Hydraulic Brake

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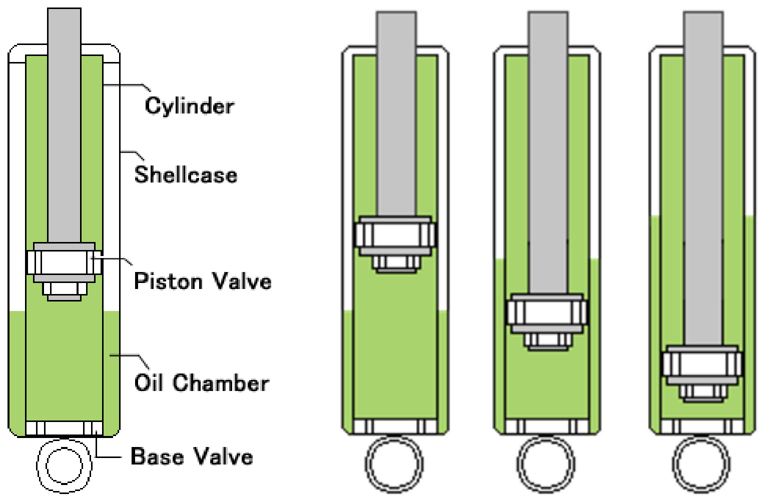
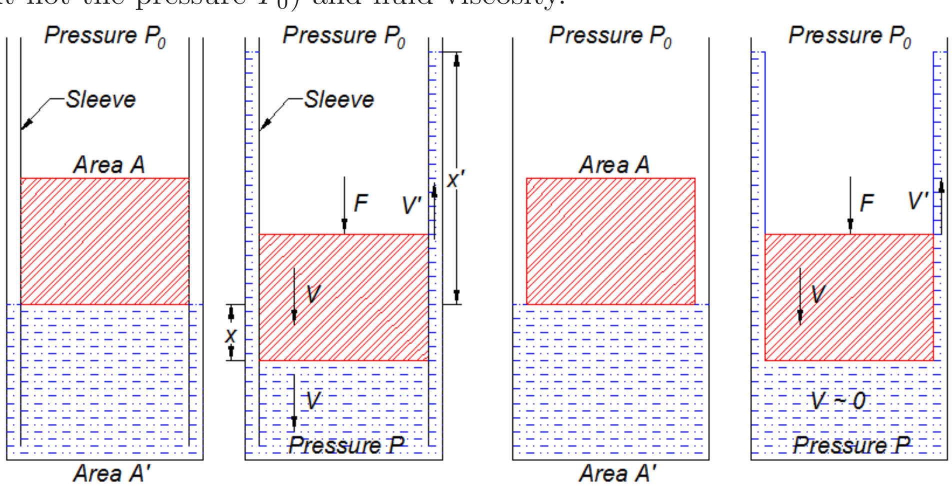
(June 26, 2014)

1 Problem

This problem considers a simplified version of an automobile shock absorber, one variant of which is sketched below.



In the simplified model, the piston valve is replaced by a solid cylinder that moves inside a cylindrical sleeve of area A, shown in the left two figures below. The sleeve does not quite extend to the bottom of the outer cylinder, of area A', such that the (incompressible) fluid can pass around the bottom of the sleeve and rise in the annular volume between the sleeve and the outer cylinder as the piston moves down. Both the sleeve and the outer cylinder are open at their tops, and so subject to atmospheric pressure P_0. Deduce the force F that must be exerted downwards on the piston such that it moves with constant speed V. Ignore gravity (but not the pressure P_0) and fluid viscosity.



Compare with a variant in which no sleeve is present, as shown in the right two figures above.

This problem was suggested by Johann Otto, and is a variant of prob. 1 at https://engineering.purdue.edu/~wassgren/notes/LME_PracticeProblems.pdf.

2 Solution

2.1 Variant with Sleeve

When the piston first pushes on the fluid, which was previously at rest, it must accelerate the fluid within the sleeve to speed V, and the fluid outside the sleeve to speed V'. For this to occur in a short time there must be a large transient force F on the piston. We consider the system only after this transient is over, assuming the piston has reached speed V while having moved a negligible distance x below the initial fluid of the surface.

When the piston of area A has moved distance x below the initial fluid level, the fluid in the annular region of area $A' - A$ outside the sleeve has risen by height x' given by,

$$xA = x'(A' - A), \tag{1}$$

so that the volume of the incompressible fluid remains constant. The fluid within the sleeve has velocity $V = dx/dt$, and the fluid outside it has velocity $V' = dx'/dt$, which are therefore related by

$$VA = V'(A' - A). \tag{2}$$

We now deduce the force F on the piston needed to maintain constant fluid speeds by an energy method, and the by consideration of fluid momentum.

2.1.1 Energy Analysis

When the piston has moved distance x the kinetic energy of the fluid, of mass density ρ, has changed by,

$$\Delta\text{KE} = \frac{\rho}{2}\left[x'(A' - A)V'^2 - xAV^2\right] = \frac{\rho x A V^2}{2}\left[\frac{A^2}{(A' - A)^2} - 1\right] = \frac{\rho x V^2}{2}\frac{AA'(2A - A')}{(A' - A)^2}. \tag{3}$$

The change in the kinetic energy is equal to the work W done on the fluid,

$$W = (F + P_0 A)x - P_0(A' - A)x' = Fx. \tag{4}$$

Hence, the force required for steady motion of the piston is,

$$F = \rho V^2 \frac{AA'(2A - A')}{2(A' - A)^2}. \tag{5}$$

2.1.2 Momentum Analysis

The upward momentum p of the fluid is,

$$p = p_0 + \rho x'(A' - A)V' - \rho x A V = p_0 + \rho x A V \left(\frac{A}{A' - A} - 1 \right) = p_0 + \rho x V \frac{A(2A - A')}{A' - A}\,, \qquad (6)$$

where p_0 is the momentum of the fluid just after the transient force has set it in motion. The rate of change of the fluid momentum equals to total upward force on the fluid,

$$\frac{dp}{dt} = \rho V^2 \frac{A(2A - A')}{A' - A} = F_{\rm up} = F_{\rm bottom} - F - P_0 A'. \qquad (7)$$

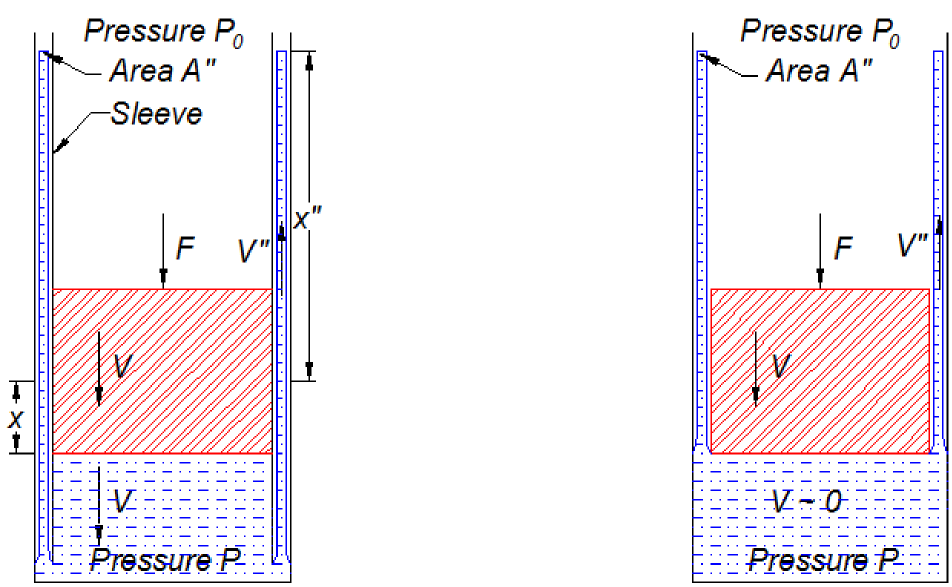
The upward force on the bottom of the fluid is area A' times the fluid pressure P at the bottom, which we approximate as the pressure $P_0 + F/A$ exerted by the piston on the fluid. Then,

$$\rho V^2 \frac{A(2A - A')}{A' - A} = F \frac{A' - A}{A}\,, \qquad F = \rho V^2 \frac{A^2(2A - A')}{(A' - A)^2}\,. \qquad (8)$$

2.1.3 Vena Contracta

The force (8) deduced via the momentum analysis is twice that, eq. (5), deduced via the energy analysis. This type of apparent conflict between energy and momentum analyses is an indication of a bad assumption, that the cross section of the fluid flow in the annular region has the same area $A' - A$ as that region. However, as noted by Toricelli in 1643, flow through apertures results in a contraction (the *vena contracta* [1]) of the area of the fluid flow to a smaller area A''. Correspondingly, the length x'' and the velocity V'' of the contracted fluid flow are given by,

$$x'' = x \frac{A}{A''}\,, \qquad V'' = V \frac{A}{A''}\,, \qquad (9)$$



The revised energy analysis is,

$$\begin{aligned} W &= (F + P_0 A)x - P_0 A'' x'' = Fx \\ &= \Delta\text{KE} = \frac{\rho}{2}\left[x'' A'' V''^2 - x A V^2\right] = \frac{\rho x A V^2}{2}\left(\frac{A^2}{A''^2} - 1\right) = \frac{\rho x V^2}{2}\frac{A(A^2 - A''^2)}{A''^2}, \end{aligned} \tag{10}$$

such that,

$$F = \rho V^2 \frac{A(A^2 - A''^2)}{2A''^2}. \tag{11}$$

In the revised momentum analysis the force on the bottom of the fluid is still $F + P_0 A'$, so we now have,

$$\frac{dp}{dt} = \rho V^2 \frac{A(A - A'')}{A''} = F\frac{A' - A}{A}, \qquad F = \rho V^2 \frac{A^2(A - A'')}{A''(A' - A)}. \tag{12}$$

Requiring the two force computations (11)-(12) to be the same, we find that,

$$\frac{A''}{A' - A} = \frac{A}{3A - A'} \quad \left(\approx \frac{1}{2} \text{ if } A' \approx A\right), \qquad F = 2\rho V^2 \frac{A^2(2A - A')}{(A' - A)^2}. \tag{13}$$

Note that the result (13) for F taking the *vena contracta* in account is the twice that of eq. (8) from the momentum analysis ignoring it.

A peculiar result holds when $A' > 2A$, that once the downward transient force has given downward velocity V to the piston, an upward force must be applied to the piston to maintain this downward velocity.

The braking force F is large when the outer area $A' = A(1 + \epsilon)$ is only slightly larger that the area A of the piston,

$$F \to 2\rho V^2 \frac{A}{\epsilon^2}. \tag{14}$$

2.2 Variant without Sleeve

In the variant without the sleeve we approximate the velocity of the fluid below the bottom face of the piston as being at rest, while the fluid in the annular area $A' - A$ outside the piston is subject to the *vena contracta*, and has height $h = x + x'' = x(A + A'')/A''$, and speed $V'' = VA/A''$ (except close to the bottom face of the piston where the *vena contracta* is not yet complete).

2.2.1 Energy Analysis

When the piston has moved distance x the kinetic energy of the fluid, of mass density ρ, has increased by,

$$\Delta\text{KE} = \frac{\rho}{2} h A'' V''^2 = \frac{\rho x V^2}{2}\frac{A^2(A + A'')}{A''^2}. \tag{15}$$

The change in the kinetic energy is equal to the work W done on the fluid, which is again,

$$W = (F + P_0 A)x - P_0 A'' x'' = Fx. \tag{16}$$

Hence, the force required for steady motion of the piston is,

$$F = \rho V^2 \frac{A^2(A + A'')}{2A''^2}\,. \tag{17}$$

2.2.2 Momentum Analysis

In the approximation that the fluid below the bottom of the piston is essentially at rest, the upward momentum p of the fluid is,

$$p = \rho h A'' V'' = \rho x V \frac{A(A + A'')}{A''}\,. \tag{18}$$

The rate of change of the fluid momentum equals to total upward force on the fluid,

$$\begin{aligned} \frac{dp}{dt} &= \rho V^2 \frac{A(A + A'')}{A''} \\ &= F_{\rm up} = F_{\rm bottom} - F - P_0 A' \approx (P_0 + F/A)A' - F - P_0 A' = F\frac{A' - A}{A}\,. \end{aligned} \tag{19}$$

Then,

$$F = \rho V^2 \frac{A^2(A + A'')}{A''(A' - A)}\,. \tag{20}$$

Requiring the two force calculations, eqs. (17) and (20) to give the same results, we find,

$$\frac{A''}{A' - A} = \frac{1}{2}\,, \qquad \frac{V''}{V} = \frac{2A}{A' - A}\,, \qquad F = \rho V^2 \frac{A^2(A + A')}{(A' - A)^2}\,. \tag{21}$$

When the outer area $A' = A(1 + \epsilon)$ is only slightly larger that the area A of the piston, the braking force is large, and the same as for the variant with the sleeve,

$$F \to 2\rho V^2 \frac{A}{\epsilon^2}\,. \tag{22}$$

References

[1] K.T. McDonald, *Vena Contracta*, (Feb. 16, 2005), http://kirkmcd.princeton.edu/examples/vena_contracta.pdf