classes of waves,

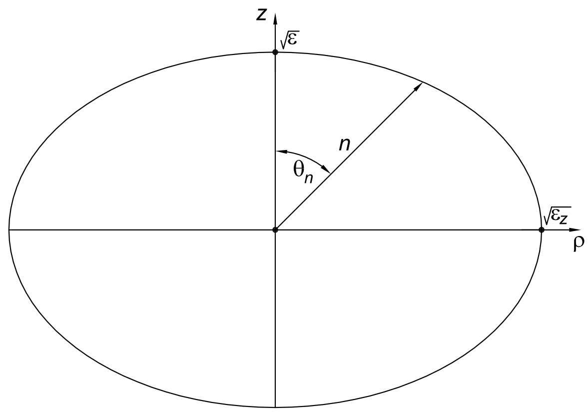
$$n^2 = \epsilon \qquad \text{(ordinary)}, \tag{13}$$

$$\frac{n_x^2 + n_y^2}{\epsilon_z} + \frac{n_z^2}{\epsilon} = \frac{n_\rho^2}{\epsilon_z} + \frac{n_z^2}{\epsilon} = n^2\left(\frac{\sin^2\theta_n}{\epsilon_z} + \frac{\cos^2\theta_n}{\epsilon}\right) = 1 \qquad \text{(extraordinary)}, \tag{14}$$

introducing cylindrical and spherical coordinate systems, (ρ, ϕ, z) and (r, θ, ϕ), in eq. (14).

For ordinary waves the magnitude of the wave vector $\mathbf{n}$ is $\sqrt{\epsilon}$, and the phase velocity, $v_p = c/n = c/\sqrt{\epsilon}$, is the same in all directions, so that a point source leads to ordinary spherical radiation.

For extraordinary waves the magnitude n of the wave vector $\mathbf{n}$ depends on its direction according to eq. (14), as sketched below in cylindrical coordinates for a uniaxial medium where $\epsilon_z > \epsilon$. We also learn that vectors $\mathbf{n}$, $\mathbf{D}$, $\mathbf{E}$ and $\langle \mathbf{S} \rangle$ all lie in the ρ-z plane for extraordinary waves.



In particular, an extraordinary plane wave that propagates in the z direction does so with phase velocity $v_{p,z} = c/n_z(n_\rho = 0) = c/\sqrt{\epsilon}$ (*i.e.*, with the same phase velocity as an ordinary wave), while an extraordinary plane wave that propagates in the ρ direction does so with phase velocity $v_{p,\rho} = c/n_\rho(n_z = 0) = c/\sqrt{\epsilon_z}$.[4]

If a point source at the origin generates an extraordinary wave, a wavefront whose normal is in the z direction will be at position $(\rho, \phi, z) = (0, 0, ct/\sqrt{\epsilon})$ at time t after that wavefront was emitted, while at the same time a wavefront whose normal is in the ρ direction will be at position $(ct/\sqrt{\epsilon_z}, \phi, 0)$. Accepting that for a point source at the origin an extraordinary wavefront is a quadratic surface, its form must therefore be,

$$\frac{\rho^2}{c^2t^2/\epsilon_z} + \frac{z^2}{c^2t^2/\epsilon} = 1, \qquad \text{or} \qquad \frac{\rho^2}{\epsilon} + \frac{z^2}{\epsilon_z} = \frac{c^2t^2}{\epsilon\epsilon_z} = K. \tag{15}$$

The figure on the next page sketches a set of extraordinary wavefront surfaces for a point source at the origin inside a uniaxial medium with $\epsilon_z > \epsilon$. Also shown are representative wavefront normal "rays", which, however, are not radial.

A related question is direction and velocity of propagation of energy in the plane wave. The direction of energy flow is, of course, that of the (time-average) Poynting vector $\hat{\mathbf{S}}$,

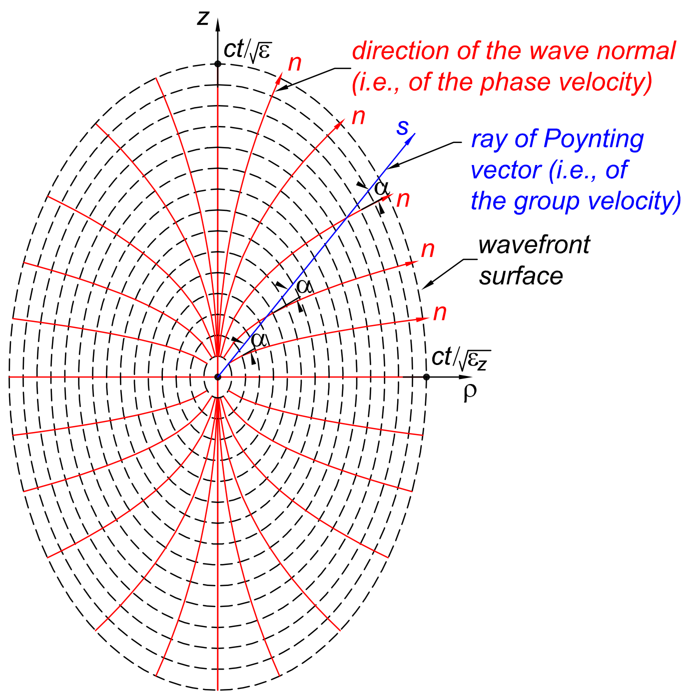
$$\langle \mathbf{S} \rangle = \frac{c}{8\pi} Re(\mathbf{E} \times \mathbf{B}^\star) \equiv \langle S \rangle \; \hat{\mathbf{S}}. \tag{16}$$

The velocity v_E of energy flow is taken to be the rate of energy flow divided by the energy

[4]For these two extremes cases the normal to the wavefront is clearly radial, so the electric field is transverse. Then, for propagation in the z direction the relevant electric fields are $D_\rho = \epsilon E_\rho$, and the relevant index of refraction is $n_z = \sqrt{\epsilon}$. Similarly, for propagation in the ρ direction the relevant electric fields are $D_z = \epsilon_z E_z$, and the relevant index of refraction is $n_\rho = \sqrt{\epsilon_z}$.

density,

$$v_E = \frac{\langle S \rangle}{\langle u \rangle}\,, \tag{17}$$



where the time-average energy density $\langle u \rangle$ in the wave is given by,

$$\begin{aligned} \langle u \rangle &= \frac{Re[\mathbf{E}\cdot\mathbf{D}^\star + \mathbf{B}\cdot\mathbf{H}^\star]}{16\pi} = \frac{Re[\mathbf{E}\cdot(\mathbf{B}^\star\times\mathbf{n}) + (\mathbf{n}\times\mathbf{E})\cdot\mathbf{B}^\star]}{16\pi} = \frac{Re[\mathbf{E}\times\mathbf{B}^\star\cdot\mathbf{n})}{8\pi} \\ &= \frac{\langle S \rangle}{c}\hat{\mathbf{S}}\cdot\mathbf{n} = \frac{\langle S \rangle}{c} n\cos\alpha, \end{aligned} \tag{18}$$

and α is the angle between $\mathbf{n}$ and $\mathbf{S}$. Thus,

$$v_E = \frac{c}{n\cos\alpha} = \frac{v_p}{\cos\alpha}\,. \tag{19}$$

An amusing and instructive game is to define a vector $\mathbf{s}$ that points in the direction of energy flow,

$$\mathbf{s} \equiv \frac{\hat{\mathbf{S}}}{n\cos\alpha}\,, \qquad \text{so that} \qquad \mathbf{n}\cdot\mathbf{s} = ns\cos\alpha = 1, \tag{20}$$

where $s = |\mathbf{s}|$. This permits an analysis for the vector $\mathbf{s}$ that is very similar to that of eqs. (10)-(14) for the vector $\mathbf{n}$. Namely,

$$s^2[\mathbf{D} - (\hat{\mathbf{S}}\cdot\mathbf{D})\,\hat{\mathbf{S}}] = s^2 D\cos\alpha\,\hat{\mathbf{E}} = s^2 n^2 E\cos^2\alpha\,\hat{\mathbf{E}} = \mathbf{E}. \tag{21}$$

Using eq. (1) together with eq. (21) we obtain the matrix equation,

$$\begin{pmatrix} s^2 - s_x^2 - 1/\epsilon & -s_x s_y & -s_x s_z \\ -s_x s_y & s^2 - s_y^2 - 1/\epsilon & -s_y s_z \\ -s_x s_z & -s_y s_z & s^2 - s_z^2 - 1/\epsilon_z \end{pmatrix} \begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{22}$$

Comparing with eq. (12), the vanishing of the determinant of the matrix leads to ,

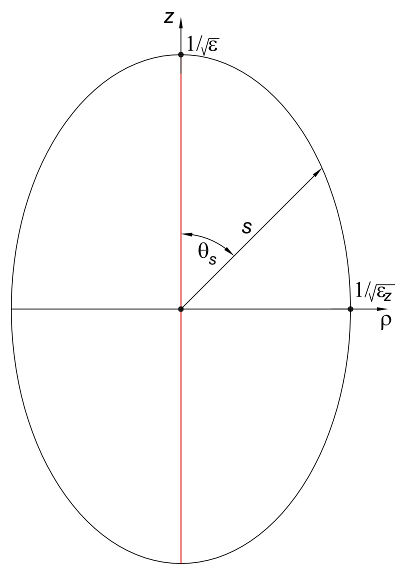
$$(s^2 - 1/\epsilon)[(s_x^2 + s_y^2)/\epsilon + s_z^2/\epsilon_z - 1/\epsilon\epsilon_z] = 0. \tag{23}$$

Thus, the vector $\mathbf{s}$ obeys,

$$s^2 = 1/\epsilon \qquad \text{(ordinary)}, \tag{24}$$

$$\epsilon_z(s_x^2 + s_y^2) + \epsilon s_z^2 = \epsilon_z s_\rho^2 + \epsilon s_z^2 = s^2(\epsilon_z \sin^2\theta_s + \epsilon\cos^2\theta_s) = 1 \qquad \text{(extraordinary)}. \tag{25}$$

For extraordinary waves the magnitude s of the vector $\mathbf{s}$ depends on its direction according to eq. (25), as sketched below for a uniaxial medium where $\epsilon_z > \epsilon$.



We note that the vectors $\mathbf{n}$ and $\mathbf{s}$ are dual in the sense that $\mathbf{s}$ is normal to the surface (15) on which $\mathbf{n}$ lies, while $\mathbf{n}$ is normal to the surface (25) on which $\mathbf{s}$ lies.[5] For example, writing the surface (25) as $f(\rho, z) = 0$ for $f = \epsilon_z s_\rho^2 + \epsilon s_z^2 - 1$, the gradient,

$$\boldsymbol{\nabla} f = (\nabla f_\rho, \nabla f_\phi, \nabla f_z) = (2\epsilon_z s_\rho, 0, 2\epsilon s_z) \tag{26}$$

obeys,

$$\frac{(\nabla f_\rho/2)^2}{\epsilon_z} + \frac{(\nabla f_z/2)^2}{\epsilon} = 1 \tag{27}$$

on the surface $f = 0$, so that

$$\boldsymbol{\nabla} f/2 = \mathbf{n}. \tag{28}$$

For extraordinary waves the angles θ_n and θ_s differ by α. Another relation between θ_n and θ_s in this case can be obtained from eqs. (26) and (28) noting that $\tan\theta_n = n_\rho/n_z$,

$$\epsilon_z s_\rho = n_\rho, \qquad \epsilon s_z = n_z, \qquad \text{so that} \qquad \tan\theta_s = \frac{s_\rho}{s_z} = \frac{\epsilon n_\rho}{\epsilon_z n_z} = \frac{\epsilon}{\epsilon_z}\tan\theta_n. \tag{29}$$

[5]Another aspect of this duality is that energy-flow vector $\mathbf{s}$ is parallel to the group velocity $\mathbf{v}_g$. Any medium for which the wave velocity is not c must have dispersion (see footnote 2 of [5]), *i.e.*, $\omega = \omega(\mathbf{k})$. Writing the wave-vector surface (13) or (14) as $F(\omega, \mathbf{k}) = 0$, the group velocity is given by, $\mathbf{v}_g = \partial\omega/\partial\mathbf{k} = (\partial F/\partial\mathbf{k})/(\partial F/\partial\omega) = \boldsymbol{\nabla} F/(\partial F/\partial\omega) \propto \mathbf{s}$.

If an extraordinary wave is generated by a point source at the origin, then the wavefront surfaces have the form (15), so at $(\rho, 0, z)$ the vector $\mathbf{n}$ is parallel to the gradient of the surface $g(\rho, z) = \rho^2/\epsilon + z^2/\epsilon_z - K$,

$$\mathbf{n} \propto \left(\frac{\rho}{\epsilon}, 0, \frac{z}{\epsilon_z}\right), \qquad \text{and} \qquad \mathbf{n} = \frac{(\epsilon_z \rho, 0, \epsilon z)}{\sqrt{\epsilon_z \rho^2 + \epsilon z^2}} = \frac{(\epsilon_z \sin\theta, 0, \epsilon\cos\theta)}{\sqrt{\epsilon_z \sin^2\theta + \epsilon\cos^2\theta}}, \tag{30}$$

using eq. (14). Then, by eq. (29),

$$\tan\theta_s = \frac{\epsilon}{\epsilon_z}\frac{\epsilon_z \rho}{\epsilon z} = \frac{\rho}{z} = \tan\theta, \tag{31}$$

where θ is the polar angle of the point $(\rho, 0, z)$. That is, $\theta_s = \theta$, and the energy flow vector $\mathbf{s}$ is radial. The figure on p. 6 shows one radial line of the Poynting vector, which can be seen to make the same angle α to three different lines of the wave vector $\mathbf{n}$.

In the extraordinary wave, the electric field lies in the plane of vectors $\mathbf{n}$ and $\mathbf{s}$, and is therefore polarized in the θ direction (transverse to $\mathbf{r}$) for the example shown on p. 6.

An ordinary wave generated by a point source at the origin must have (transverse) electric polarization orthogonal to that of the extraordinary wave, so the ordinary wave has electric field $\mathbf{E}$ in the ϕ direction.

2.2 Wave Equation for the Vector Potential

Returning to the problem of the 3-dimensional fields generated in a uniaxial medium by Hertzian dipoles, we note that a wave equation for the vector potential $\mathbf{A}$ can be obtained from the fourth Maxwell equation (2), also using eqs. (1), (3) and the third Maxwell equation,

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{B}|_j &= [\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A})]_j = \frac{\partial}{\partial x_j}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2 A_j = \frac{1}{c}\frac{\partial}{\partial t}\epsilon_j E_j + \frac{4\pi}{c} J_j \\ &= \frac{\epsilon_j}{c}\frac{\partial}{\partial t}\left(-\frac{\partial V}{\partial x_j} - \frac{1}{c}\frac{\partial A_j}{\partial t}\right) + \frac{4\pi}{c} J_j = -\frac{\epsilon_j}{c}\frac{\partial^2 V}{\partial x_j \partial t} - \frac{\epsilon_j}{c^2}\frac{\partial^2 A_j}{\partial t^2} + \frac{4\pi}{c} J_j, \end{aligned} \tag{32}$$

and hence,

$$\nabla^2 A_j - \frac{\epsilon_j}{c^2}\frac{\partial^2 A_j}{\partial t^2} = -\frac{4\pi}{c} J_j + \frac{\partial}{\partial x_j}\left(\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{\epsilon_j}{c}\frac{\partial V}{\partial t}\right). \tag{33}$$

The usual procedure at this step is to take advantage of the arbitrariness of the potentials (*i.e.*, their gauge invariance) to set the last term in eq. (33) to zero. However, because the dielectric constants ϵ_j have two different values for a uniaxial medium, we have two choices:

$$\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{\epsilon}{c}\frac{\partial V}{\partial t} = 0, \qquad \text{or} \qquad \boldsymbol{\nabla} \cdot \mathbf{A} + \frac{\epsilon_z}{c}\frac{\partial V}{\partial t} = 0, \tag{34}$$

which will turn out to correspond to transverse magnetic and transverse electric waves, respectively, where the nominal transverse field has only a ϕ component. For Hertzian dipoles oriented along the z axis the transverse electric waves are ordinary, while the transverse magnetic waves are extraordinary.[6]

[6]In a transverse magnetic wave only the magnetic field need be transverse to the direction between source and observer. Such waves could have a component of the electric field that is longitudinal with respect to this direction, even though the electric field is transverse to the direction of energy flow.

2.3 Transverse Magnetic Waves

Our solution for the transverse magnetic waves is a simplified version of that given in [6] for dipole antennas in stratified uniaxial media, with approximations to the Fourier integrals following [7]. For solutions based on the use of dyadic Green functions, see [9, 10].

We first set,

$$\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{\epsilon}{c}\frac{\partial V}{\partial t} = 0. \tag{35}$$

This choice is particularly appropriate if the current density has only a z component, as in this case the x and y components of eq. (33) admit the trivial solution $A_x = A_y = 0$. Then,

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{\partial A_z}{\partial z} = -\frac{\epsilon}{c}\frac{\partial V}{\partial t}\,, \tag{36}$$

so that,

$$\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{\epsilon_z}{c}\frac{\partial V}{\partial t} = \frac{\partial A_z}{\partial z}\left(1 - \frac{\epsilon_z}{\epsilon}\right), \tag{37}$$

and the only nontrivial component of eq. (33) becomes,

$$\nabla_\perp^2 A_z + \frac{\epsilon_z}{\epsilon}\frac{\partial^2 A_z}{\partial z^2} - \frac{\epsilon_z}{c^2}\frac{\partial^2 A_z}{\partial t^2} = -\frac{4\pi}{c}J_z, \tag{38}$$

where,

$$\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}. \tag{39}$$

This suggests that the solution will be found most easily in a cylindrical coordinate system (ρ, ϕ, z).[7]

The source current J_z has axial symmetry, so we expect that lines of magnetic field circle around the z axis, and the solution can be characterized as transverse magnetic (TM).

The vector potential will have the same time dependence, $e^{-i\omega t}$, as the drive currents of the Hertzian electric dipole, $pe^{-i\omega t}\,\hat{\mathbf{z}}$, and the axial symmetry of the source implies that the vector potential has this symmetry also. Thus, in cylindrical coordinates we can write,

$$\mathbf{A} = A_z(\rho, z)\, e^{-i\omega t}\,\hat{\mathbf{z}}. \tag{40}$$

Using eq. (40) in the wave equation (38), we obtain,

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial A_z}{\partial \rho}\right) + \frac{\epsilon_z}{\epsilon}\frac{\partial^2 A_z}{\partial z^2} + \frac{\epsilon_z \omega^2}{c^2}A_z = -\frac{4\pi}{c}J_z. \tag{41}$$

We seek solutions to the homogeneous, linear partial differential equation that are sum/integrals of terms of the form,

$$A_z(\rho, z) = R(\rho)Z(z), \tag{42}$$

for which eq. (41) implies that,

$$\frac{1}{\rho R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{\epsilon_z}{\epsilon}\frac{Z''}{Z} = -\frac{\epsilon_z}{\epsilon}k^2, \tag{43}$$

[7] The discussion of sec. 2.1 also suggests merit in the use of spheroidal coordinates. See, for example, [8].

where,

$$k = \frac{\sqrt{\epsilon}\omega}{c}. \tag{44}$$

We introduce a separation constant κ_z (which is not $\sqrt{\epsilon_z}\omega/c$) to write,

$$Z'' = -\kappa_z^2 Z, \tag{45}$$

so that,

$$Z = e^{\pm i\kappa_z z}. \tag{46}$$

The homogeneous equation (43) now reduces to the radial equation,

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) = -\frac{\epsilon_z}{\epsilon}(k^2 - \kappa_z^2)R \equiv -\kappa_\rho^2 R, \tag{47}$$

where,

$$\kappa_\rho = \pm\sqrt{\frac{\epsilon_z}{\epsilon}}\sqrt{k^2 - \kappa_z^2} \tag{48}$$

is a second separation constant. Equation (47) is a form of Bessel's equation of order 0.

We seek solutions $R(\kappa_\rho \rho)$ that are cylindrical waves for large values of ρ, so that the radial function should approach the form $e^{i\kappa_\rho \rho}/\sqrt{\kappa_\rho \rho}$ (times $e^{-i\omega t}$), where the factor $1/\sqrt{\kappa_\rho \rho}$ is required so that the absolute square of the wave function is consistent with conservation of energy in the expanding cylindrical wave. Therefore, among the various solutions to Bessel's equation, we use the so-called Hankel function $H_0^{(1)}(\kappa_\rho \rho)$ of type 1 and order 0.[8] See, for example, secs. 19 and 21 of [11]. The asymptotic expansion of this function for large $\kappa_\rho \rho$ is (eq. (55), sec. 19 of [11])

$$H_0^{(1)}(\kappa_\rho \rho) \to \sqrt{\frac{-2i}{\pi}}\frac{e^{i\kappa_\rho \rho}}{\sqrt{\kappa_\rho \rho}}. \tag{49}$$

An awkwardness in the use of the function $H_0^{(1)}(\rho)$ is that it has a logarithmic divergence for small ρ. However, we are mainly concerned with the far-zone fields for which the approximation (49) will be used.

There is no constraint on the separation constants κ_ρ and κ_z other than eq. (48), so they take on continuous values. In particular, $-\infty < \kappa_\rho < \infty$. The vector potential A_z can then expressed as a Fourier integral over κ_ρ,

$$A_z(\rho, z) = \int_{-\infty}^{\infty} d\kappa_\rho\, a(\kappa_\rho) H_0^{(1)}(\kappa_\rho \rho)\, e^{i\kappa_z |z|}, \tag{50}$$

where $a(\kappa_\rho)$ are the Fourier coefficients to be determined. The factor $e^{i\kappa_z |z|}$ insures that the waves propagate away from the source located at the origin.

We can make a connection with the isotropic case, $\epsilon_z = \epsilon$, using a particular integral of the Hankel function that relates it to a spherical wave (see eq. (14a) of sec. 31 of [11] or eq. (40) of [6]),

$$\frac{e^{ikr}}{r} = \frac{e^{ik\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}} = \frac{i}{2}\int_{-\infty}^{\infty} d\kappa_\rho\, \frac{\kappa_\rho}{\kappa_z} H_0^{(1)}(\kappa_\rho \rho)\, e^{i\kappa_z |z|}, \tag{51}$$

[8]If the oscillatory time dependence were taken to be $e^{i\omega t}$, as in [9], then we would use the Hankel function $H_0^{(2)}(\kappa_\rho \rho)$.

where $\kappa_z = \sqrt{k^2 - \kappa_\rho^2}$ in eq. (51). For a Hertzian dipole $pe^{-i\omega t}\,\hat{\mathbf{z}}$ in an isotropic dielectric medium it is well known (see, for example, sec. 9.2 of [14]) that the vector potential is given by

$$A_z = -i\frac{kp}{\sqrt{\epsilon}}\frac{e^{ikr}}{r}\,. \tag{52}$$

Thus, setting $a(\kappa_\rho) = kp\kappa_\rho/2\sqrt{\epsilon}\kappa_z$ in eq. (50) gives a representation in cylindrical coordinates of the vector potential (52) when $\epsilon_z = \epsilon$.

We argue that the vector potential can be obtained when ϵ_z does not equal ϵ by using the Fourier coefficient, $a(\kappa_\rho) = \sqrt{\epsilon}kp\kappa_\rho/2\epsilon_z\kappa_z$, where now $\kappa_z = \sqrt{k^2 - \kappa_\rho^2\epsilon/\epsilon_z}$ relates the separation constants κ_ρ and κ_z.[9] Then, the desired vector potential is,

$$A_z(\rho, z) = \frac{\sqrt{\epsilon}kp}{2\epsilon_z}\int_{-\infty}^{\infty} d\kappa_\rho\,\frac{\kappa_\rho}{\kappa_z}H_0^{(1)}(\kappa_\rho\rho)\,e^{i\kappa_z|z|}. \tag{53}$$

Since the vector potential (53) is independent of ϕ, the magnetic field $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ has only a ϕ component,

$$B_\phi = -\frac{\partial A_z}{\partial\rho} = -\frac{\sqrt{\epsilon}kp}{2\epsilon_z}\int_{-\infty}^{\infty} d\kappa_\rho\,\frac{\kappa_\rho^2}{\kappa_z}H_0^{(1)'}(\kappa_\rho\rho)\,e^{i\kappa_z|z|}, \tag{54}$$

where for large values of $\kappa_\rho\rho$,

$$H_0^{(1)'}(\kappa_\rho\rho) = \frac{dH_0^{(1)}(\kappa_\rho\rho)}{d\kappa_\rho\rho} \approx iH_0^{(1)}(\kappa_\rho\rho), \tag{55}$$

recalling eq. (49). The magnetic field is transverse to both directions $\hat{\boldsymbol{\rho}}$ and $\hat{\mathbf{r}}$ where $r = \sqrt{\rho^2 + z^2}$, so the radiation is indeed transverse magnetic.

We can now deduce the electric displacement $\mathbf{D}$ from the magnetic field according to, eq. (4),

$$D_\rho = -\frac{ic}{\omega}\frac{\partial B_\phi}{\partial z} = \mp\frac{\epsilon p}{2\epsilon_z}\int_{-\infty}^{\infty} d\kappa_\rho\,\kappa_\rho^2 H_0^{(1)'}(\kappa_\rho\rho)\,e^{i\kappa_z|z|}, \tag{56}$$

$$D_z = \frac{ic}{\omega\rho}\frac{\partial(\rho B_\phi)}{\partial\rho}\,, \tag{57}$$

although this gives a relatively simple form only for D_ρ. We can also obtain the electric field $\mathbf{E}$ from eq. (3), once we have the scalar potential V. Recalling eq. (36), we find

$$V = \frac{i}{\omega}\frac{\partial V}{\partial t} = -\frac{ic}{\epsilon\omega}\frac{\partial A_z}{\partial z}\,. \tag{58}$$

Thus,

$$E_\rho = -\frac{\partial V}{\partial\rho} = \frac{ic}{\epsilon\omega}\frac{\partial^2 A_z}{\partial\rho\partial z} = \mp\frac{p}{2\epsilon_z}\int_{-\infty}^{\infty} d\kappa_\rho\,\kappa_\rho^2 H_0^{(1)'}(\kappa_\rho\rho)\,e^{i\kappa_z|z|}, \tag{59}$$

[9] Equation (52) indicates that the Fourier coefficient a for a uniaxial medium should vary as $1/\sqrt{\epsilon^p\epsilon_z^q}$ where $p + q = 1$. That the correct choice is $p = -1$, $q = 2$ is not self evident, but is confirmed in sec. 3.1.1.

$$
\begin{aligned}
E_z &= -\frac{\partial V}{\partial z} - \frac{1}{c}\frac{\partial A_z}{\partial t} = \frac{ic}{\epsilon\omega}\frac{\partial^2 A_z}{\partial z^2} + \frac{i\omega}{c}A_z = \frac{ip}{2\epsilon_z}\int_{-\infty}^{\infty} d\kappa_\rho\,\frac{\kappa_\rho}{\kappa_z}(k^2-\kappa_z^2)H_0^{(1)}(\kappa_\rho\rho)\,e^{i\kappa_z|z|} \\
&= \frac{i\epsilon p}{2\epsilon_z^2}\int_{-\infty}^{\infty} d\kappa_\rho\,\frac{\kappa_\rho^3}{\kappa_z}H_0^{(1)}(\kappa_\rho\rho)\,e^{i\kappa_z|z|}, \qquad (60)
\end{aligned}
$$

using eq. (48). Then, from eqs. (1), (56) and (59)-(60), the nonzero electric field components are, in cylindrical coordinates,

$$
\begin{aligned}
E_\rho &= \mp\frac{p}{2\epsilon_z}\int_{-\infty}^{\infty} d\kappa_\rho\,\kappa_\rho^2 H_0^{(1)\prime}(\kappa_\rho\rho)\,e^{i\kappa_z|z|}, && (61)\\
E_z &= \frac{i\epsilon p}{2\epsilon_z^2}\int_{-\infty}^{\infty} d\kappa_\rho\,\frac{\kappa_\rho^3}{\kappa_z}H_0^{(1)}(\kappa_\rho\rho)\,e^{ik_z|z|}, && (62)\\
D_\rho &= \mp\frac{\epsilon p}{2\epsilon_z}\int_{-\infty}^{\infty} d\kappa_\rho\,\kappa_\rho^2 H_0^{(1)\prime}(\kappa_\rho\rho)\,e^{i\kappa_z|z|}, && (63)\\
D_z &= \frac{i\epsilon p}{2\epsilon_z}\int_{-\infty}^{\infty} d\kappa_\rho\,\frac{\kappa_\rho^3}{\kappa_z}H_0^{(1)}(\kappa_\rho\rho)\,e^{i\kappa_z|z|}. && (64)
\end{aligned}
$$

To gain insight as to the behavior of the fields in the far zone, we approximate the integrals (61)-(64) using the so-called method of steepest descent. We note that spherical coordinates (r,θ,ϕ) are related to the cylindrical coordinates (ρ,ϕ,z) by,

$$
\rho = r\sin\theta, \qquad z = r\cos\theta, \tag{65}
$$

and that the relation (48) between the separation constants κ_ρ and κ_z can be rewritten as,

$$
\frac{\epsilon}{\epsilon_z}\kappa_\rho^2 + \kappa_z^2 = k^2, \tag{66}
$$

which suggests a change of variables [7],

$$
\sqrt{\frac{\epsilon}{\epsilon_z}}\kappa_\rho = k\sin\alpha, \qquad \kappa_z = k\cos\alpha. \tag{67}
$$

We will approximate the integrals for large values of $\kappa_\rho\rho$, where the asymptotic expansion (49) of the Hankel function can now be written as,

$$
H_0^{(1)}(\kappa_\rho\rho) \to \sqrt{\frac{-2i}{\pi}}\frac{e^{i\sqrt{\epsilon_z/\epsilon}kr\sin\alpha\sin\theta}}{\sqrt{\sqrt{\epsilon_z/\epsilon}kr\sin\alpha\sin\theta}}. \tag{68}
$$

We illustrate this technique first with the known integral (51) for the case that $\epsilon_z = \epsilon$,

$$
\frac{e^{ikr}}{r} \equiv I = \frac{i}{2}\int_{-\infty}^{\infty} d\kappa_\rho\,\frac{\kappa_\rho}{\kappa_z}H_0^{(1)}(\kappa_\rho\rho)e^{i\kappa_z|z|} \approx \sqrt{\frac{ik}{2\pi r\sin\theta}}\int d\alpha\,\sqrt{\sin\alpha}\,e^{ikr\cos(\alpha-\theta)}. \tag{69}
$$

We make a second change of variables,

$$
\cos(\alpha-\theta) = 1 + i\beta^2, \qquad \sin(\alpha-\theta) = \beta\sqrt{-2i}\sqrt{1+i\beta^2/2}, \qquad d\alpha = \frac{\sqrt{-2i}\,d\beta}{\sqrt{1+i\beta^2/2}}, \tag{70}
$$

to cast eq. (69) into the form of a Gaussian integral where the resulting Gaussian factor $e^{-kr\beta^2}$ is maximal for $\beta = 0$ at which point $\sin\alpha = \sin\theta$. Then, as expected,

$$I \approx e^{ikr}\sqrt{\frac{k}{\pi r\sin\theta}}\int d\beta\,\frac{\sqrt{\sin\alpha}}{\sqrt{1+i\beta^2/2}}e^{-kr\beta^2} \approx e^{ikr}\sqrt{\frac{k}{\pi r}}\int d\beta\, e^{-kr\beta^2} = \frac{e^{ikr}}{r}\,, \tag{71}$$

where the heart[10] of the method of steepest descent is the approximation,

$$\int d\beta\, f(\beta)e^{-kr\beta^2} \approx f(0)\int d\beta\, e^{-kr\beta^2} = \sqrt{\frac{\pi}{kr}}f(0). \tag{72}$$

Turning now to the integrals (61)-(64), we approximate E_z for $z > 0$ as,

$$\begin{aligned} E_z &= \frac{i\epsilon p}{2\epsilon_z^2}\int_{-\infty}^{\infty} d\kappa_\rho\,\frac{\kappa_\rho^3}{\kappa_z}H_0^{(1)}(\kappa_\rho\rho)e^{i\kappa_z z} \\ &\approx \frac{k^2 p}{\epsilon}\sqrt{\frac{ik\sqrt{\epsilon/\epsilon_z}}{2\pi r\sin\theta}}\int d\alpha\,\sin^{5/2}\alpha\, e^{i(\omega/c)r(\sqrt{\epsilon_z}\sin\alpha\sin\theta+\sqrt{\epsilon}\cos\alpha\cos\theta)}. \end{aligned} \tag{73}$$

We desire that the wave at point $\mathbf{r} = (\rho, 0, z)$ appears locally to be a plane wave with the form,

$$e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = e^{i[(\omega/c)\mathbf{n}\cdot\mathbf{r}-\omega t]} = e^{i[(\omega/c)r\sqrt{\epsilon_z\sin^2\theta+\epsilon\cos^2\theta}-\omega t]}, \tag{74}$$

recalling eqs. (7) and (30). Therefore, we make the change of variables,

$$\sqrt{\epsilon_z}\sin\alpha\sin\theta + \sqrt{\epsilon}\cos\alpha\cos\theta = \sqrt{\epsilon_z\sin^2\theta+\epsilon\cos^2\theta} + i\sqrt{\epsilon}\beta^2, \tag{75}$$

$$\sqrt{\epsilon_z}\cos\alpha\sin\theta - \sqrt{\epsilon}\sin\alpha\cos\theta = -\epsilon^{1/4}\beta\sqrt{-2i}\sqrt{(\epsilon_z\sin^2\theta+\epsilon\cos^2\theta)^{1/2} + i\sqrt{\epsilon}\beta^2/2}, \tag{76}$$

$$d\alpha = \frac{\sqrt{-2i}\epsilon^{1/4}\,d\beta}{\sqrt{(\epsilon_z\sin^2\theta+\epsilon\cos^2\theta)^{1/2} + i\sqrt{\epsilon}\beta^2/2}}\,. \tag{77}$$

When $\beta = 0$, we have,

$$\cos\alpha = \frac{\sqrt{\epsilon}\cos\theta}{\sqrt{\epsilon_z\sin^2\theta+\epsilon\cos^2\theta}}\,, \qquad \text{and} \qquad \sin\alpha = \frac{\sqrt{\epsilon_z}\sin\theta}{\sqrt{\epsilon_z\sin^2\theta+\epsilon\cos^2\theta}}\,. \tag{78}$$

Then, eq. (73) can be approximated as,

$$\begin{aligned} E_z &\approx \frac{k^2 p}{\epsilon}\sqrt{\frac{k\sqrt{\epsilon/\epsilon_z}}{\pi r}}\frac{\epsilon^{1/4}\epsilon_z^{5/4}}{(\epsilon_z\sin^2\theta+\epsilon\cos^2\theta)^{3/2}}\sin^2\theta\, e^{i(\omega/c)r\sqrt{\epsilon_z\sin^2\theta+\epsilon\cos^2\theta}}\int d\beta\, e^{-kr\beta^2} \\ &= \frac{\epsilon_z k^2 p}{\sqrt{\epsilon}r}\frac{\sin^2\theta}{(\epsilon_z\sin^2\theta+\epsilon\cos^2\theta)^{3/2}}\, e^{i(\omega/c)r\sqrt{\epsilon_z\sin^2\theta+\epsilon\cos^2\theta}}. \end{aligned} \tag{79}$$

[10] There are also issues of the contour of integration in the complex plane, which we naively ignore here. For discussion, see sec. 6.2 of [12] and the appendices of [13].

If follows that,

$$D_z = \epsilon_z E_z \approx \frac{\epsilon_z^2 k^2 p}{\sqrt{\epsilon} r} \frac{\sin^2\theta}{(\epsilon_z \sin^2\theta + \epsilon\cos^2\theta)^{3/2}}\, e^{i(\omega/c)r\sqrt{\epsilon_z \sin^2\theta+\epsilon\cos^2\theta}}. \tag{80}$$

Similarly, eqs. (55) and (61) for $z > 0$ and $\kappa_\rho \rho \gg 1$ lead to,

$$\begin{aligned}
E_\rho &= -\frac{p}{2\epsilon_z}\int_{-\infty}^{\infty} d\kappa_\rho\, \kappa_\rho^2 H_0^{(1)'}(\kappa_\rho\rho) e^{i\kappa_z z} \approx -\frac{ip}{2\epsilon_z}\int_{-\infty}^{\infty} d\kappa_\rho\, \kappa_\rho^2 H_0^{(1)}(\kappa_\rho\rho) e^{i\kappa_z z} \\
&\approx -\frac{k^2 p}{\epsilon}\sqrt{\frac{ik\sqrt{\epsilon_z/\epsilon}}{2\pi r\sin\theta}} \int d\alpha\, \cos\alpha \sin^{3/2}\alpha\, e^{i(\omega/c)r(\sqrt{\epsilon_z}\sin\alpha\sin\theta+\sqrt{\epsilon}\cos\alpha\cos\theta)} \\
&\approx -\frac{\epsilon_z k^2 p}{\sqrt{\epsilon} r}\frac{\cos\theta\sin\theta}{(\epsilon_z \sin^2\theta + \epsilon\cos^2\theta)^{3/2}}\, e^{i(\omega/c)r\sqrt{\epsilon_z \sin^2\theta+\epsilon\cos^2\theta}},
\end{aligned} \tag{81}$$

and so,

$$D_\rho = \epsilon E_\rho \approx -\frac{\epsilon_z\sqrt{\epsilon} k^2 p}{r}\frac{\cos\theta\sin\theta}{(\epsilon_z \sin^2\theta + \epsilon\cos^2\theta)^{3/2}}\, e^{i(\omega/c)r\sqrt{\epsilon_z \sin^2\theta+\epsilon\cos^2\theta}}. \tag{82}$$

For completeness, we use eqs. (9), (30), (79) and (81) to find the magnetic field in the far zone,

$$\begin{aligned}
B_\phi &\approx n_z E_\rho - n_\rho E_z \\
&\approx -\frac{\epsilon_z k^2 p}{\sqrt{\epsilon} r}\frac{\sin\theta}{\epsilon_z^2 \sin^2\theta + \epsilon\cos^2\theta}\, e^{i(\omega/c)r\sqrt{\epsilon_z \sin^2\theta+\epsilon\cos^2\theta}}.
\end{aligned} \tag{83}$$

As anticipated in sec. 2.1, the electric field of the extraordinary wave is transverse in the far zone,

$$E_r = E_\rho\sin\theta + E_z\cos\theta \approx 0, \tag{84}$$

so the (time-average) Poynting vector $\langle\mathbf{S}\rangle$ is radial there. However, the electric displacement $\mathbf{D}$ has a nonzero longitudinal component, and the wave vector $\mathbf{k}$, which points in the direction $\mathbf{D}\times\mathbf{B}$, is not radial. The wavefronts and "rays" of the wave vector have been sketched in the figure on p. 5 for a case when $\epsilon_z > \epsilon$.

The only nonzero component of the electric field in spherical coordinates in the far zone is $E_\theta = B_\phi/n_r = B_\phi/(n_\rho\sin\theta + n_z\cos\theta) = B_\phi/\sqrt{\epsilon_z\sin^2\theta+\epsilon\cos^2\theta}$ according to eqs. (9) and (30). We confirm this using eqs. (79) and (81) by noting that,

$$\begin{aligned}
E_\theta &= E_\rho\cos\theta - E_z\sin\theta \\
&\approx -\frac{\epsilon_z k^2 p}{\sqrt{\epsilon} r}\frac{\sin\theta}{(\epsilon_z \sin^2\theta + \epsilon\cos^2\theta)^{3/2}}\, e^{i(\omega/c)r\sqrt{\epsilon_z \sin^2\theta+\epsilon\cos^2\theta}}.
\end{aligned} \tag{85}$$

The time-average energy flux (Poynting vector) is purely radial in the far zone,

$$\begin{aligned}
\langle\mathbf{S}\rangle &= \frac{c}{8\pi}Re(\mathbf{E}\times\mathbf{B}^\star) = \frac{c}{8\pi}Re(E_\theta B_\phi^\star)\,\hat{\mathbf{r}} \\
&\approx \frac{c}{8\pi}\frac{\epsilon_z^2 k^4 p^2}{\epsilon r^2}\frac{\sin^2\theta}{(\epsilon_z \sin^2\theta + \epsilon\cos^2\theta)^{5/2}}\,\hat{\mathbf{r}}.
\end{aligned} \tag{86}$$

Since the magnetic field is transverse both in the near and far zones, we can say that the extraordinary wave is a TEM (transverse electromagnetic) wave in the far zone.

In the near zone only the magnetic field is transverse, so the term transverse magnetic characterizes the extraordinary wave at any distance from a point source

2.4 Transverse Electric Waves

Instead of enforcing the gauge condition (35), we can set,

$$\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{\epsilon_z}{c} \frac{\partial V}{\partial t} = 0. \tag{87}$$

This choice is particularly appropriate if the current density has no z component, as in this case the z component of eq. (33) admits the trivial solution $A_z = 0$.

When $J_z = 0$, we restrict our attention to the case that the current is purely azimuthal: $\mathbf{J} = J_\phi \hat{\boldsymbol{\phi}}$. Then, the vector potential will have only a ϕ component as well.

We further restrict our attention to the case of current loops of radius small compared to a wavelength, so that J_ϕ and A_ϕ are both independent of ϕ. Then, $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, and from eq. (87) the scalar potential V is constant in time and can be set to zero. Thus, the electric field, $\mathbf{E} = -\partial \mathbf{A}/\partial ct$, also has only a ϕ component, so the solution in this case can be characterized as transverse electric (TE).

Then, the wave equation for A_ϕ follows from eq. (33) as,

$$\nabla^2 A_\phi - \frac{\epsilon}{c^2} \frac{\partial^2 A_\phi}{\partial t^2} = -\frac{4\pi}{c} J_\phi. \tag{88}$$

This is identical to the form of the wave equation for an isotropic dielectric medium with dielectric constant ϵ that is excited by purely azimuthal currents.

We consider the limiting case of a Hertzian (point) magnetic dipole,

$$\mathbf{m} = m e^{-i\omega t}\, \hat{\mathbf{z}}, \tag{89}$$

for which eq. (88) has the well-known solution[11]

$$\mathbf{A} = k \hat{\mathbf{r}} \times \mathbf{m} \frac{e^{ikr}}{r} \left(i - \frac{1}{kr} \right), \tag{90}$$

where the wave number is given by,

$$k = \frac{\sqrt{\epsilon}\omega}{c}. \tag{91}$$

The electric and magnetic fields follow from eqs. (3) and (90) as,

$$\mathbf{E} = -\frac{k^2}{\sqrt{\epsilon}} \hat{\mathbf{r}} \times \mathbf{m} \frac{e^{ikr}}{r} \left(1 - \frac{1}{ikr} \right), \tag{92}$$

$$\mathbf{B} = -k^2 \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{m}) \frac{e^{ikr}}{r} + [3(\hat{\mathbf{r}} \cdot \mathbf{m})\hat{\mathbf{r}} - \mathbf{m}] \frac{e^{ikr}}{r} \left(\frac{1}{r^3} - \frac{ik}{r^2} \right). \tag{93}$$

Because the electric field $\mathbf{E}$ has no z component, the electric displacement is given simply by,

$$\mathbf{D} = \epsilon \mathbf{E}, \tag{94}$$

even though the medium is anisotropic.

[11] See, for example, sec. 9.3 of [14].

Since the fields $\mathbf{B}$, $\mathbf{D}$ and $\mathbf{E}$ are the same as would occur for a Hertzian magnetic dipole in an isotropic dielectric with constant ϵ, we can say that the case of a magnetic dipole oriented along the z axis leads to ordinary radiation, in the language of birefringence.

In the far zone, both $\mathbf{B}$ and $\mathbf{E}$ are purely transverse,

$$\mathbf{E}_{\text{far}} = -\frac{k^2}{\sqrt{\epsilon}} \hat{\mathbf{r}} \times \mathbf{m} \frac{e^{ikr}}{r}, \qquad \mathbf{B}_{\text{far}} = \sqrt{\epsilon} \hat{\mathbf{r}} \times \mathbf{E}_{\text{far}}, \tag{95}$$

so a Hertzian magnetic dipole oriented along the z axis of a uniaxial dielectric medium actually produces TEM (transverse electromagnetic) radiation in the far zone. The time-average energy flux (Poynting) vector,

$$\langle \mathbf{S}_{\text{far}} \rangle = \frac{c}{8\pi} Re(\mathbf{E}_{\text{far}} \times \mathbf{B}^{\star}_{\text{far}}) = \frac{c}{8\pi} \sqrt{\epsilon} \left| \mathbf{E}_{\text{far}} \right|^2 \hat{\mathbf{r}} = \frac{c}{8\pi} \frac{k^4 m^2}{\sqrt{\epsilon} r^2} \sin^2 \theta \, \hat{\mathbf{r}}, \tag{96}$$

is purely radial at large distances from the source.[12]

A Appendix: The Scaling Method of Clemmow

An interesting method of scaling solutions of Maxwell's equations in vacuum to ones in a uniform dielectric medium has been given by Clemmow [16].

We again restrict our attention to media of unit permeability.

Suppose that for a known current density $\mathbf{J}^0 e^{-i\omega t}$ (and the related charge density $\varrho^0 e^{-i\omega t} = -(i/\omega) \boldsymbol{\nabla}^0 \cdot \mathbf{J}^0 e^{-i\omega t}$) surrounded by vacuum we know the solutions $\mathbf{E}^0$ and $\mathbf{B}^0$ to Maxwell's equations (2). We develop a scaling procedure by emphasizing the third and fourth Maxwell equations. Once the method is in hand we verify that it is consistent with the first two Maxwell equations. So, we first consider the vacuum Maxwell equations,

$$\frac{\partial E_z^0}{\partial y^0} - \frac{\partial E_y^0}{\partial z^0} = i\omega B_x^0, \qquad \frac{\partial E_x^0}{\partial z^0} - \frac{\partial E_z^0}{\partial x^0} = i\omega B_y^0, \qquad \frac{\partial E_y^0}{\partial x^0} - \frac{\partial E_x^0}{\partial y^0} = i\omega B_z^0, \tag{97}$$

$$\frac{\partial B_z^0}{\partial y^0} - \frac{\partial B_y^0}{\partial z^0} = -i\omega E_x^0 + \frac{4\pi}{c} J_x^0, \qquad \frac{\partial B_x^0}{\partial z^0} - \frac{\partial B_z^0}{\partial x^0} = -i\omega E_y^0 + \frac{4\pi}{c} J_y^0,$$

$$\frac{\partial B_y^0}{\partial x^0} - \frac{\partial B_x^0}{\partial y^0} = -i\omega E_z^0 + \frac{4\pi}{c} J_z^0, \tag{98}$$

where we write the rectangular spatial coordinates in vacuum as (x^0, y^0, z^0).

The corresponding equations for a current distribution $\mathbf{J} e^{-i\omega t}$ in an anisotropic medium where $D_i = \epsilon_i E_i$ are, of course,

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = i\omega B_x, \qquad \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = i\omega B_y, \qquad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z, \tag{99}$$

[12] For a point dipole the time-average Poynting vector is purely radial in the near zone as well, as discussed in [14, 15].

$$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = -i\omega\epsilon_x E_x + \frac{4\pi}{c} J_x, \qquad \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = -i\omega\epsilon_y E_y + \frac{4\pi}{c} J_y,$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -i\omega\epsilon_z E_z + \frac{4\pi}{c} J_z, \tag{100}$$

We now seek under what conditions solutions to eqs. (97)-(98) can be scaled to provide solutions to eqs. (98)-(99) via the linear scaling relations,

$$\begin{aligned} x^0 &= \alpha x, & y^0 &= \beta y, & z^0 &= \gamma z, \\ B_x &= b_x B_x^0, & B_y &= b_y B_y^0, & B_z &= b_z B_z^0, \\ E_x &= e_x E_x^0, & E_y &= e_y E_y^0, & E_z &= e_z E_z^0, \\ J_x^0 &= j_x J_x, & J_y^0 &= j_y J_y, & J_z^0 &= j_z J_z, \end{aligned} \tag{101}$$

where α, β, γ, b_x, b_y, b_z, e_x, e_y, e_z, e_x, e_y and e_z are nonzero constants.

If both terms on the lefthand sides of each of eq. (99) are to scale the same way, we must have,

$$\beta e_z = \gamma e_y, \qquad \gamma e_x = \alpha e_z, \qquad \alpha e_y = \beta e_x, \tag{102}$$

which can be satisfied provided,

$$e_x = \alpha p, \qquad e_y = \beta p, \qquad e_z = \gamma p, \tag{103}$$

where p is a nonzero constant. Similarly, if both terms on the lefthand sides of each of eq. (100) are to scale the same way, we must have,

$$b_x = \alpha q, \qquad b_y = \beta q, \qquad b_z = \gamma q, \tag{104}$$

where q is a nonzero constant.

For all three components of eq. (99) to be the same multiple of the corresponding components of eq. (97), we must also have,

$$\frac{q}{p} = \frac{\gamma\beta}{\alpha} = \frac{\alpha\gamma}{\beta} = \frac{\alpha\beta}{\gamma} \,. \tag{105}$$

However, this implies that $\alpha = \beta = \gamma$, and the proposed scaling is isotropic. To have an anisotropic scaling, we must be able to ignore one of the three equalities in eq. (105). This is possible in case the solutions are transverse magnetic (TM) with respect to x, y or z.

A.1 Transverse Magnetic Fields

For example, suppose the vacuum solutions are transverse magnetic with respect to the z axis: $B_z^0 = 0$. Then, eq. (105) reduces to,

$$\frac{q}{p} = \frac{\gamma\beta}{\alpha} = \frac{\alpha\gamma}{\beta} \,, \tag{106}$$

which can be satisfied if

$$\alpha = \beta, \qquad \frac{q}{p} = \gamma. \tag{107}$$

Using the relations (103), (104) and (107), eq. (100) can be written as,

$$-\frac{\partial B_y^0}{\partial z^0} = -i\omega \frac{\epsilon_x}{\gamma^2} E_x^0 + \frac{4\pi}{c} \frac{J_x^0}{\alpha\gamma q j_x}\,, \qquad \frac{\partial B_x^0}{\partial z^0} = -i\omega \frac{\epsilon_y}{\gamma^2} E_y^0 + \frac{4\pi}{c} \frac{J_y^0}{\alpha\gamma q j_y},$$

$$\frac{\partial B_y^0}{\partial x^0} - \frac{\partial B_x^0}{\partial y^0} = -i\omega \frac{\epsilon_z}{\alpha^2} E_z^0 + \frac{4\pi}{c} \frac{J_z^0}{\alpha^2 q j_z}\,, \tag{108}$$

For these to be the same as eq. (98), we require that,

$$1 = \frac{\epsilon_x}{\gamma^2} = \frac{\epsilon_y}{\gamma^2} = \frac{\epsilon_z}{\alpha^2} = \alpha\gamma q j_x = \alpha\gamma q j_y = \alpha^2 q j_z. \tag{109}$$

We learn that the scaling method will not work for the most general anisotropic, linear dielectric medium, but it will succeed for an uniaxial medium where $\epsilon_x = \epsilon_y \equiv \epsilon$. Then, $\alpha = \beta = \sqrt{\epsilon_z}$ and $\gamma = \sqrt{\epsilon}$. We also see that any value of q is viable, so we take $q = 1/\alpha^2 = 1/\epsilon_z$ to make $j_z = 1$. Then, $p = 1/\epsilon_z\sqrt{\epsilon}$, and the scaling coefficients are,

$$\begin{array}{lll} \alpha = \sqrt{\epsilon_z}, & \beta = \sqrt{\epsilon_z}, & \gamma = \sqrt{\epsilon}, \\ b_x = \frac{1}{\sqrt{\epsilon_z}}\,, & b_y = \frac{1}{\sqrt{\epsilon_z}}\,, & b_z = \frac{\sqrt{\epsilon}}{\epsilon_z}\,, \\ e_x = \frac{1}{\sqrt{\epsilon\epsilon_z}}\,, & e_y = \frac{1}{\sqrt{\epsilon\epsilon_z}}\,, & e_z = \frac{1}{\epsilon_z}\,, \\ j_x = \sqrt{\frac{\epsilon_z}{\epsilon}}, & j_y = \sqrt{\frac{\epsilon_z}{\epsilon}}, & j_z = 1. \end{array} \tag{110}$$

The scaling relations for transverse magnetic fields are thus,

$$x_0 = \sqrt{\epsilon_z} x, \qquad y_0 = \sqrt{\epsilon_z} y, \qquad z_0 = \sqrt{\epsilon} z. \tag{111}$$

$$E_x(\mathbf{r}) = \frac{1}{\sqrt{\epsilon\epsilon_z}} E_x^0(\mathbf{r}^0), \qquad E_y(\mathbf{r}) = \frac{1}{\sqrt{\epsilon\epsilon_z}} E_y^0(\mathbf{r}^0), \qquad E_z(\mathbf{r}) = \frac{1}{\epsilon_z} E_z^0(\mathbf{r}^0), \tag{112}$$

$$D_x(\mathbf{r}) = \sqrt{\frac{\epsilon}{\epsilon_z}} D_x^0(\mathbf{r}^0), \qquad D_y(\mathbf{r}) = \sqrt{\frac{\epsilon}{\epsilon_z}} D_y^0(\mathbf{r}^0), \qquad D_z(\mathbf{r}) = D_z^0(\mathbf{r}^0), \tag{113}$$

$$B_x(\mathbf{r}) = \frac{1}{\sqrt{\epsilon_z}} B_x^0(\mathbf{r}^0), \qquad B_y(\mathbf{r}) = \frac{1}{\sqrt{\epsilon_z}} B_y^0(\mathbf{r}^0), \qquad B_z(\mathbf{r}) = \frac{\sqrt{\epsilon}}{\epsilon_z} B_z^0(\mathbf{r}^0) = 0, \tag{114}$$

$$J_x^0(\mathbf{r}^0) = \sqrt{\frac{\epsilon_z}{\epsilon}} J_x(\mathbf{r}), \qquad J_y^0(\mathbf{r}^0) = \sqrt{\frac{\epsilon_z}{\epsilon}} J_y(\mathbf{r}), \qquad J_z^0(\mathbf{r}^0) = J_z(\mathbf{r}). \tag{115}$$

The scaling relation for the charge density ϱ is,

$$\begin{aligned} \varrho^0(\mathbf{r}^0) &= -\frac{i}{\omega} \boldsymbol{\nabla}^0 \cdot \mathbf{J}^0 = -\frac{i}{\omega} \left(\frac{\partial J_x^0}{\partial x^0} + \frac{\partial J_y^0}{\partial y^0} + \frac{\partial J_z^0}{\partial z^0} \right) \\ &= -\frac{i}{\omega} \left(\frac{\partial \sqrt{\epsilon_z/\epsilon} J_x}{\partial \sqrt{\epsilon_z} x} + \frac{\partial \sqrt{\epsilon_z/\epsilon} J_y}{\partial \sqrt{\epsilon_z} y} + \frac{\partial J_z}{\partial \sqrt{\epsilon} z} \right) = -\frac{i}{\sqrt{\epsilon}\omega} \boldsymbol{\nabla} \cdot \mathbf{J} = \frac{\varrho(\mathbf{r})}{\sqrt{\epsilon}}\,. \end{aligned} \tag{116}$$

We now check that the first two Maxwell equations (2) also scale properly. As expected,

$$\begin{aligned}
4\pi\varrho &= 4\pi\sqrt{\epsilon}\varrho^0 = \sqrt{\epsilon}\boldsymbol{\nabla}^0\cdot\mathbf{D}^0 = \sqrt{\epsilon}\left(\frac{\partial D_x^0}{\partial x^0} + \frac{\partial D_y^0}{\partial y^0} + \frac{\partial D_z^0}{\partial z^0}\right) \\
&= \sqrt{\epsilon}\left(\frac{\partial\sqrt{\epsilon_z/\epsilon}D_x}{\partial\sqrt{\epsilon_z}x} + \frac{\partial\sqrt{\epsilon_z/\epsilon}D_y}{\partial\sqrt{\epsilon_z}y} + \frac{\partial D_z}{\partial\sqrt{\epsilon}z}\right) = \boldsymbol{\nabla}\cdot\mathbf{D}. \qquad (117)
\end{aligned}$$

The scaling of the second Maxwell equation for TM fields (but not for general magnetic fields) is also satisfactory,

$$0 = \boldsymbol{\nabla}^0\cdot\mathbf{B}^0 = \frac{\partial B_x^0}{\partial x^0} + \frac{\partial B_y^0}{\partial y^0} = \frac{\partial\sqrt{\epsilon_z}B_x}{\partial\sqrt{\epsilon_z}x} + \frac{\partial\sqrt{\epsilon_z}B_y}{\partial\sqrt{\epsilon_z}y} = \boldsymbol{\nabla}\cdot\mathbf{B}. \qquad (118)$$

The scaling transformations for cylindrical (ρ, ϕ, z) are, noting that $E_\rho = E_x\sin\phi + E_y\cos\phi$ and $E_\phi = E_x\cos\phi - E_y\sin\phi$,

$$\rho^0 = \sqrt{x^{02} + y^{02}} = \sqrt{\epsilon_z}\rho, \quad \phi^0 = \tan^{-1}\frac{y^0}{x^0} = \phi, \quad z^0 = \sqrt{\epsilon}z, \qquad (119)$$

$$E_\rho(\mathbf{r}) = \frac{1}{\sqrt{\epsilon\epsilon_z}}E_\rho^0(\mathbf{r}^0), \qquad E_\phi(\mathbf{r}) = \frac{1}{\sqrt{\epsilon\epsilon_z}}E_\phi^0(\mathbf{r}^0), \qquad E_z(\mathbf{r}) = \frac{1}{\epsilon_z}E_z^0(\mathbf{r}^0), \qquad (120)$$

$$D_\rho(\mathbf{r}) = \sqrt{\frac{\epsilon}{\epsilon_z}}D_\rho^0(\mathbf{r}^0), \qquad D_\phi(\mathbf{r}) = \sqrt{\frac{\epsilon}{\epsilon_z}}D_\phi^0(\mathbf{r}^0), \qquad D_z(\mathbf{r}) = D_z^0(\mathbf{r}^0), \qquad (121)$$

$$B_\rho(\mathbf{r}) = \frac{1}{\sqrt{\epsilon_z}}B_\rho^0(\mathbf{r}^0), \qquad B_\phi(\mathbf{r}) = \frac{1}{\sqrt{\epsilon_z}}B_\phi^0(\mathbf{r}^0), \qquad B_z(\mathbf{r}) = \frac{\sqrt{\epsilon}}{\epsilon_z}B_z^0(\mathbf{r}^0) = 0. \qquad (122)$$

The scaling relations for spherical coordinates (r, θ, ϕ) are,

$$r^0 = \sqrt{\rho^{02} + z^{02}} = r\sqrt{\epsilon_z\sin^2\theta + \epsilon\cos^2\theta}, \qquad \tan\theta^0 = \frac{\rho^0}{z^0} = \sqrt{\frac{\epsilon_z}{\epsilon}}\tan\theta, \qquad \phi^0 = \phi, \qquad (123)$$

so that,

$$\sin\theta^0 = \frac{\sqrt{\epsilon_z}\sin\theta}{\sqrt{\epsilon_z\sin^2\theta + \epsilon\cos^2\theta}}, \qquad \cos\theta^0 = \frac{\sqrt{\epsilon}\cos\theta}{\sqrt{\epsilon_z\sin^2\theta + \epsilon\cos^2\theta}}. \qquad (124)$$

Then,

$$\begin{aligned}
E_r(\mathbf{r}) &= E_\rho\sin\theta + E_z\cos\theta = \frac{E_\rho^0}{\sqrt{\epsilon\epsilon_z}}\sin\theta + \frac{E_z^0}{\epsilon_z}\cos\theta \\
&= (E_r^0\sin\theta^0 + E_\theta^0\cos\theta^0)\frac{\sin\theta}{\sqrt{\epsilon\epsilon_z}} + (E_r^0\cos\theta^0 - E_\theta^0\sin\theta^0)\frac{\cos\theta}{\epsilon_z} \\
&= E_r^0(\mathbf{r}^0)\frac{\sqrt{\epsilon_z\sin^2\theta + \epsilon\cos^2\theta}}{\epsilon_z\sqrt{\epsilon}}. \qquad (125)
\end{aligned}$$

Similarly,

$$E_\theta(\mathbf{r}) = E_\rho\cos\theta - E_z\sin\theta = \frac{E_\rho^0}{\sqrt{\epsilon\epsilon_z}}\cos\theta + \frac{E_z^0}{\epsilon_z}\sin\theta$$

$$
\begin{aligned}
&= (E_r^0 \sin\theta^0 + E_\theta^0 \cos\theta^0)\frac{\cos\theta}{\sqrt{\epsilon\epsilon_z}} - (E_r^0 \cos\theta^0 - E_\theta^0 \sin\theta^0)\frac{\sin\theta}{\epsilon_z} \\
&= \frac{E_\theta^0(\mathbf{r}^0)}{\sqrt{\epsilon_z}\sqrt{\epsilon_z \sin^2\theta + \epsilon\cos^2\theta}}\,.
\end{aligned}
\tag{126}
$$

If the righthand sides of scaling relations (120) for $\mathbf{E}$ are multiplied by $\sqrt{\epsilon}$ they take on the same forms as relations (122) for $\mathbf{B}$. This behavior persists in spherical coordinates as well, so the scaling relations for $\mathbf{E}$ and $\mathbf{B}$ in these coordinates are,

$$
E_r(\mathbf{r}) = E_r^0(\mathbf{r}^0)\frac{\sqrt{\epsilon_z \sin^2\theta + \epsilon\cos^2\theta}}{\epsilon_z\sqrt{\epsilon}}\,, \qquad E_\theta(\mathbf{r}) = \frac{E_\theta^0(\mathbf{r}^0)}{\sqrt{\epsilon_z}\sqrt{\epsilon_z \sin^2\theta + \epsilon\cos^2\theta}}\,,
$$

$$
E_\phi(\mathbf{r}) = \frac{1}{\sqrt{\epsilon\epsilon_z}} E_\phi^0(\mathbf{r}^0), \tag{127}
$$

$$
B_r(\mathbf{r}) = B_r^0(\mathbf{r}^0)\frac{\sqrt{\epsilon_z \sin^2\theta + \epsilon\cos^2\theta}}{\epsilon_z}\,, \qquad B_\theta(\mathbf{r}) = \sqrt{\frac{\epsilon}{\epsilon_z}}\frac{B_\theta^0(\mathbf{r}^0)}{\sqrt{\epsilon_z \sin^2\theta + \epsilon\cos^2\theta}}\,,
$$

$$
B_\phi(\mathbf{r}) = \frac{1}{\sqrt{\epsilon_z}} B_\phi^0(\mathbf{r}^0). \tag{128}
$$

The scaling relations (120)-(122) and (127)-(128) for $\mathbf{E}$ and $\mathbf{B}$ in both cylindrical and spherical coordinates are such that there is no mixing of components. In particular, if the TM vacuum solutions are also TEM (transverse electromagnetic) in some region, the scaled solutions will also be TEM in the corresponding region.

However, the scaling relations for the electric displacement $\mathbf{D}$ in spherical coordinates mix the r and θ components,

$$
\begin{aligned}
D_r(\mathbf{r}) &= D_\rho \sin\theta + D_z\cos\theta = \sqrt{\frac{\epsilon}{\epsilon_z}} D_\rho^0 \sin\theta + D_z^0\cos\theta \\
&= \sqrt{\frac{\epsilon}{\epsilon_z}}(D_r^0 \sin\theta^0 + D_\theta^0\cos\theta^0)\sin\theta + (D_r^0\cos\theta^0 - D_\theta^0\sin\theta^0)\cos\theta \\
&= \frac{\sqrt{\epsilon\epsilon_z}D_r^0(\mathbf{r}^0) + (\epsilon-\epsilon_z)D_\theta^0(\mathbf{r}^0)\sin\theta\cos\theta}{\sqrt{\epsilon_z}\sqrt{\epsilon_z \sin^2\theta + \epsilon\cos^2\theta}}\,,
\end{aligned}
\tag{129}
$$

$$
\begin{aligned}
D_\theta(\mathbf{r}) &= D_\rho \cos\theta - D_z\sin\theta = \sqrt{\frac{\epsilon}{\epsilon_z}} D_\rho^0 \cos\theta - D_z^0\sin\theta \\
&= \sqrt{\frac{\epsilon}{\epsilon_z}}(D_r^0 \sin\theta^0 + D_\theta^0\cos\theta^0)\cos\theta - (D_r^0\cos\theta^0 - D_\theta^0\sin\theta^0)\sin\theta \\
&= \frac{D_\theta^0(\mathbf{r}^0)(\epsilon\sin^2\theta + \epsilon_z\cos^2\theta)}{\sqrt{\epsilon_z}\sqrt{\epsilon_z \sin^2\theta + \epsilon\cos^2\theta}}\,,
\end{aligned}
\tag{130}
$$

$$
D_\phi(\mathbf{r}) = \sqrt{\frac{\epsilon}{\epsilon_z}} D_\phi^0(\mathbf{r}^0). \tag{131}
$$

In the far zone ($\omega r^0/c \gg 1$) of a source that emits TM waves, the electric displacement $\mathbf{D}^0$ is transverse to $\mathbf{r}^0$ in vacuum, but $\mathbf{D}$ has a nonzero radial component in a uniaxial medium. This is a manifestation of the extraordinary character of the TM fields.

A.1.1 Fields of a Hertzian Electric Dipole Aligned with the z Axis

An example of TM waves in vacuum are the fields of a Hertzian (point) electric dipole $\mathbf{p}^0 = p^0 e^{-i\omega t}\,\hat{\mathbf{z}}$ at the origin, with the dipole axis along the z axis. The electric and magnetic fields are (see, for example, sec. 9.2 of [14]), in spherical coordinates with $k^0 = \omega/c$,

$$
\begin{aligned}
E_r^0 &= -2ik^0p^0\frac{e^{i(k^0r^0-\omega t)}}{r^{0^2}}\left(1+\frac{i}{k^0r^0}\right)\cos\theta^0, &(132)\\
E_\theta^0 &= -k^{0^2}p^0\frac{e^{i(k^0r^0-\omega t)}}{r^0}\left(1+\frac{i}{k^0r^0}-\frac{1}{k^{0^2}r^{0^2}}\right)\sin\theta^0, &(133)\\
B_\phi^0 &= -k^{0^2}p^0\frac{e^{i(k^0r^0-\omega t)}}{r^0}\left(1+\frac{i}{k^0r^0}\right)\sin\theta^0. &(134)
\end{aligned}
$$

Using eqs. (119)-(120), we see that an electric dipole moment $\mathbf{p}^0$ scales as,

$$
\begin{aligned}
\mathbf{p}^0 &= \int \varrho^0\mathbf{r}^0\,d^3r^0 = \int \frac{\varrho}{\sqrt{\epsilon}}(\sqrt{\epsilon_z}x,\sqrt{\epsilon_z}y,\sqrt{\epsilon}z)\,\epsilon_z\sqrt{\epsilon}\,d^3r\\
&= \epsilon_z(\sqrt{\epsilon_z}p_x,\sqrt{\epsilon_z}p_y,\sqrt{\epsilon}p_z). &(135)
\end{aligned}
$$

Thus, the fields of a Hertzian electric dipole $\mathbf{p} = pe^{-i\omega t}\,\hat{\mathbf{z}}$ in a uniaxial dielectric (1) can be scaled using eqs. (123)-(124) and (127)-(128) from those of eqs. (132)-(134) with $p^0 = \epsilon_z\sqrt{\epsilon}p$,

$$
\begin{aligned}
E_r &= -2ikp\,\frac{e^{i(k^0r^0-\omega t)}}{r^2}\left(1+\frac{i\sqrt{\epsilon}}{kr^0}\right)\frac{\cos\theta}{(\epsilon_z\sin^2\theta+\epsilon\cos^2\theta)}\,, &(136)\\
E_\theta &= -\frac{\epsilon_zk^2p}{\sqrt{\epsilon}}\frac{e^{i(k^0r^0-\omega t)}}{r}\left(1+\frac{i\sqrt{\epsilon}}{kr^0}-\frac{\epsilon}{k^2r^{0^2}}\right)\frac{\sin\theta}{(\epsilon_z\sin^2\theta+\epsilon\cos^2\theta)^{3/2}}\,, &(137)\\
B_\phi &= -\frac{\epsilon_zk^2p}{\sqrt{\epsilon}}\frac{e^{i(k^0r^0-\omega t)}}{r}\left(1+\frac{i\sqrt{\epsilon}}{kr^0}\right)\frac{\sin\theta}{\epsilon_z\sin^2\theta+\epsilon\cos^2\theta}\,, &(138)
\end{aligned}
$$

where $k = \sqrt{\epsilon}\omega/c = \sqrt{\epsilon}k^0$ and $r^0 = r\sqrt{\epsilon_z\sin^2\theta+\epsilon\cos^2\theta}$. The fields E_θ and B_ϕ in the far zone agree with those found in eqs. (83) and (85), which validates the factor $\sqrt{\epsilon}/\epsilon_z$ used in eq. (53).

A.2 Transverse Electric Fields (and Fields in Isotropic Dielectrics)

An alternative evasion of the overconstraint in eq. (105) is to consider transverse electric fields, for which, say, $E_z^0 = 0$. Then, the dielectric constant ϵ_z does not appear in the Maxwell equation (100). If the medium is uniaxial with $\epsilon_x = \epsilon_y = \epsilon$, Maxwell's equations for TE fields are identical to those of an isotropic dielectric of constant ϵ. In this case we are happy to use an isotropic scaling relation where $\alpha = \beta = \gamma$ in eq. (101). The logic that led to eq. (109) now implies that,

$$
1 = \frac{\epsilon}{\alpha^2} = \alpha^2 qj_x = \alpha^2 qj_y = \alpha^2 qj_z. \qquad (139)
$$

Thus, $\alpha = \beta = \gamma = \sqrt{\epsilon}$. We take $q = 1/\epsilon$ and $p = 1/\epsilon\sqrt{\epsilon}$, so the scaling relations for transverse electric fields (as well as for fields in an isotropic dielectric) are,

$$\mathbf{r}^0 = \sqrt{\epsilon}\mathbf{r}, \qquad \mathbf{J}^0(\mathbf{r}^0) = \mathbf{J}(\mathbf{r}), \qquad \varrho^0(\mathbf{r}^0) = \frac{\varrho(\mathbf{r})}{\sqrt{\epsilon}}, \tag{140}$$

$$\mathbf{E}(\mathbf{r}) = \frac{\mathbf{E}^0(\mathbf{r}^0)}{\epsilon}, \qquad \mathbf{D}(\mathbf{r}) = \mathbf{D}^0(\mathbf{r}^0), \qquad \mathbf{B}(\mathbf{r}) = \frac{\mathbf{B}^0(\mathbf{r}^0)}{\sqrt{\epsilon}}. \tag{141}$$

An example of TE waves in vacuum are the fields of a Hertzian point magnetic dipole $\mathbf{m}^0 = m^0 e^{-i\omega t}\,\hat{\mathbf{z}}$ at the origin, with the dipole axis along the z axis. The electric and magnetic fields are (see, for example, sec. 9.3 of [14]), in spherical coordinates with $k^0 = \omega/c$,

$$\mathbf{E}^0 = k^{0\,2} m^0 \frac{e^{i(k^0 r^0 - \omega t)}}{r^0}\left(1 + \frac{i}{k^0 r^0}\right)\sin\theta^0\,\hat{\boldsymbol{\phi}}, \tag{142}$$

$$\mathbf{B}^0 = -k^{0\,2} m^0 \frac{e^{i(k^0 r^0 - \omega t)}}{r^0}\left[\frac{2i}{k^0 r^0}\left(1 + \frac{i}{k^0 r^0}\right)\cos\theta^0\,\hat{\mathbf{r}} + \left(1 + \frac{i}{k^0 r^0} - \frac{1}{k^{0\,2} r^{0\,2}}\right)\sin\theta^0\,\hat{\boldsymbol{\theta}}\right] \tag{143}$$

A magnetic dipole moment $\mathbf{m}^0$ scales as,

$$\mathbf{m}^0 = \frac{1}{2c}\int \mathbf{r}^0 \times \mathbf{J}^0\, d^3r^0 = \frac{1}{2c}\int \sqrt{\epsilon}\mathbf{r} \times \mathbf{J}\,\epsilon^{3/2}\, d^3r = \epsilon^2\mathbf{m}. \tag{144}$$

Then, the fields of a Hertzian magnetic dipole $\mathbf{m} = me^{-i\omega t}\,\hat{\mathbf{z}}$ in a uniaxial dielectric (1) can be scaled using eq. (140) from those of eqs. (142)-(143) with $m^0 = \epsilon^2 m$,

$$\mathbf{E} = \frac{k^2 m}{\sqrt{\epsilon}}\frac{e^{i(kr-\omega t)}}{r}\left(1 + \frac{i}{kr}\right)\sin\theta\,\hat{\boldsymbol{\phi}}, \tag{145}$$

$$\mathbf{B} = -k^2 m\frac{e^{i(kr-\omega t)}}{r}\left[\frac{2i}{kr}\left(1 + \frac{i}{kr}\right)\cos\theta\,\hat{\mathbf{r}} + \left(1 + \frac{ik}{kr} - \frac{1}{k^2 r^2}\right)\sin\theta\,\hat{\boldsymbol{\theta}}\right], \tag{146}$$

where $k = \sqrt{\epsilon}\omega/c$. These fields, of course, agree with those given in eqs. (92)-(93).

A.3 Hertzian Dipole with Arbitrary Orientation

If the Hertzian dipole has arbitrary orientation, its field is neither pure transverse electric nor pure transverse magnetic. To use the scaling method, the vacuum fields must first be resolved into transverse electric and transverse magnetic components. A procedure for this has been given by Clemmow [17].

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