

# Bunching of Photons

## When Two Beams Pass Through a Beam Splitter

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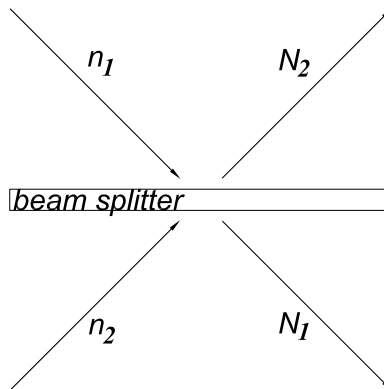
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### 1 Problem

Dirac has written [1] *Each photon then interferes only with itself. Interference between two different photons never occurs.* Indeed, a practical definition is that “classical” optics consists of phenomena due to the interference of photons only with themselves. However, photons obey Bose statistics which implies a “nonclassical” tendency for them to “bunch”.

For a simple example of nonclassical optical behavior, consider two pulses containing  $n_1$  and  $n_2$  photons of a single frequency that are simultaneously incident on two sides of a lossless, 50:50 beam splitter, as shown in the figure. Deduce the probability that  $N_1$  photons are observed in the direction of beam 1, where  $0 \leq N_1 \leq n_1 + n_2$  for a lossless splitter.



Hint: a relatively elementary argument can be given by recalling that the phase of a reflected photon (*i.e.*, of the reflected wave from a single input beam) is  $90^\circ$  different from that of a transmitted photon [2]. Consider first the cases that one of  $n_1$  or  $n_2$  is zero.

### 2 Solution

An elegant solution can be given by noting the the creation and annihilation operators relevant to a beam splitter obey an  $SU(2)$  symmetry [3, 4]. Here, we give a more elementary solution, in the spirit of Feynman (Vol. I, sec. 33-6 of [5]).

Experimental demonstration of the case where  $n_1 = n_2 = 1$  was first given in [6], and the case of  $n_1 = n_2 = 2$  has been studied in [7].

## 2.1 A Single Input Beam

We first consider the case of a single input beam with  $n_1 > 0$ . Then, of course,  $n_2 = 0$ .

In a classical view, the input beam would have energy  $u_1 = n_1 \hbar \omega$ , where  $\omega$  is the angular frequency of the photons. Then, the effect of the 50:50 beam splitter would be to create output beams of equal energies,  $U_1 = U_2 = u_1/2$ . In terms of photon numbers, the classical view would imply that the only possibility for the output beams is  $N_1 = N_2 = n_1/2$ .

But in fact, the transmitted beam can contain any number  $N_1$  of photons between 0 and  $n_1$ , while the reflected beam contains  $N_2 = n_1 - N_1$  photons.

If the photons were distinguishable, we would assign a probability of  $(1/2)^{n_1}$  to each configuration of transmitted and reflected photons in the 50:50 splitter. But the photons are indistinguishable, so that the probability that  $N_1$  out of  $n_1$  photons are transmitted is larger than  $(1/2)^{n_1}$  by the number of ways the  $n_1$  photons can be arranged into a group of  $N_1$  transmitted and  $n_1 - N_1$  reflected photons without regard to their order, *i.e.*, by the binomial coefficient,

$$C_{N_1}^{n_1} = \frac{n_1!}{N_1!(n_1 - N_1)!}. \quad (1)$$

Thus, the probability  $P(N_1, n_1 - N_1 | n_1, 0)$  that  $N_1$  out of  $n_1$  photons (in a single input beam) are transmitted by the beam splitter is,

$$P(N_1, n_1 - N_1 | n_1, 0) = C_{N_1}^{n_1} \left(\frac{1}{2}\right)^{n_1}. \quad (2)$$

The result (2) is already very nonclassical, in that there is a small, but nonzero probability that the entire input beam is transmitted, or reflected. However, in the limit of large  $n_1$  the largest probability is that the numbers of photons in the reflected and transmitted beams are very nearly equal. We confirm this by use of Stirling's approximation for large  $n$ ,

$$n! \approx e^{-n} n^n \sqrt{2\pi n}. \quad (3)$$

For large  $n$ , and  $k = (1 + \epsilon)n/2$ , we have,

$$\begin{aligned} C_k^n &\approx \frac{1}{\sqrt{2\pi n} \left(\frac{k}{n}\right)^{k+1/2} \left(1 - \frac{k}{n}\right)^{n-k+1/2}} = \frac{2^{n+1}}{\sqrt{2\pi n} (1 - \epsilon^2)^{(n+1)/2} \left(\frac{1+\epsilon}{1-\epsilon}\right)^{n\epsilon/2}} \\ &\approx \frac{2^{n+1}}{\sqrt{2\pi n} (1 + n\epsilon^2/2)}. \end{aligned} \quad (4)$$

The probability of  $k$  photons out of  $n$  being transmitted drops to 1/2 the peak probability when  $\epsilon \approx \sqrt{2/n}$ . Hence, for large  $n$  the number distribution of photons in the transmitted (and reflected) beam is essentially a delta function centered at  $n/2$ , in agreement with the classical view.

The most dramatic difference between the classical and quantum behavior of a single beam in a 50:50 beam splitter occurs when  $n_1 = 2$ ,

$$P(0, 2 | 2, 0) = \frac{1}{4}, \quad P(1, 1 | 2, 0) = \frac{1}{2}, \quad P(2, 0 | 2, 0) = \frac{1}{4}. \quad (5)$$

In the subsequent analysis we shall need to consider interference effects, so we note that the magnitude of the probability amplitude that  $k$  out of  $n$  photons in a single beam are transmitted by a 50:50 beam splitter can be obtained by taking the square root of eq. (2),

$$|A(k, n - k|n, 0)| = \sqrt{C_k^n} \left(\frac{1}{2}\right)^{n/2}. \quad (6)$$

These amplitudes have the obvious symmetries,

$$|A(k, n - k|n, 0)| = |A(n - k, k|n, 0)| = |A(k, n - k|0, n)| = |A(n - k, k|0, n)|. \quad (7)$$

We must also consider the phases of these amplitudes, or at least the relative phases. The hint is that we may consider the phase of a reflected photon to be shifted with respect to that of a transmitted photon by  $90^\circ$ , as follows from a classical analysis of waves in a 50:50 beam splitter [2] (see also the Appendix). In this problem, we define the phase of a transmitted photon to be zero, so that the probability amplitude should include a factor of  $i = \sqrt{-1}$  for each reflected photon. Thus, we have,

$$A(k, n - k|n, 0) = i^{n-k} \sqrt{C_k^n} \left(\frac{1}{2}\right)^{n/2}, \quad (8)$$

$$A(n - k, k|n, 0) = i^k \sqrt{C_k^n} \left(\frac{1}{2}\right)^{n/2}, \quad (9)$$

$$A(k, n - k|0, n) = i^k \sqrt{C_k^n} \left(\frac{1}{2}\right)^{n/2}, \quad (10)$$

$$A(n - k, k|0, n) = i^{n-k} \sqrt{C_k^n} \left(\frac{1}{2}\right)^{n/2}. \quad (11)$$

## 2.2 Two Input Beams

We now calculate the general probability  $P(N_1, n_1 + n_2 - N_1|n_1, n_2)$  that  $N_1$  output photons are observed along the direction of input beam 1 when the number of photons in the input beams is  $n_1$  and  $n_2$ .

We first give a classical wave analysis. The input waves have amplitudes  $a_{1,2} = \sqrt{n_{1,2}\hbar\omega}$ , and are in phase at the center of the beam splitter. The output amplitudes are the sums of the reflected and transmitted parts of the input amplitudes. A reflected amplitude has a phase shift of  $90^\circ$  relative to its corresponding transmitted amplitude, as discussed in sec. 2.1. In the 50:50 beam splitter, the magnitude of both the reflected and transmitted amplitudes from a single input beam are  $1/\sqrt{2}$  times the magnitude of the amplitude of that beam. Hence, the output amplitudes are,

$$A_1 = \frac{1}{\sqrt{2}}(a_1 + ia_2), \quad (12)$$

$$A_2 = \frac{1}{\sqrt{2}}(ia_1 + a_2). \quad (13)$$

Taking the absolute square of eqs. (12)-(13), we find the output beams to be described by,

$$N_{1,2} = \frac{|A_{1,2}|^2}{\hbar\omega} = \frac{a_1^2 + a_2^2}{2\hbar\omega} = \frac{n_1 + n_2}{2}. \quad (14)$$

The classical view is that a 50:50 beam splitter simply splits both input beams, when they are in phase.

For a quantum analysis, we proceed by noting that of the  $N_1$  photons in output beam 1,  $k$  of these could have come by transmission from input beam 1, and  $N_1 - k$  by reflection from input beam 2 (so long as  $N_1 - k \leq n_2$ ). The probability amplitude that  $k$  out of  $N_1$  photons are transmitted from beam 1 while  $N_1 - k$  photons are reflected from beam 2 is, to within a phase factor, the product of the amplitudes for each of these configurations resulting from a single input beam,

$$A(k, N_1 - k|n_1, 0)A(N_1 - k, n_2 - N_1 + k|0, n_2) = (-1)^{n_1 - k} \sqrt{C_k^{n_1} C_{N_1 - k}^{n_2}} \left(\frac{1}{2}\right)^{(n_1 + n_2)/2}, \quad (15)$$

referring to eqs. (8)-(11). The most dramatic nonclassical features to be found below can be attributed to the presence of the factor  $(-1)^{n_1 - k}$  that arises from the  $90^\circ$  phase shift between reflected and transmitted photons.

Since photons obey Bose statistics, we sum the sub-amplitudes (15), weighting each one by the square root of the number of ways that  $k$  out of the  $N_1$  photons in the first output beam can be assigned to input beam 1, namely  $C_k^{N_1}$ , time the square root of the number of ways that the remaining  $n_1 - k$  photons from input beam 1 can be assigned to the  $N_2$  photons in output beam 2, namely  $C_{n_1 - k}^{N_2}$  to obtain,<sup>1</sup>

$$\begin{aligned} A(N_1, n_1 + n_2 - N_1|n_1, n_2) &= \sum_k \sqrt{C_k^{N_1} C_{n_1 - k}^{N_2}} A(k, N_1 - k|n_1, 0)A(N_1 - k, n_2 - N_1 + k|0, n_2) \\ &= (-1)^{n_1} \left(\frac{1}{2}\right)^{(n_1 + n_2)/2} \sum_k (-1)^k \sqrt{C_k^{n_1} C_{N_1 - k}^{n_2} C_k^{N_1} C_{n_1 - k}^{N_2}}. \end{aligned} \quad (16)$$

When evaluating this expression, any binomial coefficient  $C_m^n$  in which  $m$  is negative, or greater than  $n$ , should be set to zero.

The desired probability is, of course,

$$P(N_1, n_1 + n_2 - N_1|n_1, n_2) = |A(N_1, n_1 + n_2 - N_1|n_1, n_2)|^2 \quad (17)$$

Some examples of the probability distributions for small numbers of input photons are given below.

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<sup>1</sup>Delicate to justify not also including factors  $C_{N_1 - k}^{N_1}$ , and  $C_{N_2 - (n_1 - k)}^{N_2}$ , these being the ways of assigning photons to output beam 2 – but these factors are the same as those already included, and so should not be counted twice...

### 2.2.1 Two Input Photons

Input $ n_1, n_2\rangle$	Output $(N_1, N_2 $		
	$(0, 2 $	$(1, 1 $	$(2, 0 $
$ 2, 0\rangle$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$ 1, 1\rangle$	$\frac{1}{2}$	0	$\frac{1}{2}$
$ 0, 2\rangle$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

When  $n_1$  or  $n_2$  is zero, the probability distribution is binomial, as found in sec. 2.1. When  $n_1 = n_2 = 1$  there is complete destructive interference between the cases where both photons are reflected (combined phase shift =  $180^\circ$ ) and when both are transmitted (combined phase shift =  $0$ ). This quantum result is strikingly different from the classical expectation that there would be one photon in each output beam.

### 2.2.2 Three Input Photons

Input $ n_1, n_2\rangle$	Output $(N_1, N_2 $			
	$(0, 3 $	$(1, 2 $	$(2, 1 $	$(3, 0 $
$ 3, 0\rangle$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
$ 2, 1\rangle$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$ 1, 2\rangle$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$ 0, 3\rangle$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

### 2.2.3 Four Input Photons

Input $ n_1, n_2\rangle$	Output $(N_1, N_2 $				
	$(0, 4 $	$(1, 4 $	$(2, 2 $	$(3, 1 $	$(4, 0 $
$ 4, 0\rangle$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$
$ 3, 1\rangle$	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$
$ 2, 2\rangle$	$\frac{3}{8}$	0	$\frac{1}{4}$	0	$\frac{3}{8}$
$ 1, 3\rangle$	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$
$ 0, 4\rangle$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

### 2.2.4 Symmetric Input Beams: $n_1 = n_2 \equiv n$

In this case there is zero probability of observing an odd number of photons in either output beam.

To see this, note that when  $n_1 = n_2 = n$ , the magnitudes of the subamplitudes are equal for having  $k$  photons appearing in output beam 1 from either input beam 1 or input beam 2. However, the phases of these two subamplitudes are  $180^\circ$  apart, so that they cancel. In particular, when  $k$  photons are transmitted into output beam 1 from input beam 1, then  $N_1 - k$  photons are reflected from input beam 2 into output beam 1; meanwhile,  $n - k$  photons are reflected from input beam 1 into output beam 2. So, the overall phase factor of this subamplitude is  $i^{N_1 - k + n - k} = (-1)^k i^{n + N_1}$ . Whereas, if  $k$  photons are reflected from input beam 2 into output beam 1, then  $N_1 - k$  photons are transmitted from input beam 1 into output beam 1, and so  $n - N_1 + k$  photons are reflected from input beam 1 into output beam 2. So, the overall phase factor of this subamplitude is  $i^{k + n - N_1 + k} = (-1)^k i^{n - N_1}$ . The phase factor between these two subamplitudes (whose magnitudes are equal) is  $i^{2N_1} = (-1)^{N_1}$ , which is  $-1$  for odd  $N_1$ , as claimed.

For the case of observing an even number of photons in the output beams, a remarkable simplification of eq. (16) holds [3]. *We have not been able to show this by elementary means. It does follow by inspection when  $m = 0$  or  $n$ , in which case eq. (16) contains only a single nonzero term. In general, the index  $k$  in eq. (16) for  $A(2m, 2n - 2m|n, n)$  runs from  $0$  to  $2m$  if  $2m \leq n$ , or from  $2m - n$  to  $n$  if  $2m \geq n$ . There are an odd number of terms, the central one having index  $k = m$ . By a strange miracle of combinatorics, the sum collapses to a simplified version of the central term of the series....* Namely,

$$A(2m, 2n - 2m|n, n) = (-1)^{n-m} \left(\frac{1}{2}\right)^n \sqrt{C_m^{2m} C_{n-m}^{2n-2m}}. \quad (18)$$

Therefore, the  $n + 1$  nonvanishing probabilities for symmetric input beams are.

$$P(2m, 2n - 2m|n, n) = \left(\frac{1}{2}\right)^{2n} C_m^{2m} C_{n-m}^{2n-2m} \approx \frac{1}{n\pi \sqrt{\frac{m}{n}(1 - \frac{m}{n})}}, \quad (19)$$

where the approximation holds for large  $m$  and large  $n$ . Note that  $\int_0^1 dx / \sqrt{x(1-x)} = \pi$ . This probability distribution peaks for  $m = 0$  or  $n$ , *i.e.*, for all photons in one or the other output beam, with value,

$$P(0, 2n|n, n) = P(2n, 0|n, n) = \left(\frac{1}{2}\right)^{2n} C_n^{2n}. \quad (20)$$

The probability of finding all output photons in a single beam when the input beams are symmetric is larger by a factor  $C_n^{2n}$  than when there is only a single input beam (of the same total number of photons), because there are  $C_n^{2n}$  ways of assigning the  $n$  photons from input beam 1 to the  $2n$  photons in the output beam. This is an extreme example of photon bunching caused by the beam splitter.

*It is noteworthy that the result (19) does not agree with the classical prediction (14) in the large  $n$  limit.*

*Of course, as pointed out by Glauber [8], a classical wave corresponds to a photon state with minimum uncertainty products  $\Delta E \Delta B$ , where  $E$  and  $B$  are the electric and magnetic field amplitudes of the wave, respectively. In case of a pulse, we expect classically that both its energy  $U$  and phase  $\phi$  are well defined, but the closest quantum equivalent is a coherent*

state with minimal uncertainty to the product  $\Delta U \Delta \phi$ . This state is a superposition of states of various photons numbers  $n$  whose expectation value for  $n$  follows a Poisson distribution with  $\langle n \rangle = U/\hbar\omega$ . For large  $n$ , the variance in photon number is  $\sqrt{n}$ .

Hence, in an experiment in which large numbers  $N_1$  and  $N_2$  of photons are observed at the two output ports of the beam splitter, we can say that the numbers  $n_1$  and  $n_2$  of photons at the input ports obeyed  $n_1 + n_2 = N_1 + N_2$ , but we cannot know  $n_1$  and  $n_2$  separately (if the inputs beams are “classical”). All we can know are the mean values  $\langle n_1 \rangle$  and  $\langle n_2 \rangle$ . Therefore, we should rewrite the probability distribution (17) as,

$$P(N_1, N_2 | \langle n_1 \rangle, \langle n_2 \rangle) = \left| \sum_{n_1, n_2} a_{n_1} a_{n_2} A(N_1, N_2 | n_1, n_2) \right|^2, \quad (21)$$

where  $a_{n_i}$  is the amplitude that input beam  $i$  contained  $n_i$  photons when the mean number of photons in this beam is  $\langle n_i \rangle$ . We conjecture that a detailed calculation of eq. (21) would agree with the classical prediction (14), but we have not confirmed this.

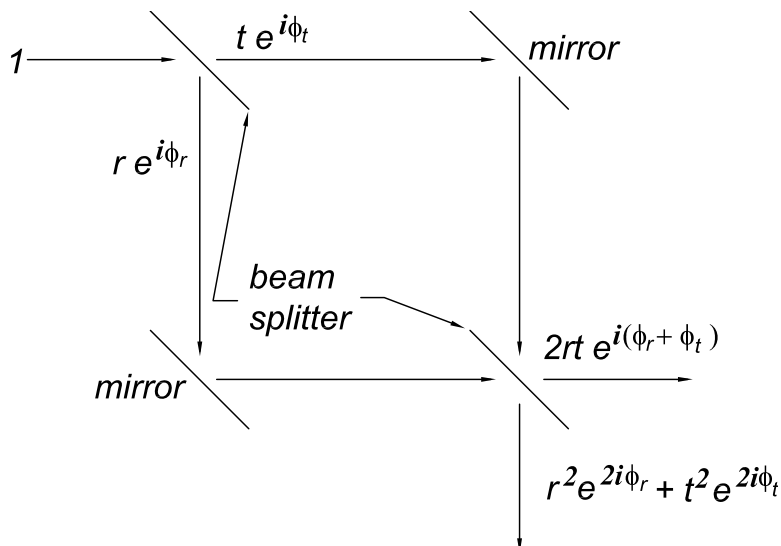
### 3 Appendix: Phase Shift in a Lossless Beam Splitter

We give a classical argument based on a Mach-Zehnder interferometer, shown in the figure below, that there is a  $90^\circ$  phase shift between the reflected and transmitted beams in a lossless, symmetric beam splitter. Then, following Dirac’s dictum [1], this result applies to a single photon.

A beam of light of unit amplitude is incident on the interferometer from the upper left. The reflected and transmitted amplitudes are  $r e^{i\phi_r}$  and  $t e^{i\phi_t}$ , where magnitudes  $r$  and  $t$  are real numbers. The condition of a lossless beam splitter is that,

$$r^2 + t^2 = 1. \quad (22)$$

The reflected and transmitted beams are reflected off mirrors and recombined in a second lossless beam splitter, identical to the first.



Then, the amplitude for transmission at the first beam splitter, followed by reflection at the second, is  $tr, e^{i(\phi_t + \phi_r)}$ , etc. Hence, the recombined beam that moves to the right has amplitude,

$$A_1 = 2rt e^{i(\phi_r + \phi_t)}, \quad (23)$$

while the recombined beam that moves downwards has amplitude,

$$A_2 = r^2 e^{2i\phi_r} + t^2 e^{2i\phi_t}. \quad (24)$$

The intensity of the first output beam is,

$$I_1 = |A_1|^2 = 4r^2t^2, \quad (25)$$

and that of the second output beam is,

$$I_2 = |A_2|^2 = r^4 + t^4 + 2r^2t^2 \cos 2(\phi_t - \phi_r). \quad (26)$$

For lossless splitters, the total output intensity must be unity,

$$I_1 + I_2 = 1 = (r^2 + t^2)^2 + 2r^2t^2[1 + \cos 2(\phi_t - \phi_r)]. \quad (27)$$

Recalling eq. (22), we must have,

$$\phi_t - \phi_r = \pm 90^\circ, \quad (28)$$

for any value of the splitting ratio  $r^2 : t^2$ .

The preceding argument does not clarify where that phase difference (28) is  $90^\circ$  or  $-90^\circ$ , but more detailed arguments [2] show the phase difference to be  $-90^\circ$ . That is,

$$\phi_r = \phi_t + 90^\circ. \quad (29)$$

Furthermore, if the beam splitter is thin compared to a wavelength, then  $\phi_t \approx 0$  and  $\phi_r \approx 90^\circ$ .

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