Stress and Momentum in a Capacitor That Moves with Constant Velocity

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1 Problem

Consider a parallel-plate capacitor whose plates are held apart by a nonconducting slab of unit (relative) dielectric constant and unit (relative) magnetic permeability.[1] Discuss the energy, momentum and stress in this (isolated) system when at rest and when moving with constant velocity parallel or perpendicular to the electric field.

Does the system contain hidden momentum, $\mathbf{P}_{\text{hidden}}$, defined for a subsystem by,

$$\mathbf{P}_{\text{hidden}} \equiv \mathbf{P} - M\mathbf{v}_{\text{cm}} - \oint_{\text{boundary}} (\mathbf{x} - \mathbf{x}_{\text{cm}})\,(\mathbf{p} - \rho\mathbf{v}_b) \cdot d\mathbf{Area}, \tag{1}$$

where $\mathbf{P}$ is the total momentum of the subsystem, $M = U/c^2$ is its total "mass," U is its total energy, c is the speed of light in vacuum, $\mathbf{x}_{\text{cm}}$ is its center of mass/energy, $\mathbf{v}_{\text{cm}} = d\mathbf{x}_{\text{cm}}/dt$, $\mathbf{p}$ is its momentum density, $\rho = u/c^2$ is its "mass" density, u is its energy density, and $\mathbf{v}_b$ is the velocity (field) of its boundary?[2]

Fringe-field effects can be ignored. The velocity can be large or small compared to the speed of light.

2 Solution

This problem is concerned with the relativistic transformation of properties of the capacitor. It represents a macroscopic application of the concepts of "Poincaré stresses" [3] that were introduced into classical models of the electron. Versions of this problem have also appeared in [4]-[10].

We first note that the isolated system of the capacitor (including the dielectric material between its plates) and its electromagnetic field has a conserved energy and 3-momentum, which form a 4-vector. Then, the system as a whole obeys $\mathbf{P} = M\mathbf{v}_{\text{cm}}$, where $M = \gamma M_0$ is the "relativistic" mass, $\gamma = 1/\sqrt{1 - v_{\text{cm}}^2/c^2}$ and M_0 is the rest mass of the system. The isolated system has no boundary, so according to eq. (1), the system as a whole has no "hidden" momentum, $\mathbf{P}_{\text{hidden}} = \mathbf{P} - M\mathbf{v}_{\text{cm}} = 0$.

In the following we consider the system to have two subsystems, its electromagnetic field, and its "mechanical" components, which may contain "hidden" momentum.

[1] The use of unit dielectric constant and unit permeability avoids entering into the interesting controversy as to the so-called Abraham and Minkowski forms of the energy-momentum-stress tensor [1].

[2] The definition (1) was inspired by a private communication from Daniel Vanzella. See also [2].

We suppose that the capacitor supports a uniform electric field $\mathbf{E}^\star = E^\star\,\hat{\mathbf{z}}$ between its plates, in its rest frame, and we ignore the fringe field.[3] *Taking the (relative) dielectric constant ϵ of the material between the plates to be unity, the electric displacement is given by $\mathbf{D}^\star = \mathbf{E}^\star$ (in Gaussian units). The macroscopic magnetic fields vanish in the capacitor's rest frame, $\mathbf{B}^\star = \mathbf{H}^\star = 0$, noting that the relative permeability is $\mu = 1$.*

Associated with these electromagnetic fields is the 4-dimensional, macroscopic (symmetric) electromagnetic energy-momentum-stress tensor (secs. 32-33 of [12], sec. 12.10B of [13]),

$$\mathsf{T}^{\mu\nu}_{\rm EM} = \left(\begin{array}{c|c} u_{\rm EM} & c\,\mathbf{p}_{\rm EM} \\ \hline c\,\mathbf{p}_{\rm EM} & -T^{ij}_{\rm EM} \end{array} \right), \tag{2}$$

where indices μ and ν take on values 0, 1, 2, 3, spatial indices i and j take on values 1, 2, 3, $u_{\rm EM}$ is the electromagnetic field energy density,

$$u_{\rm EM} = \frac{E^2 + B^2}{4\pi}\,, \tag{3}$$

$\mathbf{p}_{\rm EM}$ is the electromagnetic momentum density,

$$\mathbf{p}_{\rm EM} = \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}\,, \tag{4}$$

and $T^{ij}_{\rm EM}$ is the 3-dimensional electromagnetic stress tensor.

$$T^{ij}_{\rm EM} = \frac{E_i E_j + B_i B_j}{4\pi} - \delta_{ij}\frac{E^2 + B^2}{8\pi}\,. \tag{5}$$

In the rest frame of the capacitor the electromagnetic energy-momentum-stress tensor has components,

$$\mathsf{T}^{\star\mu\nu}_{\rm EM} = \left(\begin{array}{c|ccc} \frac{E^{\star 2}}{8\pi} & & \mathbf{0} & \\ \hline & \frac{E^{\star 2}}{8\pi} & & \\ \mathbf{0} & & \frac{E^{\star 2}}{8\pi} & \\ & & & -\frac{E^{\star 2}}{8\pi} \end{array} \right), \tag{6}$$

in the region between the capacitor plates. The nonzero diagonal elements $\mathsf{T}^{\star 11}$, $\mathsf{T}^{\star 22}$ and $\mathsf{T}^{\star 33}$, indicate that there are internal electric forces on, and stresses in, the material between the capacitor plates. Thus, we are led to consider also the mechanical energy-momentum stress tensor,

$$\mathsf{T}^{\mu\nu}_{\rm mech} = \left(\begin{array}{c|c} u_{\rm mech} & c\,\mathbf{p}_{\rm mech} \\ \hline c\,\mathbf{p}_{\rm mech} & -T^{ij}_{\rm mech} \end{array} \right), \tag{7}$$

[3] As will be seen in sec. 2.2 below, for $\mathbf{v} \perp \mathbf{E}^\star$ this approximation leads to expressions for the total energy and momentum of the system that do not form a 4-vector, so there is limited validity to the results in that section.

where the mechanical energy density is $u_{\rm mech} = \rho_{\rm m} c^2$, the mass/energy density $\rho_{\rm m}$ includes the term $U_{\rm mech}/c^2$ where $U_{\rm mech}$ is the mechanical energy density associated with nonzero mechanical stresses, the density of mechanical momentum is $\mathbf{p}_{\rm mech}$, and $T^{ij}_{\rm mech}$ is the 3-dimensional mechanical stress tensor.

The capacitor plates are attracted to one another with force/area in the $z^\star$-direction of $E^{\star 2}/8\pi = \mathsf{T}^{\star zz}$. If the material between plates is constrained not to expand transversely, then in the rest frame of the isolated capacitor, the mechanical energy-momentum-stress tensor (in the region between the capacitor plates) has components,[4]

$$\mathsf{T}^{\star\mu\nu}_{\rm mech} = \left(\begin{array}{c|ccc} \rho^\star_{\rm m} c^2 & & \mathbf{0} & \\ \hline & \frac{\sigma}{1-\sigma}\frac{E^{\star 2}}{8\pi} & & \\ \mathbf{0} & & \frac{\sigma}{1-\sigma}\frac{E^{\star 2}}{8\pi} & \\ & & & \frac{E^{\star 2}}{8\pi} \end{array} \right), \tag{8}$$

where $-1 < \sigma \le 1/2$ is the so-called Poisson ratio of the medium. $\sigma = 1/2$ corresponds to a perfect fluid/ideal gas, which requires containing walls that also have a mechanical energy-momentum stress tensor. Here, we avoid this complication by supposing that $\sigma = 0$, such that no containing box is required.[5]

The total energy-momentum-stress tensor is the sum of the electromagnetic and mechanical tensors (2) and (7). In the rest frame of the capacitor the total energy-momentum-stress tensor has components,

$$\mathsf{T}^{\star\mu\nu}_{\rm total} = \mathsf{T}^{\star\mu\nu}_{\rm EM} + \mathsf{T}^{\star\mu\nu}_{\rm mech} = \left(\begin{array}{c|ccc} \rho^\star_{\rm m} c^2 + \frac{E^{\star 2}}{8\pi} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \frac{E^{\star 2}}{8\pi} & 0 & 0 \\ \mathbf{0} & 0 & \frac{E^{\star 2}}{8\pi} & 0 \\ \mathbf{0} & 0 & 0 & 0 \end{array} \right), \tag{9}$$

We interpret the component $\mathsf{T}^{\star 00}$ as implying the total mass density in the rest frame to be,

$$\rho^\star_{\rm total} = \rho^\star_{\rm m} + \frac{E^2}{8\pi c^2}\,, \tag{10}$$

where $\rho^\star_{\rm m}$ includes a contribution to the mechanical mass due to the mechanical stress.

2.1 The Capacitor Has Velocity $\mathbf{v} \parallel \mathbf{E}^\star$

In a frame where the capacitor has constant velocity $\mathbf{v} = v\,\hat{\mathbf{z}}$, the electric and magnetic fields between its plates are given by the transformation,

$$\mathbf{E}_\parallel = \mathbf{E}^\star_\parallel = E^\star\,\hat{\mathbf{z}}, \tag{11}$$

[4]See, for example, eq. (5.13), p. 14 of [14].

[5]The case of an ideal gas in a box is discussed in [8, 15].

$$
\begin{aligned}
\mathbf{E}_\perp &= \gamma(\mathbf{E}^\star_\perp - \frac{\mathbf{v}}{c} \times \mathbf{B}^\star) = 0, && (12)\\
\mathbf{B}_\parallel &= \mathbf{B}^\star_\parallel = 0, && (13)\\
\mathbf{B}_\perp &= \gamma(\mathbf{B}^\star_\perp + \frac{\mathbf{v}}{c} \times \mathbf{E}^\star) = 0, && (14)
\end{aligned}
$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$. That is, (ignoring fringe fields) the electromagnetic fields have the same values inside the capacitor in its rest frame and in frames where the capacitor moves with velocity $\mathbf{v}$ parallel to $\mathbf{E}$. Hence, the electromagnetic energy-momentum-stress tensor has the same component values in all such frames,

$$
\mathsf{T}^{\mu\nu}_{\rm EM} = \mathsf{T}^{\star\mu\nu}_{\rm EM}. \tag{15}
$$

It is useful to confirm this result via a Lorentz transformation of the stress tensor. The transformation L_z from the rest frame to a frame in which the capacitor has velocity $v\,\hat{\mathbf{z}}$ can be expressed in tensor form as,

$$
\mathsf{L}_z^{\mu\nu} = \left(\begin{array}{c|ccc} \gamma & 0 & 0 & \gamma\beta \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{array} \right), \tag{16}
$$

where $\beta = v/c$. Then, the transform of a tensor,

$$
\mathsf{T}^{\star\mu\nu} = \left(\begin{array}{c|ccc} \mathsf{T}^{\star 00} & 0 & 0 & 0 \\ \hline 0 & \mathsf{T}^{\star 11} & 0 & 0 \\ 0 & 0 & \mathsf{T}^{\star 22} & 0 \\ 0 & 0 & 0 & \mathsf{T}^{\star 33} \end{array} \right), \tag{17}
$$

that is diagonal in the rest frame is given by,

$$
\mathsf{T}^{\mu\nu} = (\mathsf{L}_z \mathsf{T}^\star \mathsf{L}_z)^{\mu\nu} = \left(\begin{array}{c|ccc} \gamma^2 \mathsf{T}^{\star 00} + \gamma^2\beta^2 \mathsf{T}^{\star 33} & 0 & 0 & \gamma^2\beta(\mathsf{T}^{\star 00} + \mathsf{T}^{\star 33}) \\ \hline 0 & \mathsf{T}^{\star 11} & 0 & 0 \\ 0 & 0 & \mathsf{T}^{\star 22} & 0 \\ \gamma^2\beta(\mathsf{T}^{\star 00} + \mathsf{T}^{\star 33}) & 0 & 0 & \gamma^2\beta^2 \mathsf{T}^{\star 00} + \gamma^2 \mathsf{T}^{\star 33} \end{array} \right). \tag{18}
$$

In particular, the transformation of $\mathsf{T}^{\star\mu\nu}_{\rm EM}$, eq. (6), is,

$$
\mathsf{T}^{\mu\nu}_{\rm EM} = \frac{E^{\star 2}}{8\pi} \left(\begin{array}{c|ccc} \gamma^2(1-\beta^2) & 0 & 0 & \gamma^2\beta(1-1) \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma^2\beta(1-1) & 0 & 0 & -\gamma^2(1-\beta^2) \end{array} \right) = \frac{E^{\star 2}}{8\pi} \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right) = \mathsf{T}^{\star\mu\nu}_{\rm EM}, \tag{19}
$$

as found above.

Similarly, the transformation of the mechanical stress tensor $\mathsf{T}^{\star\mu\nu}_{\rm mech}$, eq. (8), is, with $\sigma = 0$,

$$\mathsf{T}^{\mu\nu}_{\rm mech} = \left(\begin{array}{c|ccc} \gamma^2 \rho^\star_{\rm m} c^2 + \gamma^2\beta^2 \frac{E^{\star 2}}{8\pi} & 0 & 0 & \gamma^2\beta\left(\rho^\star_m c^2 + \frac{E^{\star 2}}{8\pi}\right) \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma^2\beta\left(\rho^\star_{\rm m} c^2 + \frac{E^{\star 2}}{8\pi}\right) & 0 & 0 & \gamma^2\beta^2\rho^\star_{\rm m} c^2 + \gamma^2 \frac{E^{\star 2}}{8\pi} \end{array} \right), \tag{20}$$

and the transformation of the total stress tensor $\mathsf{T}^{\star\mu\nu}_{\rm total}$, eq. (9), is,

$$\mathsf{T}^{\mu\nu}_{\rm total} = \left(\begin{array}{c|ccc} \gamma^2 \rho^\star_{\rm total} c^2 & 0 & 0 & \gamma^2\beta\rho^\star_{\rm total} c^2 \\ \hline 0 & \frac{E^{\star 2}}{8\pi} & 0 & 0 \\ 0 & 0 & \frac{E^{\star 2}}{8\pi} & 0 \\ \gamma^2\beta\rho^\star_{\rm total} c^2 & 0 & 0 & \gamma^2\beta^2\rho^\star_{\rm total} c^2 \end{array} \right) = \mathsf{T}^{\mu\nu}_{\rm EM} + \mathsf{T}^{\mu\nu}_{\rm mech}, \tag{21}$$

with the total mass density $\rho^\star_{\rm total}$ given by eq. (10).

One noteworthy feature of eq (21) is the nonzero value of $\mathsf{T}^{33}_{\rm total}$. We recall that the purely spatial components of a stress-energy-momentum tensor have the dual interpretation as the momentum-flux tensor. In the present case, the flux of momentum is in the z direction, with magnitude equal to the momentum density times v, namely $(\gamma\rho^\star_{\rm total} v) \cdot v = \gamma^2\beta^2\rho^\star_{\rm total} c^2 = \mathsf{T}^{33}_{\rm total}$.

Turning to the component $\mathsf{T}^{00}_{\rm total}$, we note that the mass of the material between the capacitor plates, when moving with velocity $\mathbf{v}$, is larger than its rest mass by the factor γ. However, a moving volume element is smaller by the factor $1/\gamma$ than when that element is at rest. Hence, the mass density $\rho_{\rm m}$ is larger by a factor of γ^2 for the moving capacitor than when at rest,

$$\rho_{\rm m} = \gamma^2 \rho^\star_{\rm m}. \tag{22}$$

Thus, the component $\mathsf{T}^{00}_{\rm total}$ transforms as expected for a mass density.

Furthermore, the four components,

$$(\mathsf{T}^{00}_{\rm total}, \mathsf{T}^{01}_{\rm total}, \mathsf{T}^{02}_{\rm total}, \mathsf{T}^{03}_{\rm total}) \tag{23}$$

of the total energy-momentum-stress tensor transform as an energy-momentum-density 4-vector, although this is not the case for the sets of components,

$$(\mathsf{T}^{00}_{\rm EM}, \mathsf{T}^{01}_{\rm EM}, \mathsf{T}^{02}_{\rm EM}, \mathsf{T}^{03}_{\rm EM}) \qquad \text{or} \qquad (\mathsf{T}^{00}_{\rm mech}, \mathsf{T}^{01}_{\rm mech}, \mathsf{T}^{02}_{\rm mech}, \mathsf{T}^{03}_{\rm mech}) \tag{24}$$

separately. This illustrates a general result that within volumes that contain both electromagnetic fields and matter, the concepts of electromagnetic momentum density, $\mathsf{T}^{0i}_{\rm EM}/c = \mathbf{E} \times \mathbf{B}/4\pi c$, and mechanical momentum density, $\mathsf{T}^{0i}_{\rm mech}/c$, are not consistent with being components of a energy-momentum 4-vector; only the total momentum density, $\mathsf{T}^{0i}_{\rm total}/c$, is satisfactory in this respect. The great utility of the concept of "electromagnetic" momentum in matter-free regions leads us to attach similar significance to it in systems containing matter. However, this often results in difficulties in the interpretation of the "mechanical" part of the momentum.

As noted in [5], $\mathsf{T}^{\star 0i}_{\rm EM} = 0$, so there is no flow of electromagnetic energy along with the moving capacitor when $\mathbf{v} \parallel \mathbf{E}$. However, the flow of mechanical energy density associated with the moving capacitor is $c\mathsf{T}^{\star 0i}_{\rm mech}$, and we see in eq.(20) that $c\mathsf{T}^{\star 03}_{\rm mech}$ has a term $\gamma^2 E^{\star 2}\mathbf{v}/8\pi = \mathsf{T}^{\star 00}_{\rm EM}\mathbf{v}$, which is the value perhaps naïvely expected for $c\mathsf{T}^{\star 0i}_{\rm EM}$. That is, the flow of energy in the moving capacitor is "mechanical," not "electromagnetic." The electromagnetic field energy inside the moving capacitor is at rest, while the "bottom" plate of the capacitor "sweeps up" this energy, converts it to mechanical energy that flows up to the "top" plate inside the stressed dielectric, where it is converted back into electromagnetic energy. This is an example of the relativity of steady energy flow [16, 17, 18].

For a capacitor moving with $\mathbf{v} \parallel \mathbf{E}$ the electromagnetic-field momentum is zero,

$$\mathbf{P}_{\rm EM} = \int \mathbf{p}_{\rm EM}\, d{\rm Vol} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} = 0, \tag{25}$$

in that $\mathsf{T}^{0i}_{\rm EM} = c\, p_{{\rm EM},i} = 0$, while the electromagnetic field energy is $U_{\rm EM} = \mathsf{T}^{00}_{\rm EM} V = E^{\star 2}V^\star/8\pi\gamma = U^\star_{\rm EM}/\gamma$, where $V = V^\star/\gamma$ is the Lorentz-contracted volume of the moving capacitor.[6] The effective mass of this energy is $M_{\rm EM} = E^{\star 2}V/8\pi c^2 = E^{\star 2}V^\star/8\pi\gamma c^2 = M^\star_{\rm EM}/\gamma$, and it moves with velocity $\mathbf{v}$, so that,

$$M_{\rm EM}\mathbf{v}_{\rm cm,EM} = \frac{E^{\star 2}V}{8\pi c^2}\mathbf{v}. \tag{26}$$

Thus, according to the definition (1) the electromagnetic field of the moving capacitor possesses "hidden" momentum,

$$\mathbf{P}_{\rm hidden,EM} = \mathbf{P}_{\rm EM} - M_{\rm EM}\mathbf{v}_{\rm cm,EM} = -\frac{E^{\star 2}V}{8\pi c^2}\mathbf{v}. \tag{27}$$

This momentum is "hidden" in the sense that the electromagnetic field has no momentum, but its center of mass/energy is in motion.

The mechanical momentum density is given from eq. (20) as,

$$\mathbf{p}_{\rm mech} = \gamma^2\left(\rho^\star_{\rm m} + \frac{E^{\star 2}}{8\pi c^2}\right)\mathbf{v} = \rho_{\rm total}\mathbf{v}. \tag{28}$$

The mechanical momentum is the momentum density times the volume V,

$$\mathbf{P}_{\rm mech} = \mathbf{p}_{\rm mech}V = \gamma^2 V\left(\rho^\star_{\rm m} + \frac{E^{\star 2}}{8\pi c^2}\right)\mathbf{v}. \tag{29}$$

[6](Dec. 8, 2020) Rohrlich [11] advocated an electromagnetic energy momentum 4-vector $P_{{\rm Rohrlich},\mu} = \gamma\, U^\star_{\rm EM}(1, \mathbf{v}/c) = (U_{\rm EM,Rorhlich}, c\,\mathbf{P}_{\rm EM,Rorhlich})$, *i.e.*, $U_{\rm EM,Rorhlich} = \gamma\, U^\star_{\rm EM}$ and $\mathbf{P}_{\rm EM,Rorhlich} = \gamma\, U^\star_{\rm EM}\mathbf{v}/c^2$. This formalism makes little physical sense to the present author.

Note also that $U_{\rm EM,Rorhlich} = \gamma^2\, U_{\rm EM}$; Rohrlich's eqs. (3.25) and (3.26) should have factors of γ^2 not γ.

The mechanical mass density is,

$$\boldsymbol{\rho}_{\rm mech} = \mathsf{T}^{00}_{\rm mech} = \gamma^2 \left(\rho^\star_{\rm m} + \beta^2 \frac{E^{\star 2}}{8\pi c^2} \right), \tag{30}$$

and this moves with velocity $\mathbf{v}$ such that,

$$M_{\rm mech}\mathbf{v}_{\rm cm,mech} = \boldsymbol{\rho}_{\rm mech} V \mathbf{v} = \gamma^2 V \left(\rho^\star_{\rm m} + \beta^2 \frac{E^{\star 2}}{8\pi c^2} \right). \tag{31}$$

According to definition (1) the matter of the moving capacitor possesses "hidden" momentum,

$$\mathbf{P}_{\rm hidden,mech} = \mathbf{P}_{\rm mech} - M_{\rm mech}\mathbf{v}_{\rm cm,mech} = \gamma^2 \left(1 - \beta^2\right) \frac{E^{\star 2} V}{8\pi c^2}\mathbf{v} = \frac{E^{\star 2} V}{8\pi c^2}\mathbf{v} = -\mathbf{P}_{\rm hidden,EM}. \tag{32}$$

This result reflects that the energy and momentum of stress in a moving subsystem do not transform like a 4-vector if that subsystem interacts with another subsystem (here, the electromagnetic field).

The total "hidden" momentum, $\mathbf{P}_{\rm hidden,mech} + \mathbf{P}_{\rm hidden,EM}$, of the system is zero.[7]

2.2 The Capacitor Has Velocity $\mathbf{v} \perp \mathbf{E}^\star$

In a frame where the capacitor has constant velocity $\mathbf{v} = v\,\hat{\mathbf{x}}$, the electric and magnetic fields between its plates are given by the transformation,

$$\begin{aligned}
\mathbf{E}_\parallel &= \mathbf{E}^\star_\parallel = 0, &(33)\\
\mathbf{E}_\perp &= \gamma(\mathbf{E}^\star_\perp - \frac{\mathbf{v}}{c} \times \mathbf{B}^\star) = \gamma E^\star\,\hat{\mathbf{z}}, &(34)\\
\mathbf{B}_\parallel &= \mathbf{B}^\star_\parallel = 0, &(35)\\
\mathbf{B}_\perp &= \gamma(\mathbf{B}^\star_\perp + \frac{\mathbf{v}}{c} \times \mathbf{E}^\star) = -\gamma\frac{v}{c}E^\star\,\hat{\mathbf{y}}. &(36)
\end{aligned}$$

Using eqs. (2)-(5) together with eqs. (33)-(36), the electromagnetic energy-momentum-stress tensor of the moving capacitor is,

$$\mathsf{T}^{\mu\nu}_{\rm EM} = \frac{E^{\star 2}}{8\pi} \left(\begin{array}{c|ccc} \gamma^2(1+\beta^2) & 2\gamma^2\beta & 0 & 0 \\ \hline 2\gamma^2\beta & \gamma^2(1+\beta^2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right). \tag{37}$$

[7] If we consider the system to consist of two subsystems, A = capacitor plates and charge thereon, B = dielectric + electromagnetic fields, then subsystem B has the same properties as the entire system considered above (where we neglected the mass/energy of the capacitor plates). Hence, subsystem B has zero "hidden" momentum.

We confirm this result using the Lorentz transformation L_x from the rest frame to a frame in which the capacitor has velocity $v\,\hat{\mathbf{x}}$,

$$\mathsf{L}_x^{\mu\nu} = \left(\begin{array}{c|ccc} \gamma & \gamma\beta & 0 & 0 \\ \hline \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right). \tag{38}$$

Then, the transform of a tensor (17) that is diagonal in the rest frame is given by,

$$\mathsf{T}^{\mu\nu} = (\mathsf{L}_x \mathsf{T}^\star \mathsf{L}_x)^{\mu\nu} = \left(\begin{array}{c|ccc} \gamma^2 \mathsf{T}^{\star 00} + \gamma^2\beta^2 \mathsf{T}^{\star 11} & \gamma^2\beta(\mathsf{T}^{\star 00} + \mathsf{T}^{\star 11}) & 0 & 0 \\ \hline \gamma^2\beta(\mathsf{T}^{\star 00} + \mathsf{T}^{\star 11}) & \gamma^2\beta^2 \mathsf{T}^{\star 00} + \gamma^2 \mathsf{T}^{\star 11} & 0 & 0 \\ 0 & 0 & \mathsf{T}^{\star 22} & 0 \\ 0 & 0 & 0 & \mathsf{T}^{\star 33} \end{array}\right), \tag{39}$$

In particular, the transformation of $\mathsf{T}^{\star\mu\nu}_{\mathrm{EM}}$, eq. (6), is,

$$\mathsf{T}^{\mu\nu}_{\mathrm{EM}} = \frac{E^{\star 2}}{8\pi} \left(\begin{array}{c|ccc} \gamma^2(1+\beta^2) & 2\gamma^2\beta & 0 & 0 \\ \hline 2\gamma^2\beta & \gamma^2(1+\beta^2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right), \tag{40}$$

as found above.

Similarly, the transformation of the mechanical stress tensor $\mathsf{T}^{\star\mu\nu}_{\mathrm{mech}}$, eq. (8), is, with $\sigma = 0$,

$$\mathsf{T}^{\mu\nu}_{\mathrm{mech}} = \left(\begin{array}{c|ccc} \gamma^2 \rho^\star_{\mathrm{m}} c^2 & \gamma^2\beta\rho^\star_{m} c^2 & 0 & 0 \\ \hline \gamma^2\beta\rho^\star_{\mathrm{m}} c^2 & \gamma^2\beta^2\rho^\star_{m} c^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{E^{\star 2}}{8\pi} \end{array}\right), \tag{41}$$

the transformation of the total stress tensor $\mathsf{T}^{\star\mu\nu}_{\mathrm{total}}$, eq. (9), is,

$$\mathsf{T}^{\mu\nu}_{\mathrm{total}} = \left(\begin{array}{c|ccc} \gamma^2\rho^\star_{\mathrm{total}} c^2 + \gamma^2\beta^2\frac{E^{\star 2}}{8\pi} & \gamma^2\beta\rho^\star_{\mathrm{total}} c^2 + +\gamma^2\beta\frac{E^{\star 2}}{8\pi} & 0 & 0 \\ \hline \gamma^2\beta\rho^\star_{\mathrm{total}} c^2 + \gamma^2\beta\frac{E^{\star 2}}{8\pi} & \gamma^2\beta^2\rho^\star_{\mathrm{total}} c^2 + \gamma^2\frac{E^{\star 2}}{8\pi} & 0 & 0 \\ 0 & 0 & \frac{E^{\star 2}}{8\pi} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) = \mathsf{T}^{\mu\nu}_{\mathrm{EM}} + \mathsf{T}^{\mu\nu}_{\mathrm{mech}}, \tag{42}$$

with the total mass density $\rho^\star_{\rm total}$ again given by eq. (10), $\rho^\star_{\rm total} = \rho^\star_m + \frac{E^{\star 2}}{8\pi c^2}$.[8]

For a capacitor moving with $\mathbf{v} \perp \mathbf{E}$ the electromagnetic-field momentum is found from from eq. (40) to be,

$$\mathbf{P}_{\rm EM} = \frac{\mathsf{T}^{01}_{\rm EM} V}{c}\,\hat{\mathbf{x}} = \frac{2\gamma^2 E^{\star 2} V}{8\pi c^2}\mathbf{v}, \tag{43}$$

and,

$$M_{\rm EM}\mathbf{v}_{\rm cm,EM} = \frac{\mathsf{T}^{00}_{\rm EM} V}{c^2}\mathbf{v} = \frac{\gamma^2(1+\beta^2) E^{\star 2} V}{8\pi c^2}\mathbf{v}. \tag{44}$$

According to definition (1) the electromagnetic field of the moving capacitor possesses "hidden" momentum,

$$\mathbf{P}_{\rm hidden,EM} = \mathbf{P}_{\rm EM} - M_{\rm EM}\mathbf{v}_{\rm cm,EM} = \frac{E^{\star 2} V}{8\pi c^2}\mathbf{v}. \tag{45}$$

The mechanical momentum is given from eq. (41) as,

$$\mathbf{P}_{\rm mech} = \frac{\mathsf{T}^{01}_{\rm mech} V}{c}\,\hat{\mathbf{x}} = \gamma^2 V \rho^\star_{\rm m}\,\mathbf{v}, \tag{46}$$

and,

$$M_{\rm mech}\mathbf{v}_{\rm cm,mech} = \frac{\mathsf{T}^{00}_{\rm mech} V}{c^2}\mathbf{v} = \gamma^2 V \rho^\star_{\rm m}\,\mathbf{v}. \tag{47}$$

According to definition (1) the matter of the moving capacitor possesses no "hidden" momentum,

$$\mathbf{P}_{\rm hidden,mech} = \mathbf{P}_{\rm mech} - M_{\rm mech}\mathbf{v}_{\rm cm,mech} = 0. \tag{48}$$

The total "hidden" momentum of the system is nonzero,

$$\mathbf{P}_{\rm hidden,total} = \mathbf{P}_{\rm hidden,EM} + \mathbf{P}_{\rm hidden,mech} = \mathbf{P}_{\rm hidden,EM} = \frac{E^{\star 2} V}{8\pi c^2}\mathbf{v}. \tag{49}$$

As noted on p. 1, the total "hidden" momentum of an isolated system is zero, so the present approximation of no fringe field of the capacitor leads to unphysical results for the case the $\mathbf{v} \perp \mathbf{E}^\star$.[9]

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[8]Element $\mathsf{T}^{01}_{\rm total}$ of eq. (42) shows that the flow of total energy between the capacitor plates is proportional to the velocity $v\,\hat{\mathbf{x}}$. However, elements $\mathsf{T}^{0\nu}_{\rm total}$ do not form an energy-momentum 4-vector, as is expected to exist for an isolated system [11]. We infer that there exists (electromagnetic) energy and momentum in the fringe field of the capacitor, such that inclusion of this in the stress tensor would permit the volume integrals of the energy and momentum densities to form a 4-vector. Furthermore, the flow of energy in the fringe field has a component in the $-\hat{\mathbf{x}}$ direction, opposite to the velocity of the capacitor, as discussed in [5]. Such counterintuitive behavior of energy flow is also encountered in examples such as the belt drive considered by Taylor and Wheeler [16, 17]; see also [18].

[9]If we had followed [8, 15] in supposing that the medium between the capacitor plates is a gas, then $\mathsf{T}^{\star 11}_{\rm mech} = \mathsf{T}^{\star 22}_{\rm mech} = E^{\star 2}/8\pi$ rather than zero as assumed in this note. This change has no effect on the results of sec. 2.1 above, but in sec. 2.2 it would alter eqs. (46)-(48) such that $\mathbf{P}_{\rm hidden,mech} = \mathbf{P}_{\rm hidden,EM}$, and the total "hidden" momentum of the moving system would again be nonzero, but with a value twice that than found in eq. (49). However, this conclusion ignores the stress-energy-momentum in the box that contains the gas.

2.3 Use of a Vector Potential (July 6, 2023)

We recall that for quasistatic systems, the electromagnetic field momentum can be computed several ways [20],

$$\mathbf{P}_{\rm EM} = \int \frac{\varrho \mathbf{A}^{(\rm C)}}{c}\, d{\rm Vol} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} = \int \frac{V^{(\rm C)}\mathbf{J}}{c^2}\, d{\rm Vol} = \int \frac{\mathbf{J}\cdot\mathbf{E}}{c^2}\mathbf{r}\, d{\rm Vol}, \tag{50}$$

where the potentials $\mathbf{A}^{(\rm C)}$ and $V^{(\rm C)}$ are in the Coulomb gauge.

The electric scalar potential V and the vector potential $\mathbf{A}$ form a 4-vector $A^\mu = (V, \mathbf{A})$, and the vector potential is zero in the rest frame of the capacitor. Hence, the (coulomb-gauge) vector potential in a frame where the capacitor have velocity $\mathbf{v}$ is simply $\mathbf{A}^{(\rm C)} = \gamma \mathbf{v} V^{\star(\rm C)}/c$, where $V^{\star(\rm C)} = \pm E^\star d^\star/2$ on the capacitor plates whose separation is $d^\star$ in the rest frame, and defining the potential to be zero on the midplane of the capacitor.

Is the present example "quasistatic", such that the electromagnetic field momentum can be computed via the first form of eq. (50)?

The plates of the capacitor have electric charge $\pm Q = \pm E^\star \text{Area}^\star/4\pi$ in the approximation of uniform electric field between the plates, and zero fringe field. Then, the electromagnetic field momentum according to the first form of eq. (50) would be,

$$\begin{aligned}\mathbf{P}_{\rm EM} = \int \frac{\varrho \mathbf{A}^{(\rm C)}}{c}\, d{\rm Vol} = \frac{\gamma \mathbf{v}}{c^2}\left(QE^*\frac{d^\star}{2} + (-Q)E^*\frac{-d^\star}{2}\right) &= \frac{\gamma \mathbf{v} Q E^* d^\star}{c^2} \\ = \frac{\gamma \mathbf{v} E^{*2}\text{Area}^\star d^\star}{4\pi c^2} &= \frac{\gamma^2 \mathbf{v} E^{*2} V}{4\pi c^2}\,,\end{aligned} \tag{51}$$

for any direction of the velocity $\mathbf{v}$, recalling that the volume of the moving capacitor is $V = V^\star/\gamma$.

While eq. (51) agrees with eq. (43) for $\mathbf{v} \perp \mathbf{E}$, it disagrees with eq. (25) for $\mathbf{v} \parallel \mathbf{E}$ where $\mathbf{P}_{\rm EM} = 0$ (according to eq. (25), which we consider to be the correct expression for the electromagnetic momentum in general). Thus, we should not describe the moving capacitor as a "quasistatic" system, for which all four forms of eq. (50) would be give the same the electromagnetic momentum.

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