Flow of Energy from a Localized Source in a Uniform Anisotropic Medium

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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1 Problem

Show that the flow of electromagnetic energy far from localized, time-dependent sources in a uniform, linear anisotropic dielectric medium is purely radial, despite the complexity of the wavefronts. Thus, the waves are transverse electromagnetic (TEM) in this region, although the electric displacement **D** has a longitudinal component in general.

It suffices to consider sources with time dependence $e^{-i\omega t}$ near the origin in a medium with unit (relative) permeability and (relative) symmetric dielectric tensor ϵ_{ij} that is diagonal with respect to the axes (x, y, z). Then, the electric displacement vector **D** and electric field vector **E** are related by,

$$D_i = \epsilon_i E_i, \qquad i = x, y, z, \tag{1}$$

in Gaussian units. Consider the general case where the three dielectric constants ϵ_x, ϵ_x and ϵ_z are all different, with $\epsilon_x < \epsilon_y < \epsilon_z$.

2 Solution

This problem can be approached by the general methods of geometric optics and acoustics, which we consider in sec. 2.1, before making an electromagnetic argument in sec. 2.2.

2.1 A Geometric Argument

This section follows two papers of Whitham [1, 2], which are variants of arguments by Landau [3, 4]. All these works are elaborations of the original paper by Sommerfeld and Runge [5] that introduced the so-called eikonal *method.*[1] *See also sec. 45 of [7].*

An argument applicable to waves of all types that are far from localized sources is based on the approximation that any scalar component of the wave function can be written as,

$$\psi = A(\mathbf{r}, t)\, e^{i\varphi(\mathbf{r},t)}, \tag{2}$$

where the A is a slowly-varying (complex) amplitude and φ is a rapidly varying (real) phase (sometimes also called the eikonal). In any small region (far from the source) the form (2) is nearly a plane wave $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ with wave vector **k** and frequency ω obtained from the first-order terms in a Taylor expansion of the phase φ,

$$\varphi(\mathbf{r}, t) = \varphi_0 + \boldsymbol{\nabla}\varphi \cdot \mathbf{r} + \frac{\partial \varphi}{\partial t} t + ..., \tag{3}$$

[1]Thanks to Michael Berry for drawing the author's attention to the methods of this section. An interesting application by Berry of these methods to so-called conical diffraction is given in [6].

such that we identify,

$$\mathbf{k} = \boldsymbol{\nabla}\varphi, \qquad \text{and} \qquad \omega = -\frac{\partial \varphi}{\partial t}\,. \tag{4}$$

The locally plane wave has phase velocity,

$$\mathbf{v}_p = \frac{\omega}{k}\hat{\mathbf{k}}, \tag{5}$$

where $k = |\mathbf{k}|$. It also follows from eq. (4) that,

$$\frac{\partial \mathbf{k}}{\partial t} = -\boldsymbol{\nabla}\omega. \tag{6}$$

The plane wave $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ is at least an approximate solution to some wave equation. Using the plane wave as a trial solution to this wave equation leads to a functional relation between ω and $\mathbf{k}$ (and possibly $\mathbf{r}$ as well) called the dispersion relation, which we write as,

$$\omega = \omega(\mathbf{k}, \mathbf{r}). \tag{7}$$

The dispersion relation can be used to generate the equations of the geometric or ray approximation as follows.

We first note that since $\mathbf{k} = \boldsymbol{\nabla}\varphi$ we have that $\boldsymbol{\nabla} \times \mathbf{k} = 0$, *i.e.*, $\partial k_i/\partial x_j = \partial k_j/\partial x_i$. Then, if we use the dispersion relation in eq. (6), the ith component of that equation can be rewritten as,

$$\frac{\partial k_i}{\partial t} = -\frac{\partial \omega}{\partial x_i} - \sum_j \frac{\partial \omega}{\partial k_j}\frac{\partial k_j}{\partial x_i} = -\frac{\partial \omega}{\partial x_i} - \sum_j \frac{\partial \omega}{\partial k_j}\frac{\partial k_i}{\partial x_j} = -\frac{\partial \omega}{\partial x_i} - \sum_j v_{g,j}\frac{\partial k_i}{\partial x_j}\,, \tag{8}$$

where,

$$\mathbf{v}_g = \frac{\partial \omega}{\partial \mathbf{k}} = \boldsymbol{\nabla}_{\mathbf{k}}\omega \tag{9}$$

is the group velocity, so that,

$$\frac{d\mathbf{k}}{dt} = \frac{\partial \mathbf{k}}{\partial t} + (\mathbf{v}_g \cdot \boldsymbol{\nabla})\mathbf{k} = -\boldsymbol{\nabla}\omega. \tag{10}$$

We can interpret eq. (10) as implying that for an observer who moves with velocity,

$$\mathbf{v}_g = \frac{d\mathbf{r}}{dt} = \boldsymbol{\nabla}_{\mathbf{k}}\omega \tag{11}$$

in a homogeneous medium (*i.e.*, one for which $\boldsymbol{\nabla}\omega = 0$), the wave vector $\mathbf{k}$ remains constant. This result leads us to introduce the concept of a ray (in ordinary space) whose direction is that of the group velocity $\mathbf{v}_g$. In a homogeneous medium the wave vector $\mathbf{k}$ is constant along a ray (although $\mathbf{k}$ is not necessarily parallel to $\mathbf{v}_g$[2]).

Furthermore, in a homogeneous medium the gradient $\boldsymbol{\nabla}_{\mathbf{k}}\omega$ is constant along a ray, since ω is only a function of $\mathbf{k}$ in such a medium, and $\mathbf{k}$ is constant along a ray. Hence, the

[2]See, for example, the figure on p. 5 of [8].

group velocity vector $\mathbf{v}_g = \nabla_{\mathbf{k}}\omega$ is constant along a ray, and the rays are straight lines in a homogeneous medium.[3] This result holds even if the medium is anisotropic, and it holds whether or not the medium is linear in the sense of eq. (1).

If we suppose a ray to be associated with the Hamiltonian,

$$H = \omega(\mathbf{r}, \mathbf{k}) \tag{12}$$

then Hamilton's equations of motion,

$$\frac{d\mathbf{r}}{dt} = \nabla_{\mathbf{k}} H = \nabla_{\mathbf{k}}\omega, \qquad \text{and} \qquad \frac{d\mathbf{k}}{dt} = -\nabla H = -\nabla\omega, \tag{13}$$

lead to the same forms as eqs. (10)-(11). Thus, the ray concept, which derives from the view of Fermat that light is a particle phenomenon, together with the principles of Hamiltonian mechanics, leads us to suppose that the energy of a particle of light is proportional is its frequency. Although this argument is perhaps the most compact derivation of so-called Hamiltonian optics, it was not made by Hamilton. Rather, it was Einstein [12] who first noted the relation between frequency and energy for quanta of light, while the use of eqs. (12)-(13) as the basis for geometric optics appears to be due to Landau [3].[4,5]

If we can now show that the group velocity is the velocity of energy flow, then we will have established that energy flows in straight lines from a localized source in a homogeneous medium, including an anisotropic medium.

Although the concept of group velocity was invented by Hamilton [15], he does not appear to have related the group velocity to the energy velocity, nor did he relate it to the ray velocity in geometric (Hamiltonian) optics. When the concept of group velocity was reinvented in the 1870's, Rayleigh [16] identified it with the velocity of energy flow. Here, we follow sec. 4 of [2] to give a generic argument that wave energy flows with the group velocity.

In the absence of a detailed model of the wave, we simply suppose that the energy U of the wave depends on the square of the (complex) wave function,

$$U(t) = \frac{1}{2}\int \psi(\mathbf{r}, t)\psi^\star(\mathbf{r}, t)\, d^3\mathbf{r}. \tag{14}$$

We also suppose that the wave function can be represented in terms of plane waves,

$$\psi(\mathbf{r}, t) = \int \psi_k(\mathbf{k})\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\, d^3\mathbf{k}, \tag{15}$$

[3]Waves that carry angular momentum [9, 10] have a phase-velocity vector $\mathbf{k}$ whose streamlines follow helices, but in this case the dispersion relation $\omega(\mathbf{k}, \mathbf{r})$ depends on $\mathbf{r}$ (see, for example, sec. 2.2.1 of [11]) and the medium (even if vacuum) is not formally homogeneous in the present sense.

[4]Although Schrödinger used Hamiltonian optics to motivate his equation for the quantum behavior of particles [13], he appears not to have considered the inverse notion of the quantum relation $E = \hbar\omega$ for the energy of particles of light as a starting point for Hamiltonian optics.

[5]In [3] Landau explicitly identifies the angular frequency ω as the Hamiltonian for geometrical optics, but the only medium he considers is vacuum. In [4] he considers anisotropic media that support mechanical waves and notes that if the medium is homogeneous then the rays are straight lines; but he does not explictly identify the dispersion relation $\omega(\mathbf{k})$ with the Hamiltonian. In [14] he considers the optics/electrodynamics of anisotropic media but omits mention that the rays are straight lines in homogeneous anisotropic media, and of the connection between ray optics and Hamiltonian mechanics.

where we restrict the discussion to homogeneous media for which the frequency is only a function of the wave vector, $\omega = \omega(\mathbf{k})$. The Fourier analysis (15) does not apply to all space, but only to a region of observation far from the sources of the waves, such that it suffices to approximate spherical waves by plane waves. Then, by considering the waveform at time $t = 0$, we can invert (15) to find the Fourier amplitudes,

$$\psi_k(\mathbf{k}) = \frac{1}{(2\pi)^3} \int \psi(\mathbf{r}, 0)\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, d^3\mathbf{r}, \tag{16}$$

which are independent of time.

Using the Fourier analysis (15) in the energy equation (14), we find,

$$\begin{aligned} U(t) &= \frac{1}{2}\int\int\int \psi_k(\mathbf{k})\psi_k^\star(\mathbf{k}')\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}\, e^{-i(\omega(\mathbf{k})-\omega(\mathbf{k}'))t}\, d^3\mathbf{k}\, d^3\mathbf{k}'\, d^3\mathbf{r} \\ &= \frac{(2\pi)^3}{2}\int\int \psi_k(\mathbf{k})\psi_k^\star(\mathbf{k}')\, e^{-i(\omega(\mathbf{k})-\omega(\mathbf{k}'))t}\delta^3(\mathbf{k}-\mathbf{k}')\, d^3\mathbf{k}\, d^3\mathbf{k}' \\ &= 4\pi^3 \int \psi_k(\mathbf{k})\psi_k^\star(\mathbf{k})\, d^3\mathbf{k}. \end{aligned} \tag{17}$$

This shows that the energy density in an element $d^3\mathbf{k}$ is only a function of the wave vector $\mathbf{k}$. Since the wave vector is constant for an observer that moves with the group velocity $\mathbf{v}_g$ (in a homogeneous medium), we see that a constant energy density flows with this velocity. That is, the energy flow velocity is the group velocity, which flows along straight rays in a homogeneous medium whether the medium is isotropic or anisotropic.[6]

2.2 An Electromagnetic Argument

Maxwell's equations for the electromagnetic fields in an anisotropic linear medium for which permeability $\mu = 1$ are (in Gaussian units),

$$\boldsymbol{\nabla}\cdot\mathbf{D} = 4\pi\rho_{\rm free}, \qquad \boldsymbol{\nabla}\cdot\mathbf{B} = 0, \qquad \boldsymbol{\nabla}\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \qquad \boldsymbol{\nabla}\times\mathbf{B} = \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{J}_{\rm free}, \tag{18}$$

where $\rho_{\rm free}$ and $\mathbf{J}_{\rm free}$ are the free-charge and -current densities, respectively. The electric fields $\mathbf{D}$ and $\mathbf{E}$ are related by the constituent equations (1).

In any small region far from the sources $\rho_{\rm free}$ and $\mathbf{J}_{\rm free}$, the fields are plane waves to a good approximation. Hence, we base our analysis on consideration of plane waves, following, for example, chap. XI of [14] or chap. XIV of [21].

The spacetime dependence of a plane wave of angular frequency ω will be written as,

$$e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{19}$$

where the wave vector $\mathbf{k}$ is also written as,

$$\mathbf{k} = \frac{\omega}{c}\mathbf{n}, \tag{20}$$

[6] The (time-average) energy-flow velocity in waves in free space that carry orbital angular momentum [17, 18, 19, 20] has streamlines that are slightly twisted. However, such waves can be described in terms of families of straight rays with an azimuthal skew [20].

where $n = |\mathbf{n}|$. The phase velocity v_p of the plane wave is,

$$v_p = \frac{c}{n}. \tag{21}$$

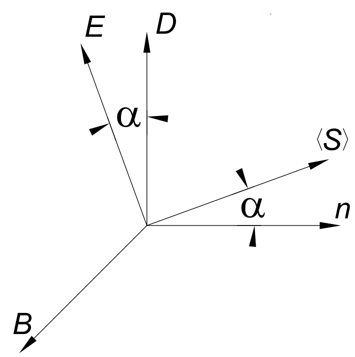
In an isotropic medium with dielectric constant ϵ, we have $\mathbf{n} = n\,\hat{\mathbf{k}}$ where $n = \sqrt{\epsilon}$ is the index of refraction.

For a plane wave (19) far from its sources, Maxwell's equations (18) imply that,

$$\mathbf{n} \cdot \mathbf{D} = 0, \qquad \mathbf{n} \cdot \mathbf{B} = 0, \qquad \mathbf{n} \times \mathbf{E} = \mathbf{B}, \qquad \mathbf{n} \times \mathbf{B} = -\mathbf{D}. \tag{22}$$

Thus, far from the sources the three vectors $\mathbf{n}$, $\mathbf{B}$ and $\mathbf{D}$ are mutually orthogonal. Vectors $\mathbf{B}$ and $\mathbf{E}$ are orthogonal, but in general the electric field $\mathbf{E}$ has a component along the direction $\mathbf{n}$ of the wave vector.

The Poynting vector $\langle \mathbf{S} \rangle = (c/8\pi) Re(\mathbf{E} \times \mathbf{B}^\star)$ is in general not parallel to the vector $\mathbf{n}$. However, the three vectors $\mathbf{B}$, $\mathbf{E}$ and $\langle \mathbf{S} \rangle$ are mutually orthogonal. The two mutually orthogonal triads, $\{\mathbf{n}, \mathbf{B}, \mathbf{D}\}$ and $\{\mathbf{B}, \mathbf{E}, \langle \mathbf{S} \rangle\}$, each include the magnetic field $\mathbf{B}$, so the four vectors $\mathbf{D}$, $\mathbf{E}$, $\mathbf{n}$ and $\langle \mathbf{S} \rangle$ all lie in a plane perpendicular to $\mathbf{B}$, as shown in the figure below.



The electric fields $\mathbf{D}$ and $\mathbf{E}$ are related by the 3rd and 4th equations of (22) according to,

$$\mathbf{D} = -\mathbf{n} \times (\mathbf{n} \times \mathbf{E}) = n^2 \mathbf{E} - (\mathbf{n} \cdot \mathbf{E})\, \mathbf{n} = n^2[\mathbf{E} - (\hat{\mathbf{n}} \cdot \mathbf{E})\, \hat{\mathbf{n}}] = n^2 E \cos\alpha\, \hat{\mathbf{D}}, \tag{23}$$

where α is the angle between $\mathbf{n}$ and $\mathbf{S}$ as well as that between $\mathbf{D}$ and $\mathbf{E}$. These fields are also related by the constituent eq. (1), which can be combined with the second form of eq. (23) to give three scalar equations, expressible in matrix form as,

$$\begin{pmatrix} n^2 - n_x^2 - \epsilon_x & -n_x n_y & -n_x n_z \\ -n_x n_y & n^2 - n_y^2 - \epsilon_y & -n_y n_z \\ -n_x n_z & -n_y n_z & n^2 - n_z^2 - \epsilon_z \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{24}$$

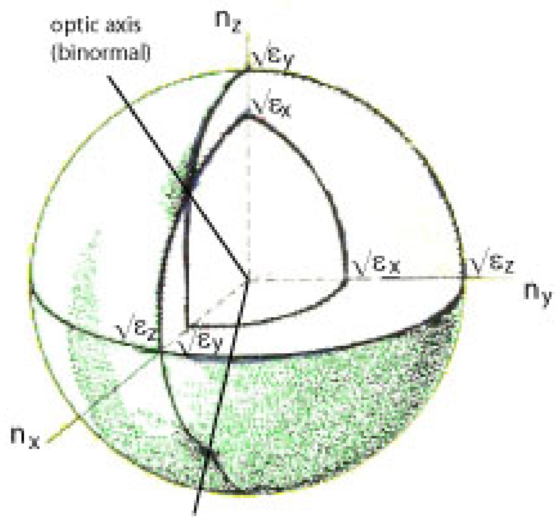
For a solution to exist, the determinant of the matrix must vanish, which after some algebra leads to the quartic equation,

$$(\epsilon_x n_x^2 + \epsilon_y n_y^2 + \epsilon_z n_z^2) n^2 - [\epsilon_x(\epsilon_y^2 + \epsilon_z^2) n_x^2 + \epsilon_y(\epsilon_x^2 + \epsilon_z^2) n_y^2 + \epsilon_z(\epsilon_x^2 + \epsilon_y^2) n_z^2] + \epsilon_x \epsilon_y \epsilon_z = 0, \tag{25}$$

which defines the wave vector surface. The intercepts of this surface with the n_x axis can be found by putting $n_y = n_z = 0$ in eq. (25), yielding the factorized form,

$$(n_x^2 - \epsilon_y)(n_x^2 - \epsilon_z) = 0. \tag{26}$$

Thus, the n_x intercepts are $\sqrt{\epsilon_y}$ and $\sqrt{\epsilon_z}$. Similarly, the n_y intercepts are $\sqrt{\epsilon_x}$ and $\sqrt{\epsilon_z}$, and the n_z intercepts are $\sqrt{\epsilon_x}$ and $\sqrt{\epsilon_y}$. The quartic wave vector surface is sketched below; the figure is adapted from [22]. This surface has two sheets that touch at four points in the x-z plane. The pair of lines that join the antipodal points of contact are called the optic axes or binormals).



The wave vector $\mathbf{n}$ is, by definition, normal to the wave front, but it is not in general normal to the wave vector surface. It is instructive to introduce a vector $\mathbf{s}$ that is normal to the wave vector surface, and whose length is chosen to satisfy,

$$\mathbf{n} \cdot \mathbf{s} = 1. \tag{27}$$

The physical significance of vector $\mathbf{s}$ is that it is parallel to the group velocity $\mathbf{v}_g$. Writing the wave-vector surface (25) as $F(\omega, \mathbf{k}) = 0$, the gradient vector $\boldsymbol{\nabla} F$ is normal to this surface and parallel to $\mathbf{s}$. Then, the group velocity is given by,

$$\mathbf{v}_g = \frac{\partial \omega}{\partial \mathbf{k}} = \frac{\partial F/\partial \mathbf{k}}{\partial F/\partial \omega} = \frac{\boldsymbol{\nabla} F}{\partial F/\partial \omega} \propto \mathbf{s}. \tag{28}$$

Furthermore, the vector $\mathbf{s}$ is parallel to the Poynting vector,

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B}, \tag{29}$$

and hence parallel to the flow of electromagnetic energy in the anisotropic medium. To verify this claim, we take derivatives (at fixed ω) of the last two equations of (22),

$$\delta \mathbf{B} = \mathbf{n} \times \delta \mathbf{E} + \delta \mathbf{n} \times \mathbf{E}, \qquad \delta \mathbf{D} = \delta \mathbf{B} \times \mathbf{n} + \mathbf{B} \times \delta \mathbf{n}, \tag{30}$$

where $\delta \mathbf{n}$ is a small displacement along the wave vector surface. Then,

$$\mathbf{B} \cdot \delta \mathbf{B} = \mathbf{B} \cdot \mathbf{n} \times \delta \mathbf{E} + \delta \mathbf{n} \times \mathbf{E} \cdot \mathbf{B} = \mathbf{B} \times \mathbf{n} \cdot \delta \mathbf{E} + \delta \mathbf{n} \cdot \mathbf{E} \times \mathbf{B} = \mathbf{D} \cdot \delta \mathbf{E} + \delta \mathbf{n} \cdot \mathbf{E} \times \mathbf{B}. \tag{31}$$

Similarly,

$$\mathbf{E} \cdot \delta \mathbf{D} = \delta \mathbf{B} \times \mathbf{n} \cdot \mathbf{E} + \mathbf{E} \cdot \mathbf{B} \times \delta \mathbf{n} = \delta \mathbf{B} \cdot \mathbf{n} \times \mathbf{E} + \mathbf{E} \times \mathbf{B} \cdot \delta \mathbf{n} = \mathbf{B} \cdot \delta \mathbf{B} + \delta \mathbf{n} \cdot \mathbf{E} \times \mathbf{B}. \tag{32}$$

Also, $\mathbf{D} \cdot \delta\mathbf{E} = \sum_i \epsilon_i E_i \delta E_i = \mathbf{E} \cdot \delta\mathbf{D}$, so that by adding eqs. (31) and (32) we learn that $\delta\mathbf{n} \cdot \mathbf{E} \times \mathbf{B} = 0$. Thus, the Poynting vector is normal to the wave surface, and therefore parallel to $\mathbf{s}$ as claimed.

Since $\mathbf{s}$ is parallel to the Poynting vector 29), $\mathbf{E}$ and $\mathbf{B}$ are both perpendicular to $\mathbf{s}$. Further, using eqs. (22 and (27) we find,

$$\mathbf{s} \times \mathbf{D} = \mathbf{s} \times (\mathbf{B} \times \mathbf{n}) = (\mathbf{s} \cdot \mathbf{n})\mathbf{B} - (\mathbf{s} \cdot \mathbf{B})\mathbf{n} = \mathbf{B}, \tag{33}$$

and,

$$\mathbf{s} \times \mathbf{B} = \mathbf{s} \times (\mathbf{n} \times \mathbf{E}) = (\mathbf{s} \cdot \mathbf{E})\mathbf{n} - (\mathbf{s} \cdot \mathbf{n})\mathbf{E} = -\mathbf{E}. \tag{34}$$

Thus, the vector $\mathbf{s}$ obeys,

$$\mathbf{s} \cdot \mathbf{E} = 0, \qquad \mathbf{s} \cdot \mathbf{B} = 0, \qquad \mathbf{s} \times \mathbf{D} = \mathbf{B}, \qquad \mathbf{s} \times \mathbf{B} = -\mathbf{E}, \tag{35}$$

which are very similar in form to the relations (22) for $\mathbf{n}$. In particular,

$$\mathbf{E} = -\mathbf{s} \times (\mathbf{s} \times \mathbf{D}) = s^2\mathbf{D} - (\mathbf{s} \cdot \mathbf{D})\,\mathbf{s} = s^2[\mathbf{D} - (\hat{\mathbf{s}} \cdot \mathbf{D})\,\hat{\mathbf{s}}] = s^2 D \cos\alpha\,\hat{\mathbf{E}}, \tag{36}$$

Using eq. (1) together with eq. (36) we obtain the matrix equation,

$$\begin{pmatrix} s^2 - s_x^2 - 1/\epsilon_x & -s_x s_y & -s_x s_z \\ -s_x s_y & s^2 - s_y^2 - 1/\epsilon_y & -s_y s_z \\ -s_x s_z & -s_y s_z & s^2 - s_z^2 - 1/\epsilon_z \end{pmatrix} \begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{37}$$

The required vanishing of the determinant of the matrix leads to the quartic equation,

$$(\epsilon_y\epsilon_y s_x^2 + \epsilon_x\epsilon_z s_y^2 + \epsilon_x\epsilon_y s_z^2)s^2 - [(\epsilon_y^2 + \epsilon_z^2)s_x^2 + (\epsilon_x^2 + \epsilon_z^2)s_y^2 + (\epsilon_x^2 + \epsilon_y^2)s_z^2] + 1 = 0. \tag{38}$$

The intercepts of the surface defined by eq. (38) with the s_x axis can be found by putting $s_y = s_z = 0$, yielding the factorized form $(n_x^2 - \epsilon_y)(n_x^2 - \epsilon_z) = 0$. Thus, the s_x intercepts are $1/\sqrt{\epsilon_y}$ and $1/\sqrt{\epsilon_z}$. Similarly, the s_y intercepts are $1/\sqrt{\epsilon_x}$ and $1/\sqrt{\epsilon_z}$, and the s_z intercepts are $1/\sqrt{\epsilon_x}$ and $1/\sqrt{\epsilon_y}$. These intercepts are equal to the wave speed divided by c.

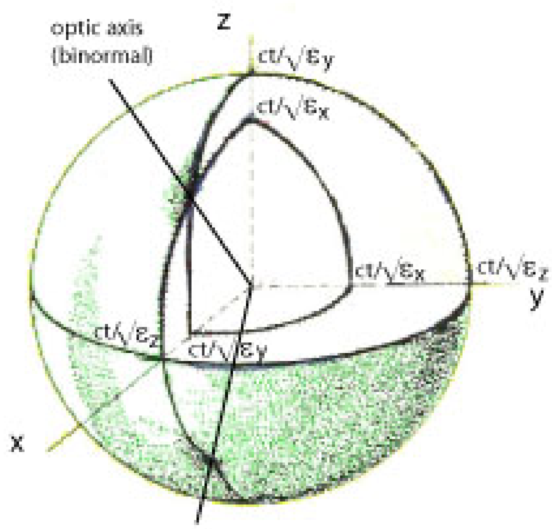
We have seen previously that vector $\mathbf{s}$ is normal to the wave vector surface defined by vector $\mathbf{n}$. Similarly, the vector $\mathbf{n}$ is normal to the surface (38) defined by vector $\mathbf{s}$. This can be verified by differentiating eq. (27) (at fixed ω) to obtain $\delta\mathbf{n} \cdot \mathbf{s} + \mathbf{n} \cdot \delta\mathbf{s} = 0$, where displacement $\delta\mathbf{n}$ lies on the wave vector surface and displacement $\delta\mathbf{s}$ on the surface (38). Then, the first term vanishes by the definition of $\mathbf{s}$, so the second term must vanish as well, which implies that $\mathbf{n}$ is normal to the surface (38). Since the wave vector $\mathbf{n}$ is by definition normal to the wavefront, we learn the wavefront surfaces have the same form as the surface (38).

Wavefronts are surfaces in ordinary space, so we convert eq. (38) into the equation of a wavefront that has traveled for time t away from its sources, whose centroid we take to be at the origin by writing,

$$(\epsilon_y\epsilon_y x^2 + \epsilon_x\epsilon_z y^2 + \epsilon_x\epsilon_y z^2)r^2 - [(\epsilon_y^2 + \epsilon_z^2)x^2 + (\epsilon_x^2 + \epsilon_z^2)y^2 + (\epsilon_x^2 + \epsilon_y^2)z^2] + c^2t^2 = 0. \tag{39}$$

Then, as expected, the intercepts of the wavefront surface with the x axis are $ct/\sqrt{\epsilon_y}$ and $ct/\sqrt{\epsilon_z}$, *etc.*

The quartic wavefront surface is sketched below.



We are finally in a position to characterize the direction of energy flow. We consider some point $\mathbf{r} = (x, y, z)$ on a wavefront far from the sources. The wave vector $\mathbf{n}$ is normal to the wavefront at $\mathbf{r}$. The direction of energy flow at point $\mathbf{r}$ is then the direction of the vector $\mathbf{s}$ that is normal to the wave vector surface at the location of vector $\mathbf{n}$. The surface (38) on which vector $\mathbf{s}$ lies is the same as the wavefront surface (39) with the substitution (s_x, s_y, s_z) for (x, y, z) (and replacing *ct* by 1). Hence, vector $\mathbf{s}$ is parallel to vector $\mathbf{r}$.

That is, far from the sources, the energy flow vector is purely radial despite the convolutions of the wavefront surface. The electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ are transverse to the radial direction, and the waves are transverse electromagnetic (TEM).

For the simpler case of uniaxial anisotropic medium, this result has been noted by Clemmow [23]; for a review by the author of this case, see [8].

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