Spinning Basketballs

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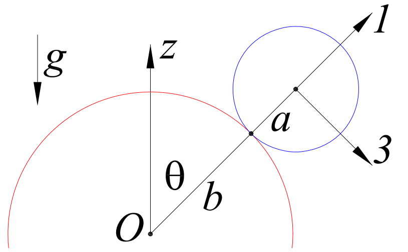
1 Problem

The Harlem Globetrotters can balance a basketball stably on a finger by spinning the ball. That stability is possible if the basketball acts like a gyroscope and precesses, rather than falling off the finger.



Consider a sphere, of mass m and radius a with moment of inertia I about its center, that rolls without slipping on a fixed sphere of radius b. Derive, and decompose into components, the (vector) equations of motion.

Show that the total angular velocity $\boldsymbol{\omega}$ obeys $\boldsymbol{\omega} \cdot d\hat{\mathbf{1}}/dt = 0 = \hat{\mathbf{1}} \cdot d\boldsymbol{\omega}/dt$, where $\hat{\mathbf{1}}$ points outward along the line of centers of the two spheres and makes angle θ to the vertical, $\hat{\mathbf{z}}$, and hence,

$$\boldsymbol{\omega} = \omega_1 \, \hat{\mathbf{1}} + \frac{a+b}{a} \, \hat{\mathbf{1}} \times \frac{d\hat{\mathbf{1}}}{dt} \, , \tag{1}$$

where $\omega_1 = \boldsymbol{\omega} \cdot \hat{\mathbf{1}} =$ constant, and that,

$$\left(I + ma^2\right) \frac{a+b}{a} \hat{\mathbf{1}} \times \frac{d^2\hat{\mathbf{1}}}{dt^2} + I\,\omega_1 \frac{d\hat{\mathbf{1}}}{dt} + mga\,\hat{\mathbf{2}} = 0. \tag{2}$$

Note that $\hat{\mathbf{1}}$ rotates about $\hat{\mathbf{z}}$ at rate $\dot{\phi}$ and about $\hat{\mathbf{2}} = \hat{\mathbf{1}} \times \hat{\mathbf{z}}$ at rate $\dot{\theta}$ (be careful with signs).

After obtaining the 3 component equations of motion, first consider steady motion, $\dot{\theta} = 0$, $\dot{\phi} = \Omega =$ constant, to show that ω_1 must satisfy,

$$\omega_1 > \frac{2}{I}\sqrt{mg(a+b)\,(I + ma^2)\cos\theta_0}, \tag{3}$$

for steady motion.

The spinning sphere will fall off the fixed sphere if the force of contact between them vanishes. Show that this happens (during steady motion) if,

$$\Omega^2 > \frac{g\cos\theta_0}{(a+b)\sin^2\theta_0} \, . \tag{4}$$

Consider nutations about steady precession,

$$\theta = \theta_0 + \epsilon \sin \alpha t, \qquad \dot{\phi} = \Omega + \delta \sin \alpha t, \tag{5}$$

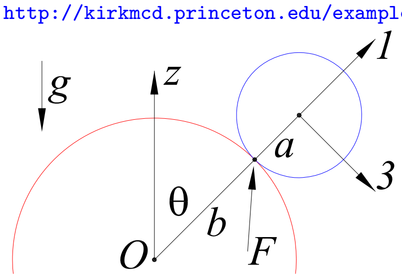
for small constants ϵ and δ to show that $\alpha^2 > 0$ for large enough ω_1, in which case the nutations are stable.

For a basketball of radius $a = 12$ cm, which is a hollow sphere with $I = 2mq^2/3$, balanced vertically on a finger of radius of curvature $b \approx 1$ cm, the spin required for gyroscopic stability is greater than 6 revolutions per second, which seems higher than in videos of "balanced", spinning basketballs. That is, their stability is due to active stabilization by horizontal motion of the support finger rather than gyroscopic effects.

One of many YouTube videos on how to spin a basketball,
https://www.youtube.com/watch?v=lLxUq6nhkb4
in which the spin seems to be only 1-2 revolutions per second.

2 Solution

This problem is the Example on p. 354, §415 of E.A. Milne, *Vectorial Mechanics* (Metheun; Interscience Publishers, 1948), http://kirkmcd.princeton.edu/examples/mechanics/milne_mechanics.pdf



We consider a sphere, of mass m and radius a with moment of inertia I about its center, that rolls without slipping on a fixed sphere of radius b. We use a set of principal axes (but not body axes) about the center of the sphere of radius a, where $\hat{\mathbf{1}}$ points outward along the line of centers of the two spheres and makes angle θ to the vertical, $\hat{\mathbf{z}}$. Also, $\hat{\mathbf{2}} = \hat{\mathbf{1}} \times \hat{\mathbf{z}} / \sin\theta$ (which is always horizontal), and $\hat{\mathbf{3}} = \hat{\mathbf{1}} \times \hat{\mathbf{2}}$ (which lies in the vertical plane of $\hat{\mathbf{1}}$ and $\hat{\mathbf{z}}$).

The center of the sphere of radius a is at position $\mathbf{r} = (a + b)\,\hat{\mathbf{1}}$ with respect to the center of the fixed sphere of radius b, which we take as the origin of coordinates in the lab frame. Then, the velocity of the center of the sphere of radius a is,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (a + b)\frac{d\hat{\mathbf{1}}}{dt}\,. \tag{6}$$

2.1 Rolling Constraint

The (nonholonomic) constraint of rolling without slipping is that the point of contact on the spinning sphere of radius a with the sphere of radius b is instantaneously at rest in the lab frame,

$$\mathbf{v}_{\rm contact} = 0 = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{a} = (a + b)\frac{d\hat{\mathbf{1}}}{dt} - a\boldsymbol{\omega} \times \hat{\mathbf{1}}, \tag{7}$$

where $\boldsymbol{\omega}$ is the total angular velocity of the sphere radius a in the lab frame, and $\mathbf{a} = -a\,\hat{\mathbf{1}}$ is the vector from the center of the sphere of radius a to the point of contact.[1]

2.2 Vector Equations of Motion

The force and torque equations of motion of (center of) the sphere of radius a are,

$$m\frac{d\mathbf{v}}{dt} = m(a+b)\frac{d^2\hat{\mathbf{1}}}{dt^2} = \mathbf{F} - mg\,\hat{\mathbf{z}}, \qquad \mathbf{F} = m(a+b)\frac{d^2\hat{\mathbf{1}}}{dt^2} + mg\,\hat{\mathbf{z}}, \tag{9}$$

$$\frac{d\mathbf{L}}{dt} = I\frac{d\boldsymbol{\omega}}{dt} = \boldsymbol{\tau} = \mathbf{a}\times\mathbf{F} = -ma(a+b)\hat{\mathbf{1}}\times\frac{d^2\hat{\mathbf{1}}}{dt^2} - mga\,\hat{\mathbf{1}}\times\hat{\mathbf{z}}. \tag{10}$$

From eq. (7) we have that,

$$\boldsymbol{\omega}\cdot\frac{d\hat{\mathbf{1}}}{dt} = 0, \tag{11}$$

while from eq. (10) we have that,

$$\hat{\mathbf{1}}\cdot\frac{d\boldsymbol{\omega}}{dt} = 0. \tag{12}$$

Hence,

$$\frac{d}{dt}(\boldsymbol{\omega}\cdot\hat{\mathbf{1}}) = \frac{d\omega_1}{dt} = 0, \tag{13}$$

and $\omega_1 = \boldsymbol{\omega}\cdot\hat{\mathbf{1}}$ is constant.

Also, we can multiply eq. (7) by $\hat{\mathbf{1}}$ to find that,

$$\boldsymbol{\omega} = \omega_1\,\hat{\mathbf{1}} + \frac{a+b}{a}\,\hat{\mathbf{1}}\times\frac{d\hat{\mathbf{1}}}{dt}\,, \qquad \frac{d\boldsymbol{\omega}}{dt} = \omega_1\frac{d\hat{\mathbf{1}}}{dt} + \frac{a+b}{a}\,\hat{\mathbf{1}}\times\frac{d^2\hat{\mathbf{1}}}{dt^2}\,, \tag{14}$$

and then rewrite the equation of motion (10) as,

$$\left(I + ma^2\right)\frac{a+b}{a}\,\hat{\mathbf{1}}\times\frac{d^2\hat{\mathbf{1}}}{dt^2} + I\,\omega_1\frac{d\hat{\mathbf{1}}}{dt} + mga\,\hat{\mathbf{1}}\times\hat{\mathbf{z}} = 0. \tag{15}$$

[1] At this point in the analysis we could also note that $\mathbf{v} = -(a+b)\dot{\phi}\sin\theta\,\hat{\mathbf{2}} + \dot{\theta}\,\hat{\mathbf{3}}$ where $\dot{\phi}$ is the angular velocity of the center of the spinning sphere about the z-axis. Then (6) implies eq. (28) below. We could also use eq. (7) to find,

$$\hat{\mathbf{1}}\times(\boldsymbol{\omega}\times\mathbf{a}) = -a\,\boldsymbol{\omega} - \omega_1\,\mathbf{a} = -\hat{\mathbf{1}}\times\mathbf{v} = (a+b)\,\dot{\theta}\,\hat{\mathbf{2}} + (a+b)\,\dot{\phi}\sin\theta\,\hat{\mathbf{3}}, \quad \boldsymbol{\omega} = \omega_1\,\hat{\mathbf{1}} - \frac{a+b}{a}\,\dot{\theta}\,\hat{\mathbf{2}} - \frac{a+b}{a}\,\dot{\phi}\,\hat{\mathbf{3}}, \tag{8}$$

where $\omega_1 = \boldsymbol{\omega}\cdot\hat{\mathbf{1}}$, in agreement with eq. (26) below.

2.3 Steady Motion

For steady motion, with $\theta = \theta_0 =$ constant, the spinning sphere, and the triad of principal axes, precess about the vertical at constant angular velocity $\mathbf{\Omega} = \boldsymbol{\omega}_{123} = \Omega\,\hat{\mathbf{z}}$, and hence,

$$\frac{d\hat{\mathbf{1}}}{dt} = \boldsymbol{\omega}_{123} \times \hat{\mathbf{1}} = \mathbf{\Omega} \times \hat{\mathbf{1}} = \Omega\,\hat{\mathbf{z}} \times \hat{\mathbf{1}}, \tag{16}$$

$$\frac{d^2\hat{\mathbf{1}}}{dt} = \mathbf{\Omega} \times \frac{d\hat{\mathbf{1}}}{dt} = \mathbf{\Omega} \times (\mathbf{\Omega} \times \hat{\mathbf{1}}) = (\mathbf{\Omega} \cdot \hat{\mathbf{1}})\mathbf{\Omega} - \Omega^2\,\hat{\mathbf{1}} = \Omega^2(\cos\theta_0\,\hat{\mathbf{z}} - \hat{\mathbf{1}}). \tag{17}$$

$$\hat{\mathbf{1}} \times \frac{d^2\hat{\mathbf{1}}}{dt} = \Omega^2 \cos\theta_0\,\hat{\mathbf{1}} \times \hat{\mathbf{z}}. \tag{18}$$

Then, all terms in the equation of motion (15) are proportional to $\hat{\mathbf{1}} \times \hat{\mathbf{z}}$, and we have,

$$\left(I + ma^2\right) \frac{a+b}{a} \Omega^2 \cos\theta_0 - I\,\omega_1\,\Omega + mga = 0, \tag{19}$$

$$\Omega = \frac{I\omega_1 \pm \sqrt{I^2\,\omega_1^2 - 4\,(I + ma^2)\,(a+b)mg\cos\theta_0}}{2\,(I + ma^2)\,\frac{a+b}{a}\cos\theta_0} = \frac{\omega_1 \pm \sqrt{\omega_1^2 - \omega_{1,\text{“min”}}^2}}{I\omega_{1,\text{“min”}}^2/2mga}\,, \tag{20}$$

where for steady precession at rate Ω to exist, we must have,

$$\omega_1 \geq \omega_{1,\text{“min”}} = \frac{2}{I}\sqrt{mg(a+b)\,(I + ma^2)\cos\theta_0}, \tag{21}$$

which (not surprisingly) limits steady motion to angle $\theta_0 < 90^\circ$.

The spinning sphere remains in contact with the fixed sphere only if the outward force of contact, $\mathbf{F} \cdot \hat{\mathbf{1}}$, is positive. From eqs. (9) and (17), we have for steady motion,

$$\mathbf{F} = m(a+b)\frac{d^2\hat{\mathbf{1}}}{dt^2} + mg\,\hat{\mathbf{z}} = m(a+b)\Omega^2(\cos\theta_0\,\hat{\mathbf{z}} - \hat{\mathbf{1}}), +mg\,\hat{\mathbf{z}}, \tag{22}$$

$$\mathbf{F} \cdot \hat{\mathbf{1}} = mg\cos\theta_0 + m(a+b)\Omega^2(\cos^2\theta_0 - 1) = mg\cos\theta_0 - m(a+b)\Omega^2\sin^2\theta_0. \tag{23}$$

Hence, the spinning sphere flies off the fixed sphere if,[2]

$$\Omega^2 > \frac{g\cos\theta_0}{(a+b)\sin^2\theta_0}\,. \tag{24}$$

In particular, if ω_1 is $\omega_{1,\text{“min”}}$ of eq. (21), the spinning sphere flies of when,

$$\frac{ma^2\sin^2\theta_0}{(I + ma^2)\cos^2\theta_0} > 0, \tag{25}$$

so only at $\theta_0 = 0$ can there be steady motion with $\omega_1 = \omega_{1,\text{“min”}}$.

That is, the true minimum of ω_1 for steady motion in contact with the fixed sphere is the root of the quartic equation obtained by combining eqs. (20) and (24). A numerical study[3] indicates that spinning sphere always flies off for Ω with the positive root in eq. (20), while for the negative root, steady motion in contact with the fixed sphere is possible for any $\theta_0 < 90^\circ$ for large enough ω_1 (much larger than $\omega_{1,\text{“min”}}$ of eq. (21) as θ_0 approaches 90°).

[2]For $\theta_0 = 0$, the spinning sphere will never fly off.

[3]http://kirkmcd.princeton.edu/examples/basketball.xlsx

2.4 Comments

For a basketball of radius $a = 12$ cm, which is a hollow sphere with $I = 2ma^2/3$ ($k = 2/3$), balanced vertically ($\theta_0 = 0$) on a finger of radius of curvature $b \approx 1$ cm $\ll a$, the minimum ω_1 required for gyroscopic stability is about 6 revolutions per second.[4] This seems higher than the rotation rates of spinning basketballs in online videos,[5] so it seems likely that their apparent stability is due to active stabilization by horizontal motion of the supporting finger, rather than gyroscopic stabilization.

2.5 Equations of Motion for Variable θ and $\dot\phi$

To discuss nutation about steady motion, we note that the angular velocity $\boldsymbol{\omega}_{123}$ of the principal axes consists of the term $-\dot\theta\,\hat{\mathbf{2}}$, together with their rotation $\dot\phi\,\hat{\mathbf{z}} = \dot\phi(\cos\theta\,\hat{\mathbf{1}} - \sin\theta\,\hat{\mathbf{3}})$.[6] Also, the total angular velocity $\boldsymbol{\omega}$ of the sphere of radius a consists of the "spin" angular velocity ω_s of the sphere about axis $\hat{\mathbf{1}}$ relative to the principal axes, together with $(a+b)/a$ times the angular velocity $\boldsymbol{\omega}_{123}$ of the principal axes relative to the lab frame (which subtle relation is inferred from eqs. (14) and (27)). Hence,

$$\boldsymbol{\omega}_{123} = \dot\phi\cos\theta\,\hat{\mathbf{1}} - \dot\theta\,\hat{\mathbf{2}} - \dot\phi\sin\theta\,\hat{\mathbf{3}}, \qquad \boldsymbol{\omega} = \omega_s\,\hat{\mathbf{1}} + \frac{a+b}{a}\left(\dot\phi\cos\theta\,\hat{\mathbf{1}} - \dot\theta\,\hat{\mathbf{2}} - \dot\phi\sin\theta\,\hat{\mathbf{3}}\right). \tag{26}$$

The time rate of change of the principal axes is related by,

$$\frac{d\hat{\mathbf{i}}}{dt} = \boldsymbol{\omega}_{123} \times \hat{\mathbf{i}}, \tag{27}$$

$$\frac{d\hat{\mathbf{1}}}{dt} = (\dot\phi\cos\theta\,\hat{\mathbf{1}} - \dot\theta\,\hat{\mathbf{2}} - \dot\phi\sin\theta\,\hat{\mathbf{3}}) \times \hat{\mathbf{1}} = -\dot\phi\sin\theta\,\hat{\mathbf{2}} + \dot\theta\,\hat{\mathbf{3}}, \tag{28}$$

$$\frac{d\hat{\mathbf{2}}}{dt} = (\dot\phi\cos\theta\,\hat{\mathbf{1}} - \dot\theta\,\hat{\mathbf{2}} - \dot\phi\sin\theta\,\hat{\mathbf{3}}) \times \hat{\mathbf{2}} = -\dot\phi\sin\theta\,\hat{\mathbf{1}} + \dot\phi\cos\theta\,\hat{\mathbf{3}}, \tag{29}$$

$$\frac{d\hat{\mathbf{3}}}{dt} = (\dot\phi\cos\theta\,\hat{\mathbf{1}} - \dot\theta\,\hat{\mathbf{2}} - \dot\phi\sin\theta\,\hat{\mathbf{3}}) \times \hat{\mathbf{3}} = -\dot\theta\,\hat{\mathbf{1}} - \dot\phi\cos\theta\,\hat{\mathbf{2}}. \tag{30}$$

$$\begin{aligned}\frac{d^2\hat{\mathbf{1}}}{dt^2} &= (-\ddot\phi\sin\theta - \dot\phi\,\dot\theta\cos\theta)\,\hat{\mathbf{2}} + \ddot\theta\,\hat{\mathbf{3}} + \dot\phi^2\sin^2\theta\,\hat{\mathbf{1}} - \dot\phi^2\sin\theta\cos\theta\,\hat{\mathbf{3}} - \dot\theta^2\,\hat{\mathbf{1}} - \dot\theta\,\dot\phi\cos\theta\,\hat{\mathbf{2}} \\ &= (\dot\phi^2\sin^2\theta - \dot\theta^2)\,\hat{\mathbf{1}} - (\ddot\phi\sin\theta + 2\dot\theta\,\dot\phi\cos\theta)\,\hat{\mathbf{2}} + (\ddot\theta - \dot\phi^2\sin\theta\cos\theta)\,\hat{\mathbf{3}},\end{aligned} \tag{31}$$

$$\hat{\mathbf{1}} \times \frac{d^2\hat{\mathbf{1}}}{dt^2} = (\dot\phi^2\sin\theta\cos\theta - \ddot\theta)\,\hat{\mathbf{2}} - (\ddot\phi\sin\theta + 2\dot\theta\,\dot\phi\cos\theta)\,\hat{\mathbf{3}}. \tag{32}$$

Using eqs. (28) and (32), and recalling that $\hat{\mathbf{1}} \times \hat{\mathbf{z}} = \sin\theta\,\hat{\mathbf{2}}$, we see that the equation of motion (15) has nonzero $\hat{\mathbf{2}}$- and $\hat{\mathbf{3}}$- components,

$$\left(I + ma^2\right)\frac{a+b}{a}(\dot\phi^2\sin\theta\cos\theta - \ddot\theta) - I\,\omega_1\dot\phi\sin\theta + mga\sin\theta = 0, \tag{33}$$

[4]According to eq. (20), the Ω corresponding to this minimum ω_1 is $3\omega_1/4$, which describes the rotation of the mathematical triad $\hat{\mathbf{1}}$-$\hat{\mathbf{2}}$-$\hat{\mathbf{3}}$. However, ω_1 (not Ω) describes the rotation of the physical sphere, as visible to observers of spinning basketballs.

[5]Many videos include remarks that higher spin makes the ball more stable.

[6]We could continue using the triad $\hat{\mathbf{1}}$, $\hat{\mathbf{z}}$, $\hat{\mathbf{1}} \times \hat{\mathbf{z}}$ as in eq. (15), but since $\hat{\mathbf{1}}$ and $\hat{\mathbf{z}}$ are not orthogonal, the algebra is somewhat more intricate.

$$\left(I + ma^2\right)\frac{a+b}{a}(\ddot{\phi}\sin\theta + 2\dot{\theta}\,\dot{\phi}\cos\theta) - I\,\omega_1\dot{\theta} = 0. \tag{34}$$

For steady motion, $\theta = \theta_0 =$ constant, $\dot{\theta} = 0$, $\dot{\phi} = \Omega =$ constant, eq. (34) is trivial, while eq. (33) leads to eq. (19).

2.6 Nutations

We now consider nutations of the form,

$$\theta = \theta_0 + \epsilon\sin\alpha t, \qquad \dot{\phi} = \Omega + \delta\sin\alpha t, \tag{35}$$

$$\sin\theta \approx \sin\theta_0 + \epsilon\cos\theta_0\sin\alpha t, \qquad \cos\theta \approx \cos\theta_0 - \epsilon\sin\theta_0\sin\alpha t, \tag{36}$$

for small constants ϵ and δ. Then, to first order in ϵ and δ, eq. (34) becomes,

$$\left(I + ma^2\right)\frac{a+b}{a}(\alpha\,\delta\sin\theta_0\cos\alpha t + 2\alpha\,\epsilon\,\Omega\cos\theta_0\cos\alpha t) - \alpha\epsilon I\,\omega_1\cos\alpha t = 0, \tag{37}$$

$$\delta = \epsilon\frac{I\omega_1}{(I+ma^2)\,\frac{a+b}{a}\sin\theta_0} - \frac{2\epsilon\,\Omega\cos\theta_0}{\sin\theta_0}, \tag{38}$$

and eq. (33) becomes, recalling eq. (19) for the 0$^{\text{th}}$-order terms,

$$\begin{aligned}\left(I + ma^2\right)\frac{a+b}{a}&[(\Omega^2 + 2\Omega\,\delta\sin\alpha t)(\sin\theta_0 + \epsilon\cos\theta_0\sin\alpha t)(\cos\theta_0 - \epsilon\sin\theta_0\sin\alpha t) + \epsilon\alpha^2\sin\alpha t] \\ &-I\,\omega_1\,(\Omega + \delta\sin\alpha t)(\sin\theta_0 + \epsilon\cos\theta_0\sin\alpha t) + mga(\sin\theta_0 + \epsilon\cos\theta_0\sin\alpha t) = 0,\end{aligned} \tag{39}$$

$$\begin{aligned}&\left(I + ma^2\right)\frac{a+b}{a}(\epsilon\Omega^2\cos 2\theta_0 + 2\delta\,\Omega\sin\theta_0\cos\theta_0 + \epsilon\alpha^2) \\ &-I\,\omega_1\,(\epsilon\Omega\cos\theta_0 + \delta\sin\theta_0) + \epsilon\,mga\cos\theta_0 = 0,\end{aligned} \tag{40}$$

$$\begin{aligned}\alpha^2 &= -\Omega^2\cos 2\theta_0 - 2\Omega\cos\theta_0\frac{I\,\omega_1}{(I+ma^2)\,\frac{a+b}{a}} + 4\Omega^2\cos^2\theta_0 \\ &+\frac{I\,\omega_1}{(I+ma^2)\,\frac{a+b}{a}}\left(\Omega\cos\theta_0 + \frac{I\,\omega_1}{(I+ma^2)\,\frac{a+b}{a}} - 2\Omega\cos\theta_0\right) - \frac{mga\cos\theta_0}{(I+ma^2)\,\frac{a+b}{a}}\end{aligned} \tag{41}$$

$$= \Omega^2(1 + 2\cos^2\theta_0) + \frac{I^2\,\omega_1^2}{(I+ma^2)^2\left(\frac{a+b}{a}\right)^2} - \frac{3I\,\omega_1\,\Omega\cos\theta_0}{(I+ma^2)\,\frac{a+b}{a}} - \frac{mga\cos\theta_0}{(I+ma^2)\,\frac{a+b}{a}}. \tag{42}$$

For sufficiently large ω_1, $\alpha^2 > 0$, and the nutations exist as ongoing, small oscillations. However, the condition for this is not simple.

We can extract a somewhat simpler condition if we restrict our attention to the "minimum" ω_1 for steady motion, as found in eq. (21) above. For this case, the associated Ω is given by eq. (20), and is called Ω_{min} here,

$$\Omega_{\text{min}} = \frac{I_1\,\omega_{1,\text{“min”}}}{2\,(I+ma^2)\,\frac{a+b}{a}\cos\theta_0}. \tag{43}$$

Using this in eq. (42), we have (after some algebra),

$$\alpha^2 = \frac{I^2\,\omega_{1,\text{“min”}}^2}{4\,(I+ma^2)^2\left(\frac{a+b}{a}\right)^2\cos^2\theta_0} - \frac{mga\cos\theta_0}{(I+ma^2)\,\frac{a+b}{a}}. \tag{44}$$

Since $\cos\theta_0 \le 1$, this condition for stable nutations is slightly weaker than the condition (21) for the existence of steady motion. That is, whenever steady motion is possible, nutations about this motion are stable.

A Appendix: A Lagrangian Approach

We consider a use of Lagrange's method, with coordinates θ, ϕ = angle of $\hat{\mathbf{2}}$ to the x-axis, and ψ = angle of rotation of the sphere about the $\hat{\mathbf{1}}$ axis.

The center of the sphere of radius a is at distance $a+b$ from the origin = center of fixed sphere of radius b, and hence the velocity of its center can be written as $\mathbf{v} = (a+b)(\dot\theta\,\hat{\mathbf{3}} - \dot\phi\sin\theta\,\hat{\mathbf{2}})$. The kinetic energy of the center-of-mass motion is,[7]

$$T_{\rm cm} = \frac{mv^2}{2} = \frac{ma^2}{2}\frac{(a+b)^2}{a^2}\left(\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right). \tag{45}$$

The kinetic energy of rotation is, recalling eq. (26) and noting that $\omega_s = \dot\psi$,

$$T_{\rm rot} = \frac{I\,\omega^2}{2} = \frac{I}{2}\left(\dot\psi + \frac{a+b}{a}\dot\phi\cos\theta\right)^2 + \frac{I}{2}\frac{(a+b)^2}{a^2}\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right), \tag{46}$$

and the potential energy is $V = mg(a+b)\cos\theta$. The Lagrangian is,

$$\mathcal{L} = T_{\rm cm} + T_{\rm rot} - V =$$
$$\frac{I}{2}\left(\dot\psi + \frac{a+b}{a}\dot\phi\cos\theta\right)^2 + \frac{(I+ma^2)}{2}\frac{(a+b)^2}{a^2}\left(\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right) - mg(a+b)\cos\theta. \tag{47}$$

The Lagrangian does not depend on ψ, so $\partial\mathcal{L}/\partial\dot\psi = I\left[\dot\psi - ((a+b)/a)\dot\phi\cos\theta\right] = I\,\omega_1$ is a conserved generalized momentum, and ω_1 is constant, as found above.

The equation of motion for coordinate θ is,

$$\frac{\partial\mathcal{L}}{\partial\theta} = -I\,\omega_1\frac{a+b}{a}\dot\phi\sin\theta + \left(I+ma^2\right)\frac{(a+b)^2}{a^2}\dot\phi^2\sin\theta\cos\theta + mg(a+b)\sin\theta$$
$$= \frac{d}{dt}\frac{\partial L}{\partial\dot\theta} = \left(I+ma^2\right)\frac{(a+b)^2}{a^2}\ddot\theta, \tag{48}$$
$$\left(I+ma^2\right)\frac{a+b}{a}\left(\dot\phi^2\sin\theta\cos\theta - \ddot\theta\right) - I\,\omega_1\,\dot\phi\sin\theta + mga\sin\theta = 0 \tag{49}$$

in agreement with eq. (33).

The equation of motion for coordinate ϕ is, recalling that ω_1 is constant,

$$\frac{\partial\mathcal{L}}{\partial\phi} = 0 = \frac{d}{dt}\frac{\partial L}{\partial\dot\phi} = \frac{d}{dt}\left[\frac{a+b}{a}I\,\omega_1\cos\theta + \left(I+ma^2\right)\frac{(a+b)^2}{a^2}\sin^2\theta\,\dot\phi\right]$$
$$= -\frac{a+b}{a}I\omega_1\,\dot\theta\sin\theta + \left(I+ma^2\right)\frac{(a+b)^2}{a^2}\left(\ddot\phi\sin^2\theta + 2\dot\theta\,\dot\phi\sin\theta\cos\theta\right), \tag{50}$$
$$\left(I+ma^2\right)\frac{a+b}{a}\left(\ddot\phi\sin\theta + 2\dot\theta\,\dot\phi\cos\theta\right) - I\omega_1\,\dot\theta = 0, \tag{51}$$

[7] As a check, we note that the rolling constraint (7) can be written as $\mathbf{v} = \mathbf{a}\times\boldsymbol{\omega} = -a\,\hat{\mathbf{1}}\times\boldsymbol{\omega}$, and hence the kinetic energy of the motion of the center of mass is $T_{\rm cm} = mv^2/2 = ma^2(\omega^2 - \omega_1^2)/2$. Using eq. (26) we again obtain eq. (45).

as found above in eq. (34).

The difficult step in the Lagrangian method is arriving at eq. (26) for the total angular velocity $\boldsymbol{\omega}$, for which the vectorial method, and awareness of the rolling constraint, is helpful (as discussed in footnote 1 above).

B Appendix: Lamb's Analysis for Small θ

It was noted in sec. 41, p. 101 of H. Lamb, *Higher Mechanics* (Cambridge U. Press, 1920), http://kirkmcd.princeton.edu/examples/mechanics/lamb_higher_mechanics.pdf that if we restrict our attention to motion in which angle θ is very small (as for balanced, spinning basketballs), we can give an analysis using only x-y-z coordinates.

The center of the spinning sphere is at $\mathbf{r} = (x, y, z)$ where $r = a+b$. The rolling constraint (7) can then be written as,

$$\mathbf{v} = \dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z}) = \mathbf{a} \times \boldsymbol{\omega} = \frac{a}{a+b}(z\,\omega_y - y\,\omega_z, x\,\omega_z - z\,\omega_x, y\,\omega_x - x\,\omega_y), \tag{52}$$

noting that $\mathbf{a} = -a\mathbf{r}/(a+b)$ and $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$.

The general equations of motion are,

$$m\ddot{\mathbf{r}} = \mathbf{F} - mg\,\hat{\mathbf{z}}, \qquad I\,\dot{\boldsymbol{\omega}} = \mathbf{a} \times \mathbf{F} = -\frac{a}{a+b}\mathbf{r} \times \mathbf{F}. \tag{53}$$

For motion with small θ, we have that $z \approx a+b$, $F_z \approx mg$, and $\omega_z \approx$ constant. The constraint relation (52) reduces to,

$$\dot{x} = a\,\omega_y - \frac{a\,\omega_z}{a+b}y, \qquad \dot{y} = -a\,\omega_x + \frac{a\,\omega_z}{a+b}x, \tag{54}$$

and the equations of motion (53) reduce to $m\ddot{x} = F_x$, $m\ddot{y} = F_y$ and,

$$I\,\dot{\omega}_x = aF_y - \frac{mga}{a+b}y = ma\ddot{y} - \frac{mga}{a+b}y, \quad I\,\dot{\omega}_y = -aF_x + \frac{mga}{a+b}x = -ma\ddot{x} + \frac{mga}{a+b}x. \tag{55}$$

Using eq. (55) in the time derivative of eq. (54), we find,

$$\ddot{x} = a\,\dot{\omega}_y - \frac{a\,\omega_z}{a+b}\dot{y} = -\frac{ma^2}{I}\ddot{x} + \frac{mga^2}{I(a+b)}x - \frac{a\,\omega_z}{a+b}\dot{y}, \tag{56}$$

$$\left(I + ma^2\right)\frac{a+b}{a}\ddot{x} + I\,\omega_z\dot{y} - magx = 0, \tag{57}$$

$$\ddot{y} = -a\,\dot{\omega}_x + \frac{a\,\omega_z}{a+b}\dot{x} = -\frac{ma}{I}\ddot{y} - \frac{mga^2}{I(a+b)}y + \frac{a\,\omega_z}{a+b}\dot{x}, \tag{58}$$

$$\left(I + ma^2\right)\frac{a+b}{a}\ddot{y} - I\,\omega_z\dot{x} - magy = 0. \tag{59}$$

Lamb noted that it is clever to introduce the complex variable $\zeta = x + iy$ where $i = \sqrt{-1}$ here. Then, eqs. (57) and (59) combine into the form,

$$\left(I + ma^2\right)\frac{a+b}{a}\ddot{\zeta} - iI\,\omega_z\,\dot{\zeta} - mag\,\zeta = 0. \tag{60}$$

We seek oscillatory behavior with $\zeta \propto e^{i\alpha t}$, which implies that,

$$\left(I + ma^2\right) \frac{a+b}{a} \alpha^2 - I\,\omega_z\,\alpha + mag = 0, \tag{61}$$

$$\alpha = \frac{I\omega_z \pm \sqrt{I^2\,\omega_z^2 - 4\,(I + ma^2)\,(a+b)mg}}{2\,(I + ma^2)\,\frac{a+b}{a}}\,. \tag{62}$$

This oscillatory behavior (nutation) exists for,

$$\left(I + ma^2\right) \frac{a+b}{a} \alpha^2 - I\,\omega_z\,\alpha + mag = 0, \tag{63}$$

$$\omega_z > \frac{2}{I}\sqrt{(I + ma^2)\,(a+b)mg}, \tag{64}$$

which is the same condition found in eq. (21) for $\theta_0 = 0$, where $\omega_1 = \omega_z$.

Lamb noted that if the real values of eq. (62) are $\alpha_\pm$, then the trajectory of the center of the spinning sphere has the form,

$$x = A_+ \cos(\alpha_+ t + \beta_+) + A_- \cos(\alpha_- t + \beta_-), \quad y = A_+ \sin(\alpha_+ t + \beta_+) + A_- \sin(\alpha_- t + \beta_-), \tag{65}$$

which describe an epicyclic curve.

He also considered a velocity-dependent friction somehow acting only on the center of the sphere, for which the mathematics is analytically tractable and implies that one of the oscillations, with angular frequency α_+ or α_-, is exponentially damped, while the other grows exponentially until the spinning sphere flies off the fixed one.

While Lamb's analysis is not as general as that of the vectorial approach in sec. 2 above, the interesting case of small θ is considerably easier it to pursue with it.[8]

[8] Lamb deduced the general equations of motion for this problem in his sec. 67, p. 160, but did not discuss their solutions there.