Čerenkov Radiation from a Short Path

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1 Problem

Discuss the character of Čerenkov radiation emitted by a charge q that traverses a short length L at velocity $v > c/n$ in a medium of index of refraction $n > 1$, where c is the speed of light in vacuum. Of course, $v < c$ for a physical charge that also has mass.

In the idealized case of infinite L the radiation flows at a single angle [1],

$$\cos\theta_C = \frac{c}{nv}, \qquad \sin\theta_C = \sqrt{1 - \frac{c^2}{n^2 v^2}}, \tag{1}$$

and the flux of energy along a "ray" falls off only as $1/r$ from its origin on the particle's trajectory. Comment on the falloff of the flux of energy with distance when the path length L is finite.

2 Solution

For a "shortcut" solution based on the Weizsäcker-Williams approximation, see sec. 5 of [2].

We imagine that the charge travels along the z-axis inside a vacuum pipe of small diameter, and this pipe has a gap $-L/2 < z < L/2$. Outside the pipe (including the gap region) the medium has index of refraction $n > 1$.

This case was considered by Tamm in 1939 [3], who noted that even if the index is 1 there is a radiation effect associated with the particle's exit from and re-entrance into the vacuum pipe. This effect is now called transition radiation, which is not the topic of the present note. In this solution, which follows Tamm, we will simply ignore the transition radiation.

In physical media, the index n is frequency dependent, and exceeds unity only over a finite ranges of frequencies. This limits the total energy emitted per unit path length to a finite amount, as is physically reasonable. Hence, it is both mathematically convenient and physically preferable to consider a Fourier analysis of the present problem.

2.1 Potentials and Fields

We find the electromagnetic fields from the retarded potentials of Lorenz [4] (in Gaussian units),

$$V(\mathbf{x}, t) = \frac{1}{\epsilon} \int \frac{\rho(\mathbf{x}', t' = t - nR/c)}{R}\, d^3\mathbf{x}', \qquad \mathbf{A}(\mathbf{x}, t) = \frac{\mu}{c} \int \frac{\mathbf{J}(\mathbf{x}', t' = t - nR/c)}{R}\, d^3\mathbf{x}', \tag{2}$$

for a medium of (relative) permittivity ϵ, (relative) permeability μ, and index of refraction $n = \sqrt{\epsilon\mu}$, where $R = |\mathbf{x} - \mathbf{x}'|$. We consider time dependence at angular frequency ω of the form $e^{-i\omega t}$, so temporal Fourier transforms have the form,

$$\rho(\mathbf{x},t) = \int \rho_\omega(\mathbf{x})\, e^{-i\omega t}\, d\omega, \qquad \rho_\omega(\mathbf{x}) = \frac{1}{2\pi}\int \rho(\mathbf{x},t)\, e^{i\omega t}\, dt, \ \ etc. \tag{3}$$

The Fourier transforms of the retarded potentials (2) are,

$$V_\omega(\mathbf{x}) = \frac{1}{\epsilon}\int \frac{\rho_\omega(\mathbf{x}')\, e^{ikR}}{R}\, d^3\mathbf{x}', \qquad \mathbf{A}_\omega(\mathbf{x}) = \frac{\mu}{c}\int \frac{\mathbf{J}_\omega(\mathbf{x}')\, e^{ikR}}{R}\, d^3\mathbf{x}', \tag{4}$$

where $k = n\omega/c$ is the (frequency-dependent) wave number.

The charge and current density, ρ and $\mathbf{J}$ of the charge when it is in the gap can be written as,

$$\rho = q\delta(x)\delta(y)\delta(z - vt), \qquad \mathbf{J} = \rho\mathbf{v} = \rho v\,\hat{\mathbf{z}} \qquad \left(-\frac{L}{2} < z < \frac{L}{2}\,,\ -\frac{L}{2v} < t < \frac{L}{2v}\right), \tag{5}$$

and zero otherwise. Their Fourier transforms are,

$$\rho_\omega = \frac{q\delta(x)\delta(y)\, e^{i\omega z/v}}{2\pi v}\,, \qquad \mathbf{J}_\omega = \rho_\omega v\,\hat{\mathbf{z}} \qquad \left(-\frac{L}{2} < z < \frac{L}{2}\right), \tag{6}$$

and zero otherwise.

We consider only a distant observer at $(r \gg L, \theta, \phi)$ in a spherical coordinate system whose polar axis is the z-axis. Then, we approximate the distance R in the denominators of eqs. (4) by r, while in the numerator we approximate $R \approx r - z\cos\theta$ for the charge along the z-axis in the gap. The retarded potentials are now approximated at large distances as,

$$V_\omega(\mathbf{x}) \approx \frac{q}{2\pi\epsilon r v}\int_{-L/2}^{L/2} e^{i\omega z'/v}\, e^{ik(r - z'\cos\theta)}\, dz' = \frac{qL\, e^{ikr}}{2\pi\epsilon r v}\frac{\sin u}{u}\,, \tag{7}$$

$$\mathbf{A}_\omega \approx \frac{\epsilon\mu v}{c}V_\omega\,\hat{\mathbf{z}} = \frac{\mu q L\, e^{ikr}}{2\pi c r}\frac{\sin u}{u}(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}), \tag{8}$$

where,

$$u = \frac{\omega L}{2v}\left(1 - \frac{nv}{c}\cos\theta\right) = \frac{\omega L}{2v}\left(1 - \frac{\cos\theta}{\cos\theta_C}\right). \tag{9}$$

The function $\sin u/u$ has a full width at half maximum of about 4.

For L large compared to a wavelength λ, we expect that significant radiation is detected by the observer only along direction θ_C, where the Čerenkov angle is given in eq. (1). In the case of a finite source length L, radiation will be observed only at angles θ close to θ_C. So, we write $\delta\theta = \theta - \theta_C$, and,

$$\cos\theta = \cos(\theta_C + \delta\theta) \approx \cos\theta_C(1 - \tan\theta_C\,\delta\theta). \tag{10}$$

Then,

$$u \approx \frac{\omega L\tan\theta_C}{2v}\,\delta\theta = \frac{n\omega L\sin\theta_C}{2c}\,\delta\theta = \frac{\pi L\sin\theta_C}{\lambda}\,\delta\theta. \tag{11}$$

The function $\sin u/u$ has a full width at half maximum of about 4, so the radiation has an angular spread about the Čerenkov angle of $\pm\delta\theta$ where,

$$\delta\theta \approx \frac{2\lambda}{\pi L \sin\theta_C} \,. \tag{12}$$

The Fourier components of the electric and magnetic fields can now be calculated from the potentials as,

$$\mathbf{E}_\omega = -\boldsymbol{\nabla} V_\omega + \frac{i\omega}{c}\mathbf{A}_\omega, \qquad \mathbf{B}_\omega = \boldsymbol{\nabla} \times \mathbf{A}_\omega. \tag{13}$$

The various terms in the fields fall of with distance as $1/r$ or faster, and the terms that fall off as $1/r$ are,

$$E_{\omega,\theta}(r,\theta) = \frac{i\omega}{c}A_{\omega,\theta} \approx -\frac{i\mu q L\omega \sin\theta_C}{\pi c^2}\frac{e^{ikr}}{r}\frac{\sin u(\omega,\theta)}{u(\omega,\theta)}\,, \qquad B_{\omega,\phi} = \frac{1}{r}\frac{\partial}{\partial r}(rA_{\omega,\theta}) \approx nE_{\omega,\theta}. \tag{14}$$

These fields are symmetric in ϕ and large only for θ within $\pm\delta\theta \approx \pm 2\lambda/\pi L \sin\theta_C$ of the Čerenkov angle θ_C.

Since the fields fall off with distance as $1/r$, the energy flux falls off as $1/r^2$.

It is noteworthy that as the path length L grows large the angular spread of the radiation in θ goes to zero, but the leading terms in the fields always fall off as $1/r$. The mathematical case of an infinite path length with fields that fall off as $1/\sqrt{r}$ is not a limit of physically possible cases of large but finite L.[1]

2.2 Energy Radiated to "Infinity"

As usual for nonmonochromatic fields, a Fourier analysis of the Poynting vector cannot be made as $(c/4\pi)\mathbf{E}_\omega \times \mathbf{H}_\omega$. Rather, we note that the entire energy U radiated to "infinity" can be calculated from the leading terms in the fields at large r according to,

$$U = \int dt \int d\Omega\, r^2 S_{r,\text{leading}}(r,t) = 2\pi r^2 \int dt \int \sin\theta\, d\theta\, \frac{c}{4\pi n} E_\theta(r,t) H_\phi(r,t) \tag{15}$$

$$\begin{aligned}
&= \frac{cr^2}{2}\int dt \int \sin\theta\, d\theta\, Re\left(\int E_{\omega,\theta}\, e^{-i\omega t}\, d\omega\right) Re\left(\int \frac{B_{\omega',\theta}}{\mu'}\, e^{-i\omega' t}\, d\omega'\right) \\
&\approx \frac{q^2L^2}{2\pi^2c^3}\int \sin\theta\, d\theta \int d\omega\, \frac{\mu\,\omega \sin\theta_C \sin u}{u} \int d\omega'\, \frac{n'\omega' \sin\theta'_C \sin u'}{u'} \\
&\qquad \int dt\, \sin(kr-\omega t)\sin(k'r-\omega' t) \qquad\qquad (16) \\
&= \frac{q^2L^2}{2\pi^2c^3}\int \sin\theta\, d\theta \int d\omega\, \frac{\mu\,\omega \sin\theta_C \sin u}{u} \int d\omega'\, \frac{n'\omega' \sin\theta'_C \sin u'}{u'}\, \pi\delta(\omega-\omega') \\
&= \frac{q^2L^2}{2\pi c^3}\int d\omega\, \mu n\omega^2 \sin^2\theta_C \int \sin\theta\, d\theta\, \frac{\sin^2 u}{u^2}
\end{aligned}$$

[1]A variation of Čerenkov radiation with a finite path length is a charged particle that moves in a circle with uniform speed $v > c/n$. As discussed on p. 4 of [5], the fields here fall off as $1/r$ at large r, despite claims to the contrary [6].

$$
\begin{aligned}
&\approx \frac{q^2 L}{\pi c^2} \int d\omega\, \mu\, \omega \sin^2 \theta_C \int du\, \frac{\sin^2 u}{u^2} \\
&= \frac{q^2 L}{c^2} \int d\omega\, \mu\, \omega \left(1 - \frac{c^2}{n^2 v^2}\right),
\end{aligned} \tag{17}
$$

where the final integral is evaluated only for those frequencies at which $n(\omega)c/v > 1$. The total energy U that is radiated to "infinity" is finite because the condition $n(\omega)c/v$ is satisfied only for a finite range of frequencies.

The result can also be written as the energy radiated per unit path length and per unit frequency interval,

$$
\frac{dU}{d\omega\, dL} \approx \frac{\mu q^2 \omega}{c^2} \sin^2 \theta_c(\omega). \tag{18}
$$

3 $v > c$ in Vacuum

A physical charge cannot travel with speed v greater than c. However, the current density $\mathbf{J}$ need not be that of a single charge moving with velocity $\mathbf{v}$, as considered in sec. 2. If the current density is due to a collection of charges it can be arranged that while the speed of every charge is less than c the charge distribution has features that move with a velocity larger than c.[2]

Such a superluminal current density can exist in vacuum. Then, we can carry over the analysis of sec. 2 by setting ϵ, μ and the index n to unity. We see immediately that for a source current density that is confined to a finite spatial region the fields fall off as $1/r$ at large distance, and the radiated energy density falls off as $1/r^2$. This is true for current densities that follow a curved path, as well as those that follow a straight path.[3]

In vacuum, the condition that the source distribution has an effective velocity greater than c in independent of frequency, so if all the analysis of sec. 2 applied here the total radiated energy per unit path length would be infinite. However, the superluminal current density must be constructed from a collection of charges whose distribution has some characteristic length scale a, rather than from a single "point" charge as considered in sec. 2. Then, pointlike Čerenkov radiation is observed only for frequencies up to $\omega \approx c/a$, and only a finite energy is radiated per unit path length.

References

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http://kirkmcd.princeton.edu/examples/EM/cerenkov_pr_52_378_37.pdf

[2] M.S. Zolotorev and K.T. McDonald, *Classical Radiation Processes in the Weizsacker-Williams Approximation* (Aug. 25, 1999), http://kirkmcd.princeton.edu/examples/weizsacker.pdf

[2]One such example is the swept electron beam in an oscilloscope, when the beam spot on its face moves faster than the speed of light such that the transition radiation emitted by the beam as it enters the face takes on much of the character of Čerenkov radiation [7].

[3]Thus, the discussion of Ardavan *et al.* [6] is misguided.

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