Stability of Transverse Oscillations in a Betatron

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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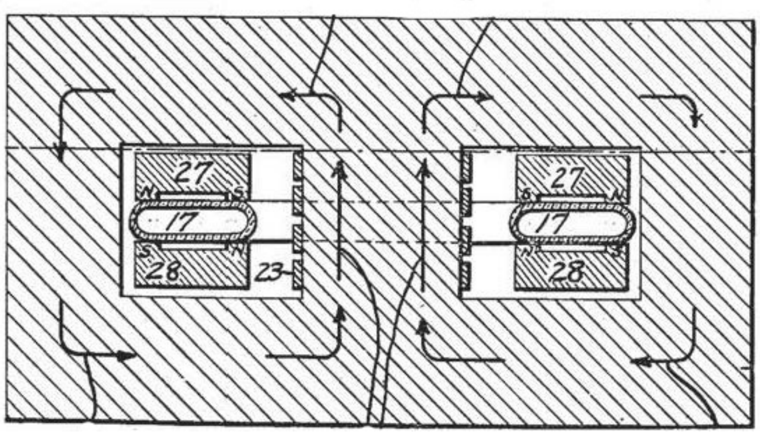
1 Problem

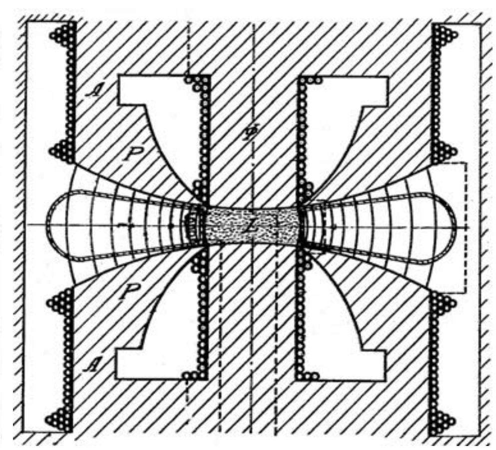
A betatron[1] is a magnetic-induction accelerator of electrons about a circular orbit of radius a, in which a time-dependent magnetic field, nominally perpendicular to the plane of the orbit, induces an azimuthal electric field that accelerates the electrons. Deduce conditions on the form of the magnetic field such that motion of (relativistic) electrons transverse to the central orbit is stable.

2 Solution

2.1 Condition for a Central Orbit with Fixed Radius

A condition for the existence of a central circular orbit with a fixed radius follows (for relativistic electrons) from the basic conception of a betatron as having a time-dependent magnetic field $B_z(r, z, t)$, in cylindrical coordinates. Early sketches [4, 10] of apparati to generate such a field are shown below (although these efforts considered only nonrelativistic electrons).





A relativistic electron has energy U simply related to the magnitude p of its momentum by $U = pc$, where c is the speed of light in vacuum. For the electron, of charge $-e$ and speed $v \approx c$, to move in a circle of radius a (with angular velocity $\omega_0 \approx c/a$) under the influence of a magnetic field $B_0(t) = B_z(a, 0, t)$ perpendicular to the plane of the orbit, Newton's law for the centripetal force F_r tells us that (in Gaussian units),

$$F_r = \omega p = \frac{pc}{a} = \frac{U}{a} = \frac{evB_0}{c} = eB_0. \tag{1}$$

[1] The name betatron was given by Kerst [1] to the circular induction accelerator after its first successful demonstration [2, 3], which followed a long conceptual history [4]-[10].

In a betatron, the electron accelerates (increases its energy U) due to the azimuthal electric field E_ϕ induced by the changing magnetic flux Φ through the orbit of radius a, which is related by Faraday's law,

$$2\pi a E_\phi = -\frac{\dot{\Phi}}{c}\,, \tag{2}$$

so that,

$$\frac{dU}{dt} = F_\phi v = -eE_\phi c = \frac{e\dot{\Phi}}{2\pi a}\,. \tag{3}$$

From eq. (1) we also have that for relativistic electrons,

$$\frac{dU}{dt} = ea\dot{B}_0, \tag{4}$$

so the magnetic flux through the orbit must be related to the magnetic field at the orbit by,

$$\dot{\Phi} \approx 2\pi a^2 \dot{B}_0, \qquad i.e., \qquad \Phi(t) = 2\pi \int_0^a B_z(r,0,t)\, r\, dr = 2\pi a^2 B_0(t). \tag{5}$$

This condition (first given in [8]) for a circular orbit of fixed radius as the relativistic electron accelerates is often stated as requiring the field at the orbit to be one half the average field in its interior,

$$B_0(t) = \frac{\langle B_z \rangle}{2} = \frac{1}{2a^2} \int_0^a B_z(r,0,t)\, r\, dr. \tag{6}$$

2.2 Stability of Motion in the Plane of the Central Orbit

The radial equation of motion is, approximately,

$$\gamma m a_r = \frac{eB_0(t)a}{c^2}\left(\ddot{r} - \frac{v_\phi^2}{r}\right) \approx -\frac{ev_\phi}{c} B_z(r,0,t) \approx -\frac{ev_\phi}{c}\left(B_z(a,0,t) + \frac{\partial B_z(a,0,t)}{\partial r}(r-a)\right), \tag{7}$$

noting that $\gamma m = U/c^2 = eB_0 a/c^2$ according to eq. (1), where $B_0 = B_z(a)$. In this we assume that electrons whose trajectories are close to the central orbit have the central energy U. Defining $x \equiv r - a$, so that $1/r \approx (1/a)(1 - x/a)$, we have that for relativistic electrons with $v_\phi \approx c$,

$$\frac{eB_0 a}{c^2}\ddot{x} \approx eB_0\left(1 - \frac{x}{a}\right) - eB_0 - e\frac{\partial B_z(a,0,t)}{\partial r} x = -x\,\frac{eB_0}{a}\left(1 + \frac{a}{B_0}\frac{\partial B_z(a,0,t)}{\partial r}\right) \tag{8}$$

$$\ddot{x} \approx -\omega_0^2\, x \left(1 + \frac{a}{B_0}\frac{\partial B_z(a,0,t)}{\partial r}\right), \tag{9}$$

where,

$$\omega_0 = \frac{c}{a} \tag{10}$$

is the angular frequency of the relativistic electrons in the central orbit.

The radial motion will be stable, simple-harmonic motion if,

$$\frac{a}{B_0(t)} \frac{\partial B_z(a,0,t)}{\partial r} > -1. \tag{11}$$

This condition for radial stability of relativistic electrons was first given in eq. (2) of [10]; an earlier argument in [6] for nonrelativistic electrons was cumbersome. See also [11, 12].[2]

2.3 Stability of Motion Perpendicular to the Plane of the Central Orbit

The equation of motion in the z-direction for electrons is approximately,

$$\gamma m \ddot{z} = \frac{eB_0(t)a}{c^2} \ddot{z} \approx \frac{ev_\phi}{c} B_r(a,z,t) \approx eB_r(a,z,t) \approx ez \frac{\partial B_z(a,z,t)}{\partial r}\,, \tag{12}$$

$$\ddot{z} \approx \omega_0^2\, z \frac{a}{B_0(t)} \frac{\partial B_z(a,0,t)}{\partial r}\,, \tag{13}$$

noting that the radial component of the magnetic field $\mathbf{B}$ vanishes in the symmetry plane $z = 0$, and $\boldsymbol{\nabla} \times \mathbf{B} = 0$ in the region of the orbits, so,

$$\frac{\partial B_r(a,z,t)}{\partial z} = \frac{\partial B_z(a,z,t)}{\partial r}\,, \tag{14}$$

$$B_r(a,z,t) = \int_0^z \frac{\partial B_r(a,z',t)}{\partial z'}\, dz' = \int_0^z \frac{\partial B_z(a,z',t)}{\partial r}\, dz' \approx z \frac{\partial B_z(a,0,t)}{\partial r}\,. \tag{15}$$

The z-motion described by eq. (13) is stable, simple-harmonic oscillation only if,

$$\frac{a}{B_0(t)} \frac{\partial B_z(a,0,t)}{\partial r} < 0. \tag{16}$$

This condition for "vertical" stability of relativistic electrons was also first given in eq. (2) of [10].

[2]Sec. 11-1 (p. 405) of [13] ends with the peculiar statement that since the Lagrangian and Hamiltonian of a charged particle in an electromagnetic field is expressed in terms of the potentials V and $\mathbf{A}$ rather than the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, it must be that knowledge of the potentials, rather than the fields, is required to characterize the stability of charged-particle orbits. The present analysis, based entirely on the magnetic field $\mathbf{B}$, with no mention of the vector potential $\mathbf{A}$, shows this not to be the case. And since the potentials are not gauge invariant, it cannot be that they, rather than their derivatives (the electromagnetic fields) determine the motion of a charged particle. Possibly, the paper [11], which deduces the equations of motion from a Hamiltonian involving the vector potential, gave a misimpression that the vector potential, and not the magnetic field, governs the motion of the charged particle.

2.4 Weak Focusing

The condition (6) for the central orbit implies that the magnetic field $B_z(r)$ will, in general, be decreasing with increasing r. Hence, it has become customary to define the field (gradient) index as,[3]

$$n = -\frac{a}{B_0(t)} \frac{\partial B_z(a,0,t)}{\partial r}\,, \tag{17}$$

such that the stability conditions (11) and (16) can be summarized as,

$$0 < n < 1 \qquad \text{(betatron stability condition)}. \tag{18}$$

The approximate radial and vertical equations of motion can now be written as,

$$\ddot{x} \approx -\omega_0^2 (n-1)\, x = -\omega_x^2\, x, \qquad \ddot{z} \approx -\omega_0^2\, n\, z = -\omega_z^2\, z, \tag{19}$$

such that the radial and vertical betatron oscillation frequencies,

$$\omega_x = \omega_0 \sqrt{1-n}, \qquad \omega_z = \omega_0 \sqrt{n}, \tag{20}$$

are equal only in the special case of $n = 1/2$, and are always lower than the orbital angular frequency ω_0. The latter behavior is characterized as weak focusing (in contrast to strong focusing [14, 15, 16] in so-called synchrotrons where the frequency of transverse oscillations is higher than the orbital frequency).

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