“Backflow” in the Interference of Two Electromagnetic Plane Waves

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1 Problem

Compute the time-average Poynting vector, $\langle \mathbf{S} \rangle = (c/8\pi) Re(\mathbf{E} \times \mathbf{B}^\star)$, for two circularly polarized, electromagnetic plane waves in vacuum, given by the real parts of,

$$\mathbf{E}_j = E_j(\hat{\mathbf{e}}_{1,j} + i\hat{\mathbf{e}}_{2,j})\, e^{i(\mathbf{k}_j \cdot \mathbf{r} - \omega t)}, \qquad \mathbf{B}_j = \hat{\mathbf{k}}_j \times \mathbf{E}_j, \tag{1}$$

in Gaussian units, where the wave vectors $\mathbf{k}_j$ (with $k_j = k = \omega/c$ where c is the speed of light in vacuum) point in the directions (θ_j, ϕ_j) in a spherical coordinate system (r, θ, ϕ) with,

$$\theta_1 = \theta_0, \qquad \phi_1 = 0, \qquad \text{and} \qquad \theta_2 = \theta_0, \qquad \phi_2 = \pi/2. \tag{2}$$

Show that $\langle S_z \rangle$ has negative values in some spatial regions, although $\langle S_{z,j} \rangle$ is positive for each of the component plane waves.

This phenomenon is called “backflow”, and was first understood in a quantum context [1]-[3], which led many to suppose (incorrectly) that it is a “nonclassical” behavior. It has been demonstrated in two experiments with laser beams [4, 5].

2 Solution

The solution follows sec. 3 of [6].

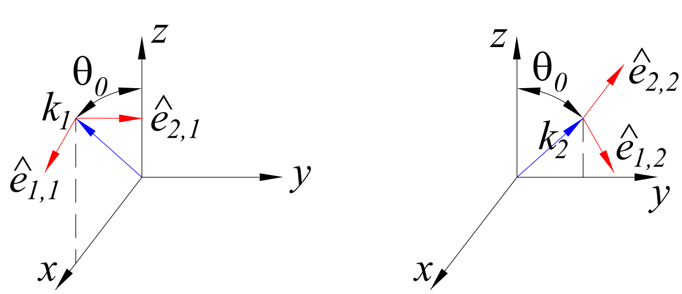
The wave vectors $\mathbf{k}_j$ for $j = 1, 2$ are, in rectangular coordinates (x, y, z),

$$\mathbf{k}_1 = k(\sin\theta_0, 0, \cos\theta_0), \qquad \text{and} \qquad \mathbf{k}_2 = k(0, \sin\theta_0, \cos\theta_0), \tag{3}$$

and the polarization unit vectors $\hat{\mathbf{e}}_{1,j}$ and $\hat{\mathbf{e}}_{2,j}$ can be written as,

$$\hat{\mathbf{e}}_{1,1} = (\cos\theta_0, 0, -\sin\theta_0), \qquad \hat{\mathbf{e}}_{2,1} = (0, 1, 0), \qquad \hat{\mathbf{e}}_{1,1} + i\hat{\mathbf{e}}_{2,1} = (\cos\theta_0, i, -\sin\theta_0), \tag{4}$$

$$\hat{\mathbf{e}}_{1,2} = (0, \cos\theta_0, -\sin\theta_0), \qquad \hat{\mathbf{e}}_{2,2} = (-1, 0, 0), \qquad \hat{\mathbf{e}}_{1,2} + i\hat{\mathbf{e}}_{2,2} = (-i, \cos\theta_0, -\sin\theta_0). \tag{5}$$



The z-component of the time-average Poynting vector is given by, recalling that $\mathbf{E}_j \cdot \mathbf{k}_j = 0$,

$$
\begin{aligned}
\frac{4\pi}{c}\langle S_z\rangle &= Re[\mathbf{E}\times\mathbf{B}^\star]_z = Re[(\mathbf{E}_1+\mathbf{E}_2)\times(\hat{\mathbf{k}}_1\times\mathbf{E}_1^\star+\hat{\mathbf{k}}_2\times\mathbf{E}_2^\star)]_z \\
&= |\mathbf{E}_1|^2 k_{1,z} - Re[(\mathbf{E}_1\cdot\hat{\mathbf{k}}_1)\mathbf{E}_{1,z}^\star] + |\mathbf{E}_2|^2 k_{2,z} - Re[(\mathbf{E}_2\cdot\hat{\mathbf{k}}_2)\mathbf{E}_{2,z}^\star] \\
&\quad + Re[\mathbf{E}_1\cdot\mathbf{E}_2^\star]\hat{\mathbf{k}}_{2,z} - Re[(\mathbf{E}_1\cdot\hat{\mathbf{k}}_2)\mathbf{E}_{2,z}^\star] + Re[\mathbf{E}_2\cdot\mathbf{E}_1^\star]\hat{\mathbf{k}}_{1,z} - Re[(\mathbf{E}_2\cdot\hat{\mathbf{k}}_1)\mathbf{E}_{1,z}^\star] \\
&= 2E_1^2\cos\theta_0 + 2E_2^2\cos\theta_0 \\
&\quad + E_1E_2 Re[(\hat{\mathbf{e}}_{1,1}+i\hat{\mathbf{e}}_{2,1})\cdot(\hat{\mathbf{e}}_{1,2}-i\hat{\mathbf{e}}_{2,2})\hat{\mathbf{k}}_{2,z}\, e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{r}}] \\
&\quad - E_1E_2 Re[((\hat{\mathbf{e}}_{1,1}+i\hat{\mathbf{e}}_{2,1})\cdot\hat{\mathbf{k}}_2)\,(\hat{\mathbf{e}}_{1,2,z}-i\hat{\mathbf{e}}_{2,2,z})\, e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{r}}] \\
&\quad + E_1E_2 Re[(\hat{\mathbf{e}}_{1,2}+i\hat{\mathbf{e}}_{2,2})\cdot(\hat{\mathbf{e}}_{1,1}-i\hat{\mathbf{e}}_{2,1})\hat{\mathbf{k}}_{1,z}\, e^{i(\mathbf{k}_2-\mathbf{k}_1)\cdot\mathbf{r}}] \\
&\quad - E_1E_2 Re[((\hat{\mathbf{e}}_{1,2}+i\hat{\mathbf{e}}_{2,2})\cdot\hat{\mathbf{k}}_1)\,(\hat{\mathbf{e}}_{1,1,z}-i\hat{\mathbf{e}}_{2,1,z})\, e^{i(\mathbf{k}_2-\mathbf{k}_1)\cdot\mathbf{r}}] \\
&= 2E_1^2\cos\theta_0 + 2E_2^2\cos\theta_0 \\
&\quad + E_1E_2 Re[(\sin^2\theta_0 + i\cos\theta_0)\cos\theta_0\, e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{r}}] \\
&\quad - E_1E_2 Re[\sin^2\theta_0(\cos\theta_0 - i)\, e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{r}}] \\
&\quad + E_1E_2 Re[(\sin^2\theta_0 - i\cos\theta_0)\cos\theta_0\, e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{r}}] \\
&\quad - E_1E_2 Re[\sin^2\theta_0(\cos\theta_0 + i)\, e^{i(\mathbf{k}_2-\mathbf{k}_1)\cdot\mathbf{r}}] \\
&= 2E_1^2\cos\theta_0 + 2E_2^2\cos\theta_0 \\
&\quad + 2E_1E_2\cos\theta_0[\sin^2\theta_0\cos(k\sin\theta_0(x-y)) - \cos\theta_0\sin(k\sin\theta_0(x-y))] \\
&\quad - 2E_1E_2\sin^2\theta_0[\cos\theta_0\cos(k\sin\theta_0(x-y)) - \sin(k\sin\theta_0(x-y))].
\end{aligned}
\tag{6}
$$

Of particular interest is the case that θ_0 approaches $\pi/2$, say $0 < \cos\theta_0 = \epsilon \ll 1$, $\sin\theta_0 \approx 1$, for which,

$$
\frac{4\pi}{c}\langle S_z\rangle \approx 2\epsilon(E_1^2+E_2^2) + 2E_1E_2\sin(k(x-y)). \tag{7}
$$

The dominant term, $2E_1E_2\sin(k(x-y))$, oscillates in the variable $x-y$ and is negative over half of the x-y plane. That is, the Poynting vector of the combination of the two waves of eqs. (1)-(5) has a negative-z component in almost half of all space even though the Poynting vectors of each of the component waves have positive z-components. This counterintuitive behavior is the (classical) "backflow".

A Appendix. The Variant of Daniel *et al.*

We follow eq. (5), p. 10 of [5] and consider the two plane waves,

$$
\mathbf{E}_j = E_j\,\hat{\mathbf{e}}_j\, e^{i(\mathbf{k}_j\cdot\mathbf{r}-\omega t)}, \qquad \mathbf{B}_j = \hat{\mathbf{k}}_j\times\mathbf{E}_j, \tag{8}
$$

in Gaussian units, where the wave vectors $\mathbf{k}_j$ (with $k_j = k = \omega/c$ where c is the speed of light in vacuum) point in the directions (θ_j, ϕ_j) in a spherical coordinate system (r, θ, ϕ) with,

$$
\theta_1 = \theta_0, \qquad \phi_1 = 0, \qquad \text{and} \qquad \theta_2 = \theta_0, \qquad \phi_2 = \pi. \tag{9}
$$

The wave vectors $\mathbf{k}_j$ for $j = 1, 2$ are, in rectangular coordinates (x, y, z),

$$\mathbf{k}_1 = k(\sin\theta_0, 0, \cos\theta_0), \qquad \mathbf{k}_2 = k(-\sin\theta_0, 0, \cos\theta_0), \qquad \mathbf{k}_1 - \mathbf{k}_2 = 2k(\sin\theta_0, 0, 0), \quad (10)$$

and the polarization unit vectors $\hat{\mathbf{e}}_j$ are taken to be,

$$\hat{\mathbf{e}}_1 = (\cos\theta_0, 0, -\sin\theta_0), \qquad \hat{\mathbf{e}}_2 = (-\cos\theta_0, 0, -\sin\theta_0), \qquad (11)$$

$$\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2 = -\cos 2\theta_0, \qquad \hat{\mathbf{e}}_1 \cdot \mathbf{k}_1 = 0 = \hat{\mathbf{e_2}} \cdot \mathbf{k_2}, \qquad \hat{\mathbf{e}}_1 \cdot \mathbf{k}_2 = -\sin 2\theta_0 = \hat{\mathbf{e_2}} \cdot \mathbf{k_1}, \qquad (12)$$

The z-component of the time-average Poynting vector is given by,

$$\begin{aligned}
\frac{4\pi}{c}\langle S_z\rangle = Re[\mathbf{E}\times\mathbf{B}^\star]_z &= Re[(\mathbf{E}_1 + \mathbf{E}_2)\times(\hat{\mathbf{k}}_1\times\mathbf{E}_1^\star + \hat{\mathbf{k}}_2\times\mathbf{E}_2^\star)]_z \\
&= |\mathbf{E}_1|^2\,k_{1,z} - Re[(\mathbf{E}_1\cdot\hat{\mathbf{k}}_1)\mathbf{E}_{1,z}^\star] + |\mathbf{E}_2|^2\,k_{2,z} - Re[(\mathbf{E}_2\cdot\hat{\mathbf{k}}_2)\mathbf{E}_{2,z}^\star] \\
+Re[\mathbf{E}_1\cdot\mathbf{E}_2^\star]\hat{\mathbf{k}}_{2,z} - Re[(\mathbf{E}_1\cdot\hat{\mathbf{k}}_2)\mathbf{E}_{2,z}^\star] &+ Re[\mathbf{E}_2\cdot\mathbf{E}_1^\star]\hat{\mathbf{k}}_{1,z} - Re[(\mathbf{E}_2\cdot\hat{\mathbf{k}}_1)\mathbf{E}_{1,z}^\star] \\
&= E_1^2\cos\theta_0 + E_2^2\cos\theta_0 \\
+E_1E_2Re[(\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_2)\hat{\mathbf{k}}_{2,z}\,e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{r}}] &- E_1E_2Re[(\hat{\mathbf{e}}_1\cdot\hat{\mathbf{k}}_2)\,\hat{\mathbf{e}}_{1,z}\,e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{r}}] \\
+E_1E_2Re[(\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1)\hat{\mathbf{k}}_{1,z}\,e^{i(\mathbf{k}_2-\mathbf{k}_1)\cdot\mathbf{r}}] &- E_1E_2Re[(\hat{\mathbf{e}}_2\cdot\hat{\mathbf{k}}_1)\,\hat{\mathbf{e}}_{1,z}\,e^{i(\mathbf{k}_2-\mathbf{k}_1)\cdot\mathbf{r}}] \\
&= E_1^2\cos\theta_0 + E_2^2\cos\theta_0 \\
-2E_1E_2\cos 2\theta_0\cos\theta_0\cos(2k\sin\theta_0\,x) &+ 2E_1E_2\sin 2\theta_0\cos\theta_0\cos(2k\sin\theta_0\,x). \qquad (13)
\end{aligned}$$

we again consider the case that θ_0 approaches $\pi/2$, say $0 < \cos\theta_0 = \epsilon \ll 1$, $\sin\theta_0 \approx 1$, $\cos 2\theta_0 \approx -1$, $\sin 2\theta_0 \approx 2\epsilon$, for which,

$$\frac{4\pi}{c}\langle S_z\rangle \approx \epsilon(E_1^2 + E_2^2) + 2\epsilon E_1E_2\cos(2kx). \qquad (14)$$

There is no "backflow" here, although for $E_1 = E_2$ it is almost present.[1]

The example of [5] was actually for scalar waves. For vector/electromagnetic waves, it seems that circular polarization is needed to demonstrate "backflow".

References

[1] G.R. Allcock, *The Time of Arrival in Quantum Mechanics I. Formal Considerations*, Ann. Phys. **53**, 253 (1969), http://kirkmcd.princeton.edu/examples/QM/allcock_ap_53_253_69.pdf

[2] G.R. Allcock, *The Time of Arrival in Quantum Mechanics II. The Individual Measurement*, Ann. Phys. **53**, 286 (1969), http://kirkmcd.princeton.edu/examples/QM/allcock_ap_53_286_69.pdf

[3] G.R. Allcock, *The Time of Arrival in Quantum Mechanics III. The Measurement Ensemble*, Ann. Phys. **53**, 311 (1969), http://kirkmcd.princeton.edu/examples/QM/allcock_ap_53_311_69.pdf

[4] Y. Eliezer, T. Zacharias and A. Bahabad, *Observation of optical backflow*, Optica **7**, 72 (2020), http://kirkmcd.princeton.edu/examples/optics/eliezer_optica_7_72_20.pdf

[1]Taking $\hat{\mathbf{e}}_2 = (0, 1, 0)$ instead does not help.

[5] A. Daniel *et al.*, *Demonstrating backflow in classical two beams' interference* (June 13, 2022), https://arxiv.org/abs/2206.05242

[6] I. Bialynicki-Birula, Z. Bialynicka-Birula and S. Augustynowicz, *Backflow in relativistic wave equations*, J. Phys. A **55**, 255702 (2022),
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