Energy Flow in a Moving Bimetallic Strip

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(October 28, 2007)

1 Problem

If the point of application of a force $\mathbf{F}$ is moving with velocity $\mathbf{v}$, we say that the force does work at the rate,

$$\frac{dE}{dt} = \mathbf{F} \cdot \mathbf{v}. \tag{1}$$

This statement readily lends itself to the interpretation that the force creates a flow of energy into whatever system exists beyond the point of application. If $\mathbf{F}$ and $\mathbf{v}$ are along the same direction, and the system has a cross sectional area A perpendicular to that direction, we are led to speak of a mechanical (vector) energy flux in the system given by,

$$\mathbf{S}_{\rm mech} = \frac{\mathbf{F} \cdot \mathbf{v}}{A} \,. \tag{2}$$

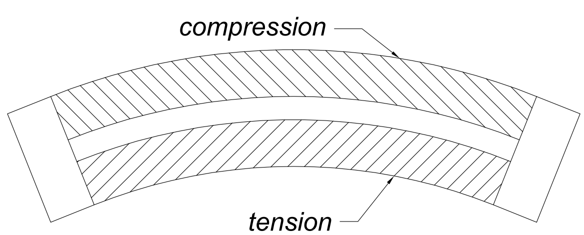
This energy flux can be parallel or antiparallel to the velocity $\mathbf{v}$.

We recall that a mechanical energy flux $\mathbf{S}_{\rm mech}$ implies a mechanical momentum density $\mathbf{p}_{\rm mech}$ which is related by,

$$\mathbf{p}_{\rm mech} = \frac{\mathbf{S}_{\rm mech}}{c^2} = \frac{\mathbf{F} \cdot \mathbf{v}}{Ac^2} \,. \tag{3}$$

This relation follows from the equivalence of energy and mass, $E = mc^2$, on writing the energy flux $\mathbf{S}_{\rm mech}$ as the product of an energy density $u = \rho_u c^2$ times the velocity $\mathbf{v}_E$ of the energy flow, where ρ_u is the mass density equivalent to energy density u. Hence, there exists a momentum density $\mathbf{p}_{\rm mech} = \rho_u \mathbf{v}_E = u\mathbf{v}_E/c^2 = \mathbf{S}_{\rm mech}/c^2$.

Consider a bimetallic strip, made of two metals of unequal coefficients of thermal expansion. At a temperature different from nominal, the differential expansion of the two strips puts one into compression and the other into tension. The strip then takes on a curvature, as illustrated in the sketch below. The strip is at rest in the lab frame.



According to an observer who moves parallel to the long axis of the strip, the strip has velocity $\mathbf{v}$ and the compressive and tensile forces inside the strip are associated with momentum densities given by eq. (3). The two momentum densities are in opposite directions and are offset transversely with respect to one another. Hence, there appears to be a nonzero angular momentum associated with the moving strip. The magnitude of this angular momentum depends on the temperature, which seems to imply nonconservation of angular momentum.

The problem is to clarify whether or not the bimetallic strip possesses angular momentum, according to moving observers.

The insights (2) and (3) of nonrelativistic mechanics can be embedded in a relativistic context via the (symmetric) mechanical energy-momentum-stress tensor of the system, which can be written as,

$$\mathsf{T}^{\mu\nu}_{\rm mech} = \left(\begin{array}{c|c} u_{\rm mech} & c\mathbf{p}_{\rm mech} \\ \hline \mathbf{S}_{\rm mech}/c & -T^{ij}_{\rm mech} \end{array} \right), \tag{4}$$

where the mechanical energy density is $u_{\rm mech} = \rho_m c^2$, the mass density is ρ_m, the speed of light is c, and $T^{ij}_{\rm mech}$ is the 3-dimensional mechanical stress tensor.

For example, if the velocity $\mathbf{v}$, where $v \ll c$, is along the z axis (axis 3), and the only stress in the system is due to the force $\mathbf{F} = F\,\hat{\mathbf{z}}$, then the energy-momentum-stress tensor in the rest frame of the system is given by,

$$\mathsf{T}^{\star\mu\nu} = \left(\begin{array}{c|ccc} \rho^\star_m c^2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & F/A \end{array} \right), \tag{5}$$

where $\rho^\star_m$ is not simply the rest mass density in the absence of the stress F/A, but includes the relativistic correction $u^\star_{\rm elastic}/c^2$ where $u^\star_{\rm elastic}$ is the density of elastic energy stored in the stressed system. In eq. (5), F is positive for a compressive stress and negative for a tensile stress.

The Lorentz transformation L_z from the rest frame to a frame in which the system has velocity $v\,\hat{\mathbf{z}}$ can be expressed in tensor form as,

$$\mathsf{L}^{\mu\nu}_z = \left(\begin{array}{c|ccc} \gamma & 0 & 0 & \gamma\beta \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{array} \right), \tag{6}$$

where $\beta = v/c$. Hence, the energy-momentum-stress tensor in that frame is given by,

$$\mathsf{T}^{\mu\nu}_{\rm mech} = (\mathsf{L}_z \mathsf{T}^\star_{\rm mech} \mathsf{L}_z)^{\mu\nu} = \left(\begin{array}{c|ccc} \gamma^2\rho^\star_m c^2 + \gamma^2\beta^2 F/A & 0 & 0 & \gamma^2\beta(\rho^\star_m c^2 + F/A) \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma^2\beta(\rho^\star_m c^2 + F/A) & 0 & 0 & \gamma^2\beta^2\rho^\star_m c^2 + \gamma^2 F/A \end{array} \right). \tag{7}$$

The total mechanical energy flux has only a z component,

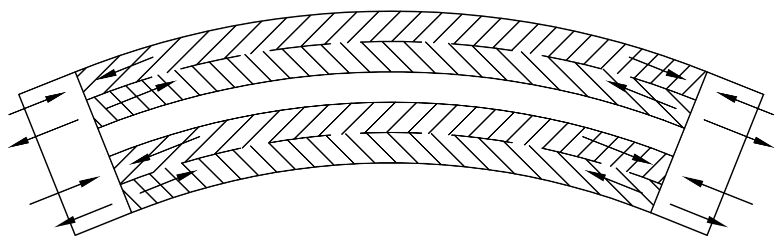
$$S_{{\rm mech},z} = \gamma^2 v(\rho^\star_m c^2 + F/A) \approx v(\rho^\star_m c^2 + F/A), \tag{8}$$

where the approximation holds for $v \ll c$. As expected, the energy flux $S_{\mathrm{mech},z}$ consists of a the transport of energy/mass density $\rho_m^\star c^2$ at velocity v plus the energy flux Fv/A associated with the compressive or tensile force whose point of application is moving with velocity v.

2 Solution

This problem illustrates aspects of the relativity of steady energy flow [1], and was inspired by the solution to problem 59 of the first edition of *Spacetime Physics* by Taylor and Wheeler [2] (but not by the problem itself).

The resolution of the apparent paradox follows from a slightly more realistic model of the stress in the bent bimetallic strip. In particular, the longitudinal stresses in a bent beam are not uniform over a cross section of the beam. The compressive stress is greatest in the region of smallest radius of curvature, and the tensile stress is greatest in the region of largest radius of curvature. There may or may not be a radius at which the stress is zero, but such is assumed in the sketch below.



The thickness of the tensile region in the upper strip is equal to that of the compressive region in the lower strip. The energy fluxes and momentum densities in these two regions imply an angular momentum whose sense is counterclockwise in the sketch. Similarly, the thickness of the compressive region in the upper strip is equal to that of the tensile region in the lower strip. The energy fluxes and momentum densities in these two regions imply an angular momentum whose sense is clockwise in the sketch. Since the stress in the upper strip is predominantly compressive, while that in the lower strip is largely tensile, the thickness of the second pair of regions greater than that of the first pair. The moment arm is, however, greater for the first pair than for the second pair, such that the angular momenta associated with the two pairs of regions are equal and opposite.

The total angular momentum of the system is zero according to all observers.

3 Comments

This problem illustrates the consistency of the notion that stress in a moving body is associated with an internal energy flux and an internal momentum density. Yet, one can question to what extent this energy flux is "real", since there is no net energy/mass transfer.

Furthermore, we cannot identify a well-defined velocity $\mathbf{v}_E$ of the energy flow, since no energy is actually flowing relative to the body itself.

A more formal statement of this difficulty is that the energy density component T^{00} of the energy-momentum-stress tensor and the momentum density components T^{0i}, $i = 1, 2, 3$

do not constitute an energy-momentum four-vector $(u, c\mathbf{p})$ that has a velocity $\mathbf{v}_E = c^2\mathbf{p}/u$, if the medium has nonzero stress in its rest frame.

Indeed, the concept of the 3-dimensional mechanical stress tensor T^{ij} for objects at rest is based on a simplified view of matter as rigid bodies with instantaneous propagation of force and energy. Despite this 3-dimensional stress tensor being embedded in the 4-dimensional energy-momentum-stress $T^{\mu\nu}$, it retains the underlying assumption of instantaneous propagation of mechanical effects, which is, of course, inconsistent with the speed of light as a limiting velocity.

In a medium, whether at rest or in motion, through which net energy flows, it must be possible to identify an energy flow velocity $\mathbf{v}_E$ that is bounded by the speed of light. Mechanical disturbances in an elastic medium propagate at the velocity of sound, which is independent of frequency in the first approximation, so that the energy flow velocity, which is equal to the group velocity, is close to the phase velocity of sound. At the beginning of a net energy flow in an elastic medium, transient sound waves are present and the initial energy flow velocity is evidently the speed of sound. But once the transients die away, the energy flow velocity is much less evident, and it is reasonably consistent to describe the medium by the energy-momentum-stress tensor $T^{\mu\nu}$ which gives no indication of the energy flow velocity or the path of that flow.[1]

4 Acknowledgment

The author thanks Edwin Taylor and Vladimir Hnizdo for e-conversations about this problem.

References

[1] K.T. McDonald, *Relativity of Steady Energy Flow* (Nov. 3, 2007),
http://kirkmcd.princeton.edu/examples/1dgas.pdf

[2] E.F. Taylor and J.A. Wheeler, *Spacetime Physics*, 1st ed. (WH Freeman, 1966). The relevant part of the solution to prob. 59 is reproduced on p. 4 of,
http://kirkmcd.princeton.edu/examples/mechanics/taylor_spacetime_physics_p147.pdf

[3] K.T. McDonald, *Energy, Momentum and Stress in a Belt Drive* (Oct. 20, 2007),
http://kirkmcd.princeton.edu/examples/belt_drive.pdf

[1] Another example that highlights the ambiguity of the description of net energy flow in elastic medium is problem 65 of [2]. See also [3].