Regarding the energy Lorentz transformation of classical particles and waves bouncing in a box

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(January 10, 2021)

Abstract

It is shown that in a rectangular box with perfectly reflecting walls, the field energy and momentum of a classical electromagnetic normal mode do not transform as a Lorentz 4-vector. Moreover, an exact expression is found for the transformed energy, and it is found to have an oscillating component. Likewise, the total energy and momentum of particles bouncing in a box do not form a Lorentz 4-vector, but a model of the energy and momentum in the walls of the box associated with the stress of the particles bouncing off them shows that the total system energy and momentum do form a Lorentz 4-vector.

Introduction

In this note, we discuss the classical problem of a box (or cavity) which contains bouncing particles or standing electromagnetic waves. In the case of waves, the box has "perfectly" reflecting walls (a.k.a. mirrors) and in the case of particles, the box has "perfectly" elastic walls. In both cases, we compute the energy and momentum of the interior subsystem (that is, the waves or particles) in the rest frame of the box, and the energy in a moving reference frame. What seems surprising (at least to me), is that subsystem energies and momenta **do not** transform as components of Lorentz 4-vectors. Upon further examination of the particle case, the root of this problem is that the box walls participate in the containment of waves or particles, and the wall energy and momentum transform in such a way that the total energy and momentum of the system **do** form a Lorentz 4-vector.

What these results illustrate, is that in relativistic systems, it isn't always possible to cleanly identify subsystems with energy-momentum vectors that transform as Lorentz 4-vectors.[1] This observation isn't new and has been previously noted in papers by Rohrlich [1] and McDonald [2], as well as in sec. 12.10 of [3]. What those authors have explored is the question of which Lorentz vectors and second-order tensors can be constructed from subsystem parameters, to have properties resembling energy and momentum.[2] The analysis below approaches things a bit differently. For the subsystems, we will take their energy and momentum variables to be given by the expressions that would apply if those subsystems

[1]In this paper, when we refer to the 4-momentum in a particular frame, that should not be taken to imply anything about the frame transformation rules to a different frame. Rather, for the purposes of this paper, the frame-dependent 4-momenta should be regarded simply as a 4-tuplet. We then will explore how the components of that 4-tuplet transform, and in cases where they transform as a Lorentz 4-vector, that will be noted explicitly.

[2]One difficulty in constructing the "usual" momentum components, is that a confined subsystem isn't translation invariant and therefore the "usual" momentum operator doesn't generate the translation-invariance symmetry.

were free (as though there were no interaction between the walls of the cavity and the substance interior to the cavity). We'll then see that for different kinds of cavity subsystems (*e.g.*, particles or EM waves),[3] the form of the energy-momentum transformations are similar but are not the Lorentz transformations for 4-vectors.

1 Particles in a box; all motion in one dimension

Imagine a box of length L and rest-mass $M_{\rm box}$, in which there are N particles of mass m and velocity $\pm u$ traveling in the $+x$ (right) and $-x$ (left) directions in the rest frame of the box. Those particles collide elastically with the walls, but do not collide with one another. The left-moving particles are uniformly spaced with separations $2L/N$, and the right-moving particles have the same separations. The interval in time between left-moving (or right-moving) particles passing a given point is $\Delta = 2L/uN$.

In the rest frame of the box, the particles in flight have 4-momentum $(E, p, 0, 0) = (Nm\gamma(u), 0, 0, 0)$, where $\gamma(u) = 1/\sqrt{1-u^2}$, in units where the speed of light is 1. Now go into a reference frame moving at velocity v to the right and compute the 4-momentum of the particles in flight in this new reference frame, which we will call the $'$ frame.[4]

Details of the calculations in the $'$ frame are in sec. 1.1 below, and are summarized as follows. The time-interval Δ is Lorentz dilated, $\Delta' = \gamma(v)\Delta$. The box is Lorentz contracted, $L' = L/\gamma(v)$. We examine only particles "in flight" – that is, we consider particle momenta and energy only during instants of time when no particles are in contact with the walls. This concept of "instant of time" is frame-dependent. Particles in flight, which are taken to be simultaneous in the $'$ frame, aren't simultaneous in the rest frame, so care must be taken to check which way the particles are going at a given time. Finally, velocities and momenta must be Lorentz transformed (differently according to whether the particles are moving left or right). The net result is that the total 4-momentum $(E', p', 0, 0)$ of the particles is related to the total rest-frame 4-momentum by $(E', p', 0, 0) = \gamma(v)E((1 + u^2v^2, -v(1 + u^2), 0, 0)$. As anticipated in the introduction, the transformed energy has a term proportional to u^2v^2 and the transformed momentum has a term proportional to u^2, neither of which would be present if the total energy-momentum of the particles were a Lorentz 4-vector.

1.1 Energy and momentum of the particle subsystem

As noted above, the total 4-momentum of the particles in the rest frame of the box is,

$$P = (E, p, 0, 0) = (Nm\gamma(u), 0, 0, 0). \tag{1}$$

In the $'$ frame, the particle velocities are,

$$u'_L = -\frac{u+v}{1+vu}\,, \qquad u'_R = \frac{u-v}{1-vu}\,. \tag{2}$$

[3] I'm grateful to Kirk McDonald for pointing out the clarifications required for our definition of subsystem energies.

[4] Quantities in the $'$ frame will be written with a $'$, while quantities in the rest frame of the box will be written without a $'$. The Lorentz transformation of coordinates from the rest frame to the $'$ frame is $x' = \gamma(v)(x - vt)$, $t' = \gamma(v)(t - vx)$, and the velocity transformation is $u'_x = (u_x - v)/(1 - u_xv)$.

Here, u'_L denotes the velocity of the particles going from the right wall towards the left wall (in the $'$ reference frame), and u'_R denotes the velocity of the particles going from the left wall towards the right wall.

What is the travel-time T'_L it takes for a particle to travel from the right wall to the left wall in the $'$ frame? Remember that the distance between walls in the $'$ frame is $L' = L/\gamma(v)$. The equation to be solved is,

$$u'_L T'_L = -vT'_L - \frac{L}{\gamma(v)} \,. \tag{3}$$

Similarly, the travel-time T'_R for a particle to travel from the left wall to the right wall is the solution to the equation,

$$u'_R T'_R = -vT'_R + \frac{L}{\gamma(v)} \,. \tag{4}$$

These are solved by,

$$T'_L = \frac{L/\gamma(v)}{(u+v)/(1+vu) - v} = \frac{L\gamma(v)(1+vu)}{u} = \frac{N\Delta\gamma(v)(1+vu)}{2} \,, \tag{5}$$

$$T'_R = \frac{L/\gamma(v)}{(u-v)/(1-vu) + v} = \frac{L\gamma(v)(1-vu)}{u} = \frac{N\Delta\gamma(v)(1-vu)}{2} \,. \tag{6}$$

Next, we calculate the time-average) number N'_L of particles traveling from the right wall to the left wall, as observed in the $'$ frame, and the number of particles N'_R traveling from the left wall to the right wall. For each direction, divide the travel-time by the bounce time-interval, $\Delta' = \gamma(v)\Delta$, as measured in the $'$ frame, to obtain,[5]

$$N'_L = \frac{T'_L}{\gamma(v)\Delta} = \frac{N(1+uv)}{2} \,, \qquad N'_R = \frac{T'_R}{\gamma(v)\Delta} = \frac{N(1-uv)}{2} \,. \tag{7}$$

It is easy to see from these equations that $N'_L + N'_R = N$, showing the Lorentz invariance of the total number of particles in flight.

The final step is to calculate the the energies and momenta of the particles in the $'$ frame. We use the Lorentz transformation $(E'(u), p'(u), 0, 0) = \gamma(v)((m\gamma(u) - vp(u)), p(u) - mv\gamma(u), 0, 0)$, where $(m\gamma(u), p(u), 0, 0)$ is the 4-momentum of the particle in the rest frame and $(E'(u), p'(u), 0, 0)$ is the 4-momentum of the particle in the $'$ frame. Each left-moving particle in the rest frame has momentum $\gamma(u)(-mu)$ and each right-moving particle in the rest frame has momentum $\gamma(u)(mu)$, so,

$$\begin{aligned}(E'_L, p'_L, 0, 0) &= N'_L m\gamma(u)\gamma(v)((1+uv), -(u+v), 0, 0)\\ &= \frac{Nm\gamma(u)\gamma(v)}{2}((1+uv)^2, -(u+v)(1+uv), 0, 0),\end{aligned} \tag{8}$$

and,

$$\begin{aligned}(E'_R, p'_R, 0, 0) &= N'_R m\gamma(u)\gamma(v)((1-uv), (u-v), 0, 0)\\ &= \frac{Nm\gamma(u)\gamma(v)}{2}((1-uv)^2, (u-v)(1-uv), 0, 0).\end{aligned} \tag{9}$$

[5] Since the number of particles cannot be fractional, all results depending on N'_L and N'_R – neither of which are generally integers – must be regarded as time-averages. That is true, for example, of both E' and p'.

As usual, the subscripts L and R denote respectively the left and right moving particles. Then, adding the two 4-vectors gives the total transformed 4-momentum as,

$$\begin{aligned} P' = (E', p', 0, 0) &= (E'_L + E'_R, p'_L + p'_R, 0, 0) \\ &= Nm\gamma(u)\gamma(v)\left(\left(1+u^2v^2\right), -v(1+u^2), 0, 0\right) \\ &= \gamma(v)E\left(\left(1+u^2v^2\right), -v(1+u^2), 0, 0\right), \end{aligned} \tag{10}$$

as claimed above, recalling the total rest-frame E from eq. (1). The terms proportional to u^2v^2 and u^2 are consequences of the fact that $N'_R \neq N'_L$.

One further remark is in order at this point. The problem was carefully set up in the rest frame so that particles collide with the two walls at precisely the same time, and therefore the same number of particles are always traveling rightwards as traveling leftwards. This was accomplished by our choice of Δ. Nothing would prevent us from picking a different value of Δ so that during some portion of the interval, there are differing numbers of particles moving leftwards and rightwards in the rest frame. Then, the total lab-frame momentum of the particles (in flight) would not always be zero, and the box enclosing the particles would not always be at rest. But, even with Δ chosen such that the rest frame total particle 4-momentum is time-independent, the $'$ frame momentum is not. As observed previously, the $'$ frame numbers of particles computed in equation (7) are generally fractional, implying a complicated time-dependence in the numbers of particles traveling in each direction. This also implies a similarly complicated time-dependence of the transformed 4-momentum. However, the time-average behavior follows eq. (10) in the $'$ frame (and eq. (1) in the rest frame).

1.2 Energy and momentum of the walls

There is no transfer of energy between particles and walls in the approximation of elastic collisions and rigid walls connected by rigid struts to form a rigid *box*. An actual box would not be rigid and the collisions would not be elastic. We make no attempt to construct a model for this, but anticipate that the physics would be challenging (see, for example, the non-interaction theorem of Currie *et al.* [4]).

Instead, we estimate the contribution of the walls' energy and momentum to the total system by imagining that the left and right walls are detached from one another. Then, the bouncing particles would push the walls apart, and in doing so the particles would gradually lose their energy. The heavier the walls are, the longer it would take for the particles to slow down.

In the calculations which follow, assume that the right and left walls each have a rest mass M_w which is much larger than the particle mass m. For convenience, also assume that side walls have 0 mass, so that in the absence of collisions (the *initial* state), the box's total rest frame 4-momentum is $P_{\text{initial,box}} = (2M_w, 0, 0, 0)$.

Start in the rest frame, and consider a right-moving particle with velocity u. Its 4-momentum is $(m\gamma(u), mu\gamma(u), 0, 0)$. Assume that the right wall (which, for the purposes of this calculation is detached from the left wall) is initially at rest. Also assume an elastic collision between the particle and the wall, by which is meant that the following energy-momentum conservation equations hold in the rest frame,

$$m\gamma(u) + M_w = m\gamma(u_f) + M_w\gamma(u_w). \tag{11}$$

$$mu\gamma(u) = mu_f\gamma(u_f) + M_w u_w\gamma(u_w), \tag{12}$$

where u_f is the final velocity of the particle, and u_w is the final velocity of the wall. It follows that,

$$u_f = -u + \mathcal{O}\left(\frac{m}{M_w}\right), \tag{13}$$

and the post-collision 4-momentum (in the rest frame) for the wall is,

$$(E_w, p_w, 0, 0) = \left(M_w\left(1 + \mathcal{O}\left(\frac{m}{M_w}\right)^2\right), 2mu\gamma(u) + m\,\mathcal{O}\left(\frac{m}{M_w}\right), 0, 0\right). \tag{14}$$

In the $'$ frame after collision, the wall's energy is transformed to $\gamma(v)(M_w - 2mvu\gamma(u)) + m\,\mathcal{O}(m/M_w)$. The wall's momentum is transformed to $\gamma(v)(2mu\gamma(u) - vM_w) + m\,\mathcal{O}(m/M_w)$. From now on, drop the terms of relative order $\mathcal{O}(m/M_w)$. Without collisions, the right wall's 4-momentum in the $'$ frame would have been $P'_{\text{initial,box}} = \gamma(v)M_w(1, -v, 0, 0)$. For each collision with the right wall, there is an extra contribution of $-2mvu\gamma(u)\gamma(v)$ to the wall's $'$-frame energy and an extra contribution of $2mu\gamma(u)\gamma(v)$ to the wall's $'$-frame momentum. So if there are N'_L particles in flight from left to right, they will have caused – during their bounces – the right wall to change its 4-momentum by $2N'_L m\gamma(u)\gamma(v)(-vu, u, 0, 0)$.

In precisely the same way, left-moving particles bouncing with the left wall, will each cause changes to the wall's energy and momentum. The left-moving particles are traveling in the opposite direction to v so some of the expressions will have sign changes. Then, following the same reasoning as before, if there are N'_R particles in flight from right to left, they will have caused – during their bounces – the left wall to change its 4-momentum by $2N'_R m\gamma(u)\gamma(v)(vu, -u, 0, 0)$.

In summary, the prior impact of particles in flight has changed the $'$-frame 4-momentum of the box from $P'_{\text{initial,box}}$ to $P'_{box} = P'_{\text{initial,box}} + 2m\gamma(u)\gamma(v)(N'_R - N'_L)(vu, -u, 0, 0)$. Of course, the actual box is rigid, so these changes of momentum are presumably not kinetic, but rather internal (to the box walls). It seems plausible to assume that the net effect on the walls should be the average energy and momentum gained from the particles in flight, thus half of the change computed above. Putting all this together, and recalling eq. (7), we obtain for this average,

$$P'_{\text{box}} = P'_{\text{initial,box}} - Nm\gamma(u)\gamma(v)uv(vu, -u, 0, 0). \tag{15}$$

When this is added to eq. (10) we get, recalling P from eq. (1),

$$\begin{aligned} &P' + P'_{\text{box}} \\ &= \gamma(v)\left(Nm\gamma(u)(1 + u^2v^2) + M_{\text{box}} - Nmu^2v^2\gamma(u), -Nm\gamma(u)v(1 + u^2) + Nm\gamma(u)u^2v, 0, 0\right) \\ &= \gamma(v)(P + P_{\text{box}}). \end{aligned} \tag{16}$$

The **total** 4-momentum of the system is thus seen to transform as a Lorentz 4-vector.

1.3 Critical illumination

Another analysis can be done which allows us to consider the total 4-momentum, but without making assumptions about the walls' internal energy or resorting to the plausibility argument given for averaging over collisions. We start off similarly to the previous section, with the left and right walls detached from one another. However, now the walls are held in place by collisions with external particles. Those external particles are emitted by two sources: one that is far to the left of the box, and one that is far to the right of the box. The frequency and momentum of the external collisions must be arranged to precisely counteract the effect of the internal particles colliding with the walls. We will describe this arrangement using the term *critical illumination*, borrowed from a paper by Fiola *et al.* on black-hole thermodynamics [5].

We will analyze this situation by picking a time interval and computing the total 4-momentum during that interval. The time interval is chosen to be very small compared to the amount of time it takes for a particle to travel from the source to the closest wall but very large compared to the travel-time from one wall to the other.[6] Under these circumstances, we can make the assumption that during the stated time interval, the sources don't emit any new particles. It therefore won't be necessary to consider the dynamics of the sources and for the remainder of this analysis, the sources will be ignored, as will the exact duration of the above-stated time interval. For convenience, we set the rest-frame x-coordinates of the two walls at 0 and L. Also, we set the number, N_E, of left external particles (*i.e.*, those to the left of the cavity) to be equal the number of right external particles.

Start in the rest frame. Consider first the right wall. The internal particles collide with the wall with a frequency $1/\Delta$, and each internal particle has a momentum $mu\gamma(u)$. Assuming all interactions are perfectly elastic, then in order for the wall to remain at rest, the external particles must collide with the wall with the same frequency as the internal balls, and also the external particles must each have an initial momentum of $-mu\gamma(u)$. Furthermore, the external particles each have the same energy $m\gamma(u)$ as the internal particles. After collision, both the internal particles and external particless reverse their momenta and maintain their energies. Similar considerations apply to the left wall. It is easy to see that at all times, the external (to the cavity) momentum is 0 and the external energy is $2N_E m\gamma(u)$. Therefore, in the rest frame, the total system momentum is 0 and the total system energy is,

$$E_s = 2N_E m\gamma(u) + E, \tag{17}$$

where E is given by eq. (1).

In the $'$ frame, we have to be particularly careful managing the non-simultaneity of events. We've already encountered that issue when computing the 4-momenta of particles inside the box, and indeed, that was the origin of the fact that the cavity energy and momentum don't transform as a 4-vector. We obtain for each external right-hand particle before collision, the 4-momentum,[7]

$$P_R'^B = m\gamma(v)\gamma(u)((1+uv), -(u+v), 0, 0), \tag{18}$$

[6] These inequalities are assumed to be true both in the rest frame and the moving frame.

[7] In obtaining these 4-momenta, it is easiest to arrange things so that the wall collisions with external particles occur at precisely the same instants as they do with the internal particles (otherwise the equations would be slightly more complicated, involving the wall energies and momenta which, when averaged over time, would vanish in the limit of large time).

and after collision,

$$P_R'^A = m\gamma(v)\gamma(u)((1-uv),(u-v),0,0), \tag{19}$$

with a net change of,

$$\Delta P_R' = 2m\gamma(v)\gamma(u)(-uv,u,0,0). \tag{20}$$

Similarly, for each external left-hand ball before collision, the energy-momentum vector is,

$$P_L'^B = m\gamma(v)\gamma(u)((1-uv),(u+v),0,0), \tag{21}$$

and after collision,

$$P_L'^A = m\gamma(v)\gamma(u)((1+uv),-(u-v),0,0), \tag{22}$$

with a net change of,

$$\Delta P_L' = 2m\gamma(v)\gamma(u)(uv,-u,0,0). \tag{23}$$

What is important to note is that for each collision, $\Delta P_L' = -\Delta P_R'$. Because of that, the net external momentum at any instant in the $'$ frame, is obtained by computing how many more collisions have occurred on the right than on the left, then multiplying by $\Delta P_R'$. Without loss of generality, let $t' = 0$ and the left side of the cavity be at $x_L' = 0$. Since the cavity is contracted in the $'$-frame, its right side will be at $x_R' = L/\gamma(v)$. Although those two points are simultaneous in the $'$ frame, they have a time difference of Lv in the rest frame. We can easily see this from the transformation equation $t = \gamma(v)(t' + vx')$ leading to the left side with $t_L = 0$ and the right side with $t_R = vL$. In the rest frame, we have previously found that the interval between internal collisions (with each wall) is $\Delta = 2L/uN$, so the total number of excess right-wall collisions in the rest-frame (corresponding to simultaneity in the $'$-frame) is $N_{\text{excess}} = vL/\Delta = Nuv/2$.

Finally, the external 4-momentum can be computed in the $'$ frame as,

$$\begin{aligned} P_E' &= (Nuv/2)\Delta P_R' = \frac{Nuv}{2} 2m\gamma(v)\gamma(u)(-uv,u) \qquad (24) \\ &= Nm\gamma(u)\gamma(v)(-u^2v^2, u^2v). \qquad (25) \end{aligned}$$

When we add this to the transformed internal cavity 4-momentum P' from eq. (10), obtaining,

$$P' + P_E' = \gamma(v)(P + P_E). \tag{26}$$

This completes the proof that the total system 4-momentum, during critical illumination, transforms as a Lorentz 4-vector.

Critical illumination can also be used to examine closed systems of length L, whose interior 4-momentum is carried by a different mechanism than that of the critical illumination. As an example to be considered shortly, the box can be replaced by a cavity in which there are standing electromagnetic waves. If this cavity is critically illuminated by particles,[8] then

[8] In this section on critical illumination, an essential assumption in the analysis was that the walls – since they are held in place by internal and external forces – have a total 4-momentum (specifically of the form $(M_w, \mathbf{0})$) which transforms as a Lorentz 4-vector. This assumption seems plausible when the six walls of the box are mechanically detached from one another and are bombarded on all sides by particles. However, perhaps more analysis is needed when some of the forces are electromagnetic in nature. In general, electric currents must then flow across the edges between adjacent walls and we may need to add an assumption that while the walls are mechanically detached from one another, they remain in electrical contact.

the transformation properties of the external 4-momentum must combine with those of the internal 4-momentum so as to preserve the Lorentz 4-vector nature of the total. Therefore, without actually doing the calculation of the transformation properties of the electromagnetic 4-momentum in the box, we can infer it from the transformation properties of the total 4-momentum of the external particles.

2 Particles in a box; three-dimensional motion

Imagine a cubical box of edge length L, with particles of mass m and velocities $(\pm u_x, \pm u_y, \pm u_z)$ bouncing elastically off the walls. We will again consider both the box rest-frame, and the $'$ frame where the observer moves in the positive x direction with speed v. From now on, in this paper, we will concentrate on the energy (and not the momentum) of the subsystem internal to the cavity. In the rest frame, the internal subsystem momentum is 0. Therefore, **if** the 4-momentum were to transform as a Lorentz 4-vector, the transformed energy E' should be $E' = \gamma(v)E$. What we will find, is that this is **not** the way the energy transforms. We will describe this phenomenon by saying that the "energy does not transform as the component of a Lorentz 4-vector".

The particles will bounce off all walls but the initial state will, as before, be set up so that in the rest frame, there are N particles in flight, bouncing off the left and right walls at a constant interval $\Delta = 2L/(u_x N)$. One minor generalization to this analysis could be to set up the problem so that, initially, multiple particles are emitted simultaneously from one of the walls. Since that would not ultimately change the relationship between rest-frame and $'$-frame energies, we won't analyze this generalization any further.

2.1 Massive particles

The calculation of particle energy proceeds similarly to the one-dimensional case. Assume in this section that $m > 0$. Each particle has a rest frame 4-momentum $m\gamma(u)(1, \pm u_x, \pm u_y, \pm u_z)$ where $u = \sqrt{u_x^2 + u_y^2 + u_z^2}$. In the rest frame, the particle subsystem total energy is then,

$$E_{3D} = Nm\gamma(u) = \frac{2Lm\gamma(u)}{u_x \Delta}\,. \tag{27}$$

In the $'$ frame, the particle's energy is $E' = \gamma(u)\gamma(v)m(1 \mp vu_x)$. It is easy to see that the numbers, $N_L'^{3D}$, of particles moving leftwards, and $N_R'^{3D}$ of particles moving rightwards, are computed just as in eq. (7), except that particle speed u_x is used instead of u. The final result in the $'$ frame for energy of particles in flight is,

$$E'_{3D} = N\gamma(v)m\gamma(u)(1 + u_x^2 v^2) = \gamma(v)E_{3D}(1 + u_x^2 v^2). \tag{28}$$

2.2 Massless particles

Much of the analysis done for massive particles, also applies to massless particles, which travel at light speed ($u = 1$). Of course, care must be taken with limits. In the case where motion is in the x direction, eq. (7) becomes,[9]

$$N_L^{\prime\text{photon}} = \frac{L}{\Delta}(1+v), \qquad N_R^{\prime\text{photon}} = \frac{L}{\Delta}(1-v). \tag{29}$$

This result was calculated by Avron *et al.* [6] who also illustrated, with a space-time diagram, the difference between N'_L and N'_R and the fact that the total number of massless particles is Lorentz invariant. Unfortunately, their analysis was incomplete and they did not deduce the anomalous Lorentz transformation behavior of energy of the massless-particle subsystem.

The problem of true (vector) photons in a cavity has also been examined by several other authors including Einstein [7], Zeldovich [8] and McDonald [9]. As observed by McDonald, photons in a box (with conducting walls) do not travel parallel to two sides of the cavity. More precisely, when using photon states to help model classical electromagnetic fields in a cavity, those photon states are superpositions representing photons whose momenta are **not** parallel to the x-axis. More on this later.

For massless particles in three dimensions, we can adapt the results of the previous section. Our starting point is the rest-frame particle 4-momentum $(\omega, \pm k_x, \pm k_y, \pm k_z)$ where $w = k = \sqrt{k_x^2 + k_y^2 + k_z^2}$. It's easy to see that the particle's x-velocity is $u_x = k_x/\omega$. In the $'$ frame, the particle energy is $E' = \gamma(v)(1 \mp v k_x)$ and finally, the $'$-frame energy of particles in flight is,

$$E'_{\text{photon}} = \gamma(v) E_{\text{photon}} \left(1 + \frac{k_x}{\omega}^2 v^2\right) = \gamma(v) E_{\text{photon}} \left(1 + u_x^2 v^2\right), \tag{30}$$

where E_{photon} is the cavity rest-frame energy of massless particles in flight.

Thus far, the massless particles discussed don't have the properties of real physical photons. In order to properly make that connection, quantum mechanics must be introduced. Furthermore, in order to relate photons to electromagnetic fields, their energy is ω, and momentum is $\mathbf{k} = (k_x, k_y, k_z)$, in units where $\hbar = 1$.

Photons are also polarized, although the polarizations have no effect on the energy relationships. What is more of a concern, is that quantum massless particles cannot be sensibly localized. This makes it difficult to convincingly make arguments about bouncing against walls, and about how many photons are traveling in each direction. In fact, there are fundamental ambiguities about how to count the photons that make up the standing waves in a cavity. One approach, described for example, in [10], is to construct coherent states. A coherent k-state $|z_\sigma(\mathbf{k})\rangle$ is defined as,

$$|z_\sigma(\mathbf{k})\rangle = e^{-|z_\sigma(\mathbf{k})|^2/2}\, e^{z_\sigma(\mathbf{k})\hat{a}_\sigma^\dagger(\mathbf{k})} |0\rangle\,, \tag{31}$$

where $\hat{a}_\sigma^\dagger(\mathbf{k})$ is the usual vector-potential creation operator, and $|0\rangle$ is the vacuum state (for example, for a cubic box of length L). This state can be thought of as a superposition of N-photon states, with momentum $\mathbf{k}$, where all values of N are weighted by the exponential operator shown. The expectation of the vector potential operator in the state consisting of the sum over $\mathbf{k}$ of coherent k-states, satisfies [10],

$$\left\langle \hat{\mathbf{A}}(t,\mathbf{x}) \right\rangle = \frac{1}{L^{3/2}} \sum_{\hat{k}} \sum_{\sigma=1}^{2} \sqrt{\frac{2\pi}{\omega}} \left[z_\sigma(\mathbf{k}) \boldsymbol{\epsilon}_\sigma(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} + z_\sigma^\star(\mathbf{k}) \boldsymbol{\epsilon}_\sigma(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{x} + i\omega t} \right]. \tag{32}$$

[9] Here we use the superscript photon to refer to scalar massless particles, and not to real photons

Since eq. (30) is independent of N, that equation applies equally well to the coherent state. However, the caveat about localization could still be of concern. One indirect way to circumvent this issue is to notice that the coherent-state expectation values of the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ can be used to model the classical Maxwell theory [10]. We can then use that classical theory to analyze the energies of standing waves both in the rest frame and $'$ frame of the box. That analysis is the subject of the next section.

3 Electromagnetic waves in a cavity

As anticipated, when analyzing classical electrodynamics of waves in a cavity, the results are expected to be similar to what is predicted with the photon analysis. However, the calculations look different. We examine a right parallelepiped (rectangular) cavity with coordinates $(0 < x_1 < L_1, 0 < x_2 < L_2, 0 < x_3 < L_3)$. Modes are indexed by $\mathbf{n} = (n_1, n_2, n_3)$. Set the mode vector[10] components and frequencies to $k_i^{\mathbf{n}} = n_i\pi/L_i$ and $\omega^{\mathbf{n}} = \sum_{i=1}^{3}(k_i^{\mathbf{n}})^2$, where the n_i are non-negative integers (of which at least 2 must be nonzero).

We will follow a convention where we describe the electric and magnetic fields as the real parts of complex vectors $\mathbf{E}$ and $\mathbf{B}$. Then, the standing waves (modes) for the i^{th} component of the electric and magnetic fields are,

$$ReE_i(t,\mathbf{x}) = \sum_{\mathbf{n}} \left[\frac{1}{\tan(k_i^{\mathbf{n}}x_i)} \prod_{j=1}^{3} \sin(k_j^{\mathbf{n}}x_j)\right] \left[\mathbf{a}_i^{\mathbf{n}} \sin(\omega^{\mathbf{n}}t) + \mathbf{b}_i^{\mathbf{n}} \cos(\omega^{\mathbf{n}}t)\right], \tag{33}$$

$$ReB_i(t,\mathbf{x}) = \sum_{\mathbf{n}} \left[\frac{\tan(k_i^{\mathbf{n}}x_i)}{\omega^{\mathbf{n}}} \prod_{j=1}^{3} \cos(k_j^{\mathbf{n}}x_j)\right] \left[-\left(\mathbf{a}^{\mathbf{n}} \times \mathbf{k}^{\mathbf{n}}\right)_i \cos(\omega^{\mathbf{n}}t) + \left(\mathbf{b}^{\mathbf{n}} \times \mathbf{k}^{\mathbf{n}}\right)_i \sin(\omega^{\mathbf{n}}t)\right], \tag{34}$$

where $\mathbf{a}^{\mathbf{n}}$ and $\mathbf{b}^{\mathbf{n}}$ are coefficient vectors each perpendicular to $\mathbf{n}$. Also, notice that although the combination of trigonometric functions appears to sometimes be ill-defined owing to either a 0 or infinite value of the tangent, it turns out that the trigonometric ratios all have perfectly finite limits as one approaches those values.

The well-known Lorentz transformations of these electromagnetic fields to the $'$ frame, with coordinates $x' = \gamma(v)(x - vt)$ and $t' = \gamma(v)(t - vx)$, are,

$$ReE_1'(t',x',y',z') = ReE_1(\gamma(v)(t'+vx'), \gamma(v)(x'+vt'), y', z'), \tag{35}$$
$$ReB_1'(t',x',y',z') = ReB_1(\gamma(v)(t'+vx'), \gamma(v)(x'+vt'), y', z'), \tag{36}$$

and for $i = 2$ or 3,

$$\begin{aligned} ReE_i'(t',x',y',z') = \gamma(v)[&ReE_i(\gamma(v)(t'+vx'), \gamma(v)(x'+vt'), y', z') \\ &+\epsilon_{i1j} v ReB_j(\gamma(v)(t'+vx'), \gamma(v)(x'+vt'), y', z')], \end{aligned} \tag{37}$$
$$\begin{aligned} ReB_i'(t',x',y',z') = \gamma(v)[&ReB_i(\gamma(v)(t'+vx'), \gamma(v)(x'+vt'), y', z') \\ &-\epsilon_{i1j} v ReE_j(\gamma(v)(t'+vx'), \gamma(v)(x'+vt'), y', z')], \end{aligned} \tag{38}$$

[10]We use the term"mode vector" to denote the vector whose components are magnitudes of the wave vectors for the oppositely moving waves that constitute the standing wave in question.

where, in these equations, there is implied summation over index j with the standard Levi-Civita epsilon symbol ϵ_{i1j}.

The electromagnetic-field energy is computed in the rest-frame as,

$$\mathcal{E}(t) = \frac{1}{2}\int_0^{L_1}\int_0^{L_2}\int_0^{L_3}\left(|Re\mathbf{E}(t,\mathbf{x})|^2 + |Re\mathbf{B}(t,\mathbf{x})|^2\right)\, d^3x, \tag{39}$$

and in the $'$ frame as,[11]

$$\mathcal{E}'(t') = \frac{1}{2}\int_{-vt'}^{\frac{L_1}{\gamma(v)}-vt'}\left[\int_0^{L_2}\int_0^{L_3}\left(|Re\mathbf{E}'(t',\mathbf{x}')|^2 + |Re\mathbf{B}'(t',\mathbf{x}')|^2\right)\, dy'\, dz'\right] dx'. \tag{40}$$

It is easily shown that, as expected, $\mathcal{E}(t)$ is time-independent so the time argument can be suppressed. However, it is not obvious that the same should be true of $\mathcal{E}'(t')$. We will return to this point later.

The details of the calculation of boosted energy will follow the elegant presentation by McDonald [9]. Rather than parameterizing waves by the $\mathbf{a}$ and $\mathbf{b}$ vectors of eqs. (33) and (34), McDonald parametrized the waves by complex vector $\mathbf{E}^0$. We will adopt McDonald's notation.[12]

Define $\mathbf{B}^0$ by,

$$\mathbf{B}^0 = \frac{\mathbf{k}}{\omega} \times \mathbf{E}^0. \tag{41}$$

The standing waves of eq. (33) and (34) are then equivalent to $\mathbf{E} = \mathbf{E}^+ + \mathbf{E}^-$ and $\mathbf{B} = \mathbf{B}^+ + \mathbf{B}^-$ where,

$$\mathbf{E}^+ = (E_1^0 \sin k_2 y \sin k_3 z, -iE_2^0 \cos k_2 y \sin k_3 z, -iE_3^0 \sin k_2 y \sin k_3 z)\frac{e^{i(k_1 x - \omega t)}}{2}\,, \tag{42}$$

$$\mathbf{E}^- = (E_1^0 \sin k_2 y \sin k_3 z, iE_2^0 \cos k_2 y \sin k_3 z, iE_3^0 \sin k_2 y \sin k_3 z)\frac{e^{-i(k_1 x + \omega t)}}{2}\,, \tag{43}$$

$$\mathbf{B}^+ = (-B_1^0 \cos k_2 y \cos k_3 z, -iB_2^0 \sin k_2 y \cos k_3 z, -iB_3^0 \cos k_2 y \sin k_3 z)\frac{e^{i(k_1 x - \omega t)}}{2}\,, \tag{44}$$

$$\mathbf{B}^- = (B_1^0 \cos k_2 y \cos k_3 z, -iB_2^0 \sin k_2 y \cos k_3 z, -iB_3^0 \cos k_2 y \sin k_3 z)\frac{e^{-i(k_1 x + \omega t)}}{2}\,, \tag{45}$$

We only are concerned with the real parts of $\mathbf{E}^\pm$ and $\mathbf{B}^\pm$. These satisfy the transformation eqs. (35)-(38). Note that a linear transformation of the real part of an equation is equal to the real part of the linear transformation. Therefore, we can apply eqs. (35)-(38) directly

[11] There is an arbitrariness to the time-dependence of the x' integral. The difference between upper and lower bounds must be equal to the contracted length L_1/γ but the choice of time-displacement depends on precisely which value of t' we choose to correspond to t. In the $'$ frame, the positions (t', x') for a fixed value of t' correspond in the rest frame to different values of t. My choice of bounds doesn't have an intuitive interpretation, but is as good as any other. All time-dependent results will be correct up to a constant shift in time.

[12] One exception on notation is that McDonald's relationship between $\mathbf{B}^0$ and $\mathbf{E}^0$ differs from mine by a factor of i.

to the complex-valued fields. Also note that the coordinates x' and t' appear only in the following situations,

$$e^{\pm i(k_1 x-\omega t)}=e^{\pm i\gamma(v)((k_1-\omega v)x'-(\omega-k_1 v)t')}, \qquad e^{\pm i(k_1 x+\omega t)}=e^{\pm i\gamma(v)((k_1+\omega v)x'+(\omega+k_1 v)t')} \tag{46}$$

The energy-density will involve products of pairs of those exponential terms. It is easy to see that when $v\neq 0$, most of those products include a term oscillating in t'. The only exceptions are products where one of the terms is multiplied by its complex conjugate. If we limit our investigation to time-averages $\langle\mathcal{E}\rangle$, then the $t'-$oscillating terms can be ignored. Consider, for example, the energy-density term $\langle Re\mathbf{E}^+\cdot Re\mathbf{E}^+\rangle$,

$$\begin{aligned} Re\mathbf{E}^+\cdot Re\mathbf{E}^+ &= \frac{1}{4}(\mathbf{E}^+ + \mathbf{E}^{\star +})(\mathbf{E}^+ + \mathbf{E}^{\star +}) \\ &= \frac{1}{4}(\mathbf{E}^+\cdot\mathbf{E}^+ + \mathbf{E}^{\star +}\cdot\mathbf{E}^{\star +} + 2\mathbf{E}^+\cdot\mathbf{E}^{\star +}). \end{aligned} \tag{47}$$

The term $\mathbf{E}^+\cdot\mathbf{E}^+$ is proportional to $e^{2i(k_1 x-\omega t)}=e^{2i\gamma(v)((k_1-\omega v)x'-(\omega-k_1 v)t')}$ so it time averages to 0. That is also the case for the term $\mathbf{E}^{\star +}\cdot\mathbf{E}^{\star +}$. As a consequence, we have,

$$\left\langle Re\mathbf{E}^+\cdot Re\mathbf{E}^+\right\rangle = \frac{\mathbf{E}^+\cdot\mathbf{E}^{\star +}}{2}\,. \tag{48}$$

Similarly, we can show that $\langle Re\mathbf{E}^+\cdot Re\mathbf{E}^-\rangle = 0$ and $\langle Re\mathbf{B}^+\cdot Re\mathbf{E}^-\rangle = 0$, *etc.*, where the time-average is with respect to t' in $'$-frame coordinates. The net result is that,

$$\langle Re\mathbf{E}\cdot Re\mathbf{E}\rangle = \frac{\mathbf{E}^+\cdot\mathbf{E}^{\star +}+\mathbf{E}^-\cdot\mathbf{E}^{\star -}}{2}\,, \tag{49}$$

and so on.

Returning to eqs. (35)-(38), we can now derive that,

$$\left\langle (ReE_1')^2+(ReB_1')^2\right\rangle \equiv U_1'^{+}+U_1'^{-}, \tag{50}$$

$$\left\langle (ReE_2')^2+(ReB_2')^2+Re(E_3')^2+Re(B_3')^2\right\rangle \equiv U_{2,3}'^{+}+U_{2,3}'^{-}, \tag{51}$$

where,

$$U_1'^{\pm} = \frac{\left|E_1^{\pm}\right|^2+\left|B_1^{\pm}\right|^2}{2}\,, \tag{52}$$

$$U_{2,3}'^{\pm} = \frac{\gamma(v)^2\left[(1+v^2)\left(\left|E_2^{\pm}\right|^2+\left|B_2^{\pm}\right|^2+\left|E_3^{\pm}\right|^2+\left|B_3^{\pm}\right|^2\right) - 4vRe\left(E_2^{\pm}B_3^{\star\pm}-E_3^{\pm}B_2^{\star\pm}\right)\right]}{2}\,. \tag{53}$$

In these equations, the fields depend only on coordinates x and y but those arguments have been suppressed for readability.

Now the energy can be computed by performing the integral in eq. (40), remembering that the fields in the integrand should be replaced by the real parts of those fields. The field components are given in eqs. (42) through (45). Since terms depend only on y and z, the x-integral becomes an overall factor of $L_1/\gamma(v)$ multiplied by the constant coefficients $|E_i^0/2|^2$ or $|B_i^0/2|^2$. Assuming for the remainder of this derivation that all wave numbers are nonzero,[13] the y and z integrals each involve the square of either a sin or cos function, and these therefore contribute a factor of $(L_1/2)(L_2/2)$. What is left are the constant coefficients. We write,

$$\langle \mathcal{E}' \rangle = \mathcal{E}'^{+} + \mathcal{E}'^{-}, \tag{54}$$

where,

$$\begin{aligned}
\mathcal{E}'^{\pm} &= \frac{L_1 L_2 L_3}{64\gamma(v)}\Big\{\left|E_1^0\right|^2 + \left|B_1^0\right|^2 + \gamma(v)^2\Big[\left(1+v^2\right)\left(\left|E_2^0\right|^2 + \left|B_2^0\right|^2 + \left|E_3^0\right|^2 + \left|B_3^0\right|^2\right) \\
&\qquad \mp 4vRe\left(E_2^0 B_3^{\star 0} - E_3^0 B_2^{\star 0}\right)\Big]\Big\} \\
&= \frac{L_1 L_2 L_3 \gamma(v)}{64}\Big\{\left(1-v^2\right)\left(\left|E_1^0\right|^2 + \left|B_1^0\right|^2\right) + \left(1+v^2\right)\left(\left|E_2^0\right|^2 + \left|B_2^0\right|^2 + \left|E_3^0\right|^2 + \left|B_3^0\right|^2\right) \\
&\qquad \mp 4vRe\left(E_2^0 B_3^{\star 0} - E_3^0 B_2^{\star 0}\right)\Big\} \\
&= \frac{L_1 L_2 L_3 \gamma(v)}{64}\Big\{\left|E_1^0\right|^2 + \left|B_1^0\right|^2 + \left|E_2^0\right|^2 + \left|B_2^0\right|^2 + \left|E_3^0\right|^2 + \left|B_3^0\right|^2 \\
&\qquad -v^2\left(\left|E_1^0\right|^2 + \left|B_1^0\right|^2 - \left|E_2^0\right|^2 - \left|B_2^0\right|^2 - \left|E_3^0\right|^2 - \left|B_3^0\right|^2\right) \mp 4vRe\left(E_2^0 B_3^{\star 0} - E_3^0 B_2^{\star 0}\right)\Big\}.
\end{aligned} \tag{55}$$

This equation can be further simplified by using eq. (41) to write the components of $\mathbf{B}^0$ in terms of the components of $\mathbf{E}^0$. Remember that $\mathbf{k}$ is perpendicular to both $\mathbf{E}^0$ and $\mathbf{B}^0$. So, $\mathbf{B}^0 \cdot \mathbf{B}^{\star 0} = \mathbf{E}^0 \cdot \mathbf{E}^{\star 0}$. This can be used to rewrite (closely following [9]) the term above which is proportional to v^2 as,

$$\begin{aligned}
&\left|E_1^0\right|^2 + \left|B_1^0\right|^2 - \left|E_2^0\right|^2 - \left|B_2^0\right|^2 - \left|E_3^0\right|^2 - \left|B_3^0\right|^2 \\
&= 2\mathbf{E}^0 \cdot \mathbf{E}^{\star 0} - 2\left|E_2^0\right|^2 - 2\left|B_2^0\right|^2 - 2\left|E_3^0\right|^2 - 2\left|B_3^0\right|^2 \\
&= 2\mathbf{E}^0 \cdot \mathbf{E}^{\star 0} - 2\left|E_2^0\right|^2 - 2\left|E_3^0\right|^2 \\
&\quad -\frac{2}{\omega^2}\left(k_3^2\left|E_1^0\right|^2 + k_1^2\left|E_3^0\right|^2 - 2k_1k_3Re\left(\left(E_1^0 E_3^{\star 0}\right)\right)\right) \\
&\quad -\frac{2}{\omega^2}\left(k_2^2\left|E_1^0\right|^2 + k_1^2\left|E_2^0\right|^2 - 2k_1k_2Re\left(E_1^0 E_2^{\star 0}\right)\right) \\
&= 2\mathbf{E}^0 \cdot \mathbf{E}^{\star 0} - 2\left|E_2^0\right|^2 - 2\left|E_3^0\right|^2 \\
&\quad -\frac{2}{\omega^2}\Big[k_1^2\left|E_3^0\right|^2 + k_1^2\left|E_2^0\right|^2 + k_1^2\left|E_1^0\right|^2 - k_1^2\left|E_1^0\right|^2 - k_1^2\left|E_1^0\right|^2 + k_1^2\left|E_1^0\right|^2 + k_2^2\left|E_1^0\right|^2 + k_3^2\left|E_1^0\right|^2 \\
&\qquad -2k_1k_3Re\left(E_1^0 E_3^{\star 0}\right) - 2k_1k_2Re\left(E_1^0 E_2^{\star 0}\right)\Big] \\
&= 2\mathbf{E}^0 \cdot \mathbf{E}^{\star 0} - 2\left|E_2^0\right|^2 - 2\left|E_3^0\right|^2 \\
&\quad -\frac{2}{\omega^2}\left[k_1^2\mathbf{E}^0 \cdot \mathbf{E}^{\star 0} + \omega^2\left|E_1^0\right|^2 - 2k_1Re\left(E_1^0\mathbf{k} \cdot \mathbf{E}^{\star 0}\right)\right] \\
&= -\frac{2k_1^2\mathbf{E}^0 \cdot \mathbf{E}^{\star 0}}{\omega^2}.
\end{aligned} \tag{56}$$

[13] When one of the wave numbers is 0, that dimension does not involve any trignometric functions, and the resultant overall factor from integration of the y and z integrals, is $(L_1/2)(L_2)$, changing that overall mode energy by a factor of 2.

Then, substituting this result back into eq. (55), and recalling that $\mathbf{B}^0 \cdot \mathbf{B}^{\star 0} = \mathbf{E}^0 \cdot \mathbf{E}^{\star 0}$, we get,

$$\begin{aligned} \mathcal{E}'^{\pm} &= \frac{L_1 L_2 L_3 \gamma(v)}{64} \left[2 \left(1 + v^2 \frac{k_1^2}{\omega^2} \right) \mathbf{E}^0 \cdot \mathbf{E}^{\star 0} \mp 4vRe \left(E_2^0 B_3^{\star 0} - E_3^0 B_2^{\star 0} \right) \right] \\ &= \frac{L_1 L_2 L_3 \gamma(v)}{32} \left(1 \mp \frac{v k_1}{\omega} \right)^2 \mathbf{E}^0 \cdot \mathbf{E}^{\star 0}, \end{aligned} \tag{57}$$

and therefore,

$$\langle \mathcal{E}' \rangle = \frac{L_1 L_2 L_3 \gamma(v)}{16} \left[1 + \left(\frac{k_1}{\omega} \right)^2 v^2 \right] \mathbf{E}^0 \cdot \mathbf{E}^{\star 0} = \gamma(v) \left[1 + \left(\frac{k_1}{\omega} \right)^2 v^2) \right] \mathcal{E} = \gamma(v) \left(1 + u_1^2 v^2 \right) \mathcal{E}, \tag{58}$$

writing $u_1 = k_1/w$ for the 1-component of the phase velocity of the right-moving waves of eqs. (42) - (44). This is consistent with the conclusion of eq. (30) for massless particles in a cavity. If it were possible to treat a standing wave as a separate subsystem, then we would have expected the boosted energy to be $\gamma(v)$ times the rest energy.[14] However, we see from eq. (58), that $\langle \mathcal{E}' \rangle$ does not transform as expected.

The focus thus far has been on the time-average of $\mathcal{E}'(t)$. A remaining question is whether $\mathcal{E}'(t')$ is, in fact, time-dependent. If this quantity were the energy of a boosted subsystem, then it should be time-independent. On the other hand, we already know that $\mathcal{E}'(t')$ isn't the Lorentz transformation of the rest energy, so there is no reason to expect time independence. This can be checked directly, but requires keeping the kinds of cross-terms that were dropped from eq. (47). The derivation is similar to what was done for the time-independent case, but considerably messier, so only the results are presented. First, since time-dependence is now explicit, we need to keep careful track of phases.

We only are concerned with the real parts of $\mathbf{E}^{\pm}$ and $\mathbf{B}^{\pm}$, so to those ends we set,

$$\mathbf{E}^0 = \left(\left| E_1^0 \right| e^{i\phi_1}, \left| E_2^0 \right| e^{i\phi_2}, \left| E_3^0 \right| e^{i\phi_3} \right). \tag{59}$$

It will also be convenient later to define ϕ by,

$$\mathbf{E}^0 \cdot \mathbf{E}^0 = \left| \mathbf{E}^0 \cdot \mathbf{E}^0 \right| e^{i\phi}. \tag{60}$$

Rewriting eq. (40), we have,

$$\mathcal{E}'(t') = \int_{-vt'}^{\frac{L_1}{\gamma(v)} - vt'} \tilde{\mathcal{H}}'(t', x') \, dx,' \tag{61}$$

where,

$$\tilde{\mathcal{H}}'(t', x') = \int_0^{L_2} \int_0^{L_3} \mathcal{H}'(t', x', y', t') \, dy' \, dz', \tag{62}$$

[14]Actually, this statement would require knowing that the rest-frame momentum of the standing wave, is zero. For that, we need the Poynting vector. However, the Poynting vector is a sum of equal but opposite contributions from the left- and right-moving waves that make up the standing wave and so is, in fact, zero.

and,

$$\mathcal{H}'(t',x',y',z') = \frac{1}{2}\left(|Re\mathbf{E}'(t',x',y',z')|^2 + |Re\mathbf{B}'(t',x',y',z')|^2\right). \tag{63}$$

$$\begin{aligned}\tilde{\mathcal{H}}'(t',x') = \frac{L_2L_3}{32}\gamma^2(v)\Bigg\{\left|\mathbf{E}^0\cdot\mathbf{E}^0\right|\Bigg[-\left(v+\frac{k_1}{\omega}\right)^2\cos(2\omega^+t'+2k_1^+x'-\phi)\\ -\left(v-\frac{k_1}{\omega}\right)^2\cos(2\omega^-t'-2k_1^-x'-\phi)+2v^2\left(1-\left(\frac{k_1}{\omega}\right)^2\right)\cos(2\gamma(v)\omega(t'+vx')-\phi)\Bigg]\\ -2\mathbf{E}^0\cdot\mathbf{E}^{\star 0}\left[\left(1-\left(\frac{k_1}{\omega}\right)^2\right)\cos\left(2\gamma(v)k_1(x'+vt')\right)-\left(1+\left(\frac{k_1}{\omega}\right)^2v^2\right)\right]\\ -4\left|E_1^0\right|^2v^2\left[\cos\left(2\gamma(v)\omega(t'+vx')-2\phi_1\right)-\cos\left(2\gamma(v)k_1(x'+vt')\right)\right]\Bigg\},\end{aligned} \tag{64}$$

where,

$$\omega^{\pm} = \gamma(v)(\omega \pm vk_1), \qquad k_1^{\pm} = \gamma(v)(k_1 \pm v\omega). \tag{65}$$

Finally, performing the integral in eq. (61),

$$\begin{aligned}\mathcal{E}'(t') = \frac{\gamma(v)L_2L_3}{64\omega^2}\Bigg\{\left|\mathbf{E}^0\cdot\mathbf{E}^0\right|\Bigg[-(k_1+v\omega)\left(\sin\left(\frac{2}{\gamma(v)}(\omega t'+k_1^+L_1)-\phi\right)-\sin\left(\frac{2\omega t'}{\gamma(v)}-\phi\right)\right)\\ +(k_1-v\omega)\left(\sin\left(\frac{2}{\gamma(v)}(\omega t'-k_1^-L_1)-\phi\right)-\sin\left(\frac{2\omega t'}{\gamma(v)}-\phi\right)\right)\\ +2v\omega\left(1-\left(\frac{k_1}{\omega}\right)^2\right)\left(\sin\left(2\omega\left(\frac{t'}{\gamma(v)}+vL_1\right)-\phi\right)-\sin\left(\frac{2\omega t'}{\gamma(v)}-\phi\right)\right)\Bigg]\\ -4\omega v^2\left|E_1^0\right|^2\left(\sin\left(2\omega\left(\frac{t'}{\gamma(v)}+vL_1\right)-2\phi_1\right)-\sin\left(\frac{2\omega t'}{\gamma(v)}-2\phi_1\right)\right)\Bigg\}\\ +\mathbf{E}^0\cdot\mathbf{E}^{\star 0}\frac{L_1L_2L_3\gamma(v)}{16}\left(1+\left(\frac{k_1}{\omega}\right)^2v^2\right).\end{aligned} \tag{66}$$

In deriving this equation, some terms integrated to 0 because of the fact that $2k_1 = 2n\pi/L_1$.

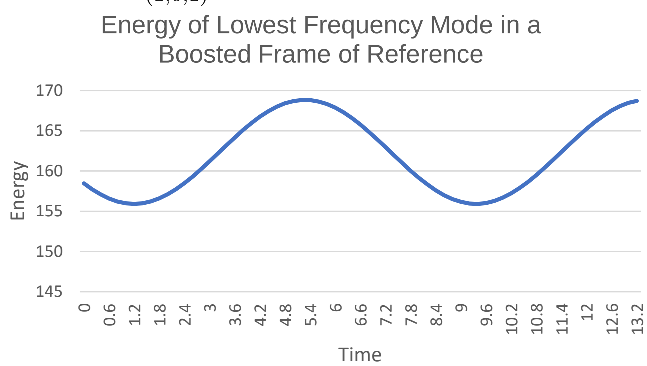
As a specific example, consider the mode with wave number $(1,0,1)$ and $E \equiv -iE_2^0$, and for simplicity consider a cubic box of length L, so that $L_i = L$. Define $k \equiv \frac{\pi}{L}$.[15] The result $\mathcal{E}'_{(1,0,1)}$ in this case is,

$$\begin{aligned}\mathcal{E}'_{(1,0,1)}(t') &= \frac{L^2}{4}\int_{-vt'}^{\frac{L}{\gamma(v)}-vt'}\tilde{\mathcal{H}}'(x',t')\,dx'\\ &= \frac{E^2L^3}{64\pi\sqrt{1-v^2}}\Bigg\{\left(1+\sqrt{2}v\right)\sin\frac{\sqrt{2}\pi\left[\left(\sqrt{2}+2v\right)L+2t'\sqrt{1-v^2}\right]}{L}\end{aligned}$$

[15] Notice the awkward convention here. Usually, the term k is set to ω but in the equations that follow, it will be convenient to define it as shown.

$$+\left(-1+\sqrt{2}v\right)\sin\frac{\sqrt{2}\pi\left[\left(-\sqrt{2}+2v\right)L+2t'\sqrt{1-v^2}\right]}{L}$$
$$-\sqrt{2}v\left[\sin\frac{2\sqrt{2}\pi\left(vL+t'\sqrt{1-v^2}\right)}{L}+\sin\frac{2\sqrt{2}\pi t'\sqrt{1-v^2}}{L}\right]$$
$$+4\pi(v^2+2)\Big\}. \tag{67}$$

Figure 1: $\mathcal{E}'_{(1,0,1)}(t)$ for $E=1$, $v=0.5$ and $L=10$.



As can be seen in Figure 1, the boosted energy indeed is time-dependent. This phenomenon shouldn't be surprising, since a similar effect was mentioned at the end of Section (1.1) with particles in a box.

Would it be possible to construct an experiment capable of detecting this oscillatory behavior, or for that matter, the transformation properties of the time-averaged field energy? This could be done in principle by inserting an appropriate detector in a cavity. Of course, in that case, the detector should be at rest relative to the moving reference frame – that is, moving relative to the cavity. For the experiment to succeed, the cavity fields should be well approximated as those of a single mode, which means that blackbody radiation in the cavity must be negligible, *i.e.*, $kT \ll \hbar\omega$ where T is the absolute temperature of the cavity.

If instead, the cavity fields were well approximated by blackbody radiation, then according to the classical analysis in sec. 2.4 of [9] and the quantum analysis (which also treats Casimir effects) of [11], eq. (58) becomes, for a cubical cavity,

$$\langle\mathcal{E}'\rangle = \gamma(v)\left(1+\frac{u_1^2 v^2}{3}\right)\mathcal{E}, \tag{68}$$

which also should be observable in an experiment.

Summary

It has been shown, through a detailed 3-dimensional calculation, that in a rectangular box with perfectly reflecting walls, the Lorentz-transformed electromagnetic-field energy, $\mathcal{E}'$, oscillates in time. We have also shown that the time-average energy of an electromagnetic normal mode satisfies the Lorentz transformation equation $\langle \mathcal{E}' \rangle = \left(1/\sqrt{1-v^2}\right)\left(1 + u_1^2 v^2\right)\mathcal{E}$, where $\mathbf{v}$ is the relative velocity of the moving reference frame (the $'$ frame) in the x-direction (of the rest frame). These results demonstrate that the electromagnetic field energy inside the box does not transform as the first component of a Lorentz 4-vector.

It was shown that a similar term is present when considering the Lorentz transformation of the total 4-momentum of particles bouncing within a box. In that case, the origin of the extra term is easily related to the non-simultaneity of opposite-wall collisions in a frame in which the box is moving. We also examined what 4-momentum is required in order to keep the box walls rigid while internal particles collide with those walls. As expected, it turns out that when adding the total internal 4-momentum to the total external 4-momentum, the resulting 4-vector transforms as a Lorentz 4-vector. For example, if the walls are held in place by collisions of external particles, those particles' 4-momenta can be compared between reference frames, leading to the above conclusion that the system total 4-momentum transforms as a Lorentz 4-vector. Since the external system is independent from the internal system, it may be possible to generalize that conclusion from the case when the total internal 4-momentum arises from bouncing particles, to the case that the total internal 4-momentum comes from the electromagnetic field. However, additional care may be required when considering contributions from the walls.

Acknowledgments

This note had its genesis in an email discussion initiated by Dale Landis and Paul Jameson, concerning how electromagnetic energy transforms in a moving reference frame. I had been thinking about standing waves in a black body and this email discussion led me to notice a potentially problematic asymmetry between the relativistic transformations of the left and right moving waves. After doing some calculations, I found that the total energy didn't transform as expected. This led me to a literature search and the papers by Avron *et al.* [6] and Kirk McDonald [9]. Although their focus was somewhat different from mine, their insights helped de-mystify the unexpected transformation properties of the standing waves, and led me to the analysis of the preceding sections. Subsequently, McDonald pointed me to other literature (for example, sec. 12.10 of Jackson [3]) addressing the general treatment of the classical electromagnetic stress tensor in the presence of sources, with specific emphasis on the construction of Lorentz covariant generalizations of the free-field expressions.

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