

Is Bernoulli's Equation Relativistically Invariant?

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1 Problem

In April 2009 the following puzzler was posted on the Advanced Physics Forum:¹

Suppose a smoker is driving in a car at a constant velocity with the windows cracked. From the smoker's point of view, the air outside the car is moving at a faster speed than the air inside the car. According to Bernoulli's equation, the faster air means that the pressure is reduced. Therefore, the smoke should flow out the window into the outside air. But according to relativity, since any inertial frame of reference is correct, a person standing at rest while the car goes by says that the air inside the car is moving faster, therefore the pressure inside the car is less and the outside air pushes inward, keeping the smoke inside the car. So what really happens?

Does the smoke stay in the car or get pushed out the window?

2 Solution

Underlying this problem is the issue whether every "law" of physics that is valid in one frame of reference is automatically valid in other frames.

The general answer is NO, in that some "laws" involve special constraints that are valid only in a particular frame.

Bernoulli's equation is an example of this.

An elegant derivation of Bernoulli's equation is given in sec. 5 of [1], starting from Euler's equation (sec. 2). Euler's equation can be expressed in a relativistic form (secs. 133-134). A key step in deducing Bernoulli's equation from Euler's equation is that the fluid flow be steady, $\partial\mathbf{v}/\partial t = 0$. Then, Bernoulli's equation,

$$P + \frac{\rho v^2}{2} + \rho g z = \text{constant}, \quad (1)$$

(for a fluid of mass density ρ with pressure P subject to the acceleration $-g\hat{\mathbf{z}}$ due to gravity) holds in any inertial frame of reference where the flow is steady.

However, when the fluid flow is steady according to an observer in one frame of reference, the flow will not, in general, be steady according to observers in frames that are in motion with respect to the first frame. That is, the validity of Bernoulli's equation in one frame of reference does not, in general, imply that it is valid in other frames of reference. Bernoulli's equation is not relativistically invariant.

In the present example, the flow of air is steady according to an observer at rest with respect to the moving car. So, Bernoulli's equation holds for this observer, and (s)he correctly

¹<http://www.advancedphysics.org/>

predicts that the air pressure inside the car is higher than that just outside the car, and hence smoke will be drawn out of the car through an open window.

However, an observer at rest with respect to the road on which the car is driving detects a pulse of wind as the car passes by. The velocity of the air (at a particular point in the “road” frame) is not steady, and Bernoulli’s equation does not apply in this frame. This observer does not think that the air pressure inside the car is lower than that outside the car, even though the velocity of air inside the car is higher than that outside the car in the “road” frame. The “road” observer must make a more complicated analysis in which the moving car continually encounters new regions of “stagnant” air, whose inertia results in forces on the air in the car (through the open window) that increase the pressure of the air in the car compared to that outside, in agreement with the observer inside the car.

Sept. 30, 2022. For another example on a nonstatic situation in which Bernoulli’s equation does not apply, see [2]. An extended version of Bernoulli’s equation applies to some nonstatic examples, as reviewed in Appendix A of [3]. The theme that Bernoulli’s equation is not relativistically invariant has also been considered in [4, 5].

References

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