Charged, Counter-Rotating Disks on a Rotating Platform

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1 Problem

A possibly surprising feature of electrodynamics in a frame that rotates with angular velocity $\boldsymbol{\omega}$ with respect to an inertial (laboratory) frame [1] is that a bulk magnetization $\mathbf{M}'$ in the rotating frame is associated with a bound charge density,

$$\rho'_{\mathrm{bound},M} = -\frac{2\boldsymbol{\omega}\cdot\mathbf{M}'}{c} + \frac{\mathbf{v}}{c}\cdot\boldsymbol{\nabla}'\times\mathbf{M}', \tag{1}$$

in Gaussian units, where c is the speed of light, $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{x}'$ is the velocity of the point of observation $\mathbf{x}'$ with respect to the lab frame, and a $'$ indicates a quantity measured in the rotating frame.

As a model of a neutral, nonconducting magnetized medium (with electrical permittivity $\epsilon = 1$), consider a pair of counter-rotating disks of radii a with common axes of rotation, and with fixed charges $\pm Q$ uniformly distributed around their circumferences, which charges rotate with tangential velocities $\pm\mathbf{u}$. The charges lie within tori of minor area A, so that the charge densities of the two disks (when at rest in an inertial frame) are $\pm\rho_0 = \pm Q/2\pi aA$.

Deduce the total charge density $\rho' = \rho'^{+} + \rho'^{-}$ in the rotating frame, and relate this to the magnetization $\mathbf{M}'$ associated with a uniform bulk distribution of pairs of such counter-rotating disks.

2 Solution

2.1 The Centers of the Rotating Disks are at Rest in an Inertial Frame

We first consider the case that the centers of the disks are at rest in an inertial frame. Quantities observed in this frame will be denoted with a superscript $\star$.

We can deduce the charge and current densities in the inertial rest frame of the centers of the disks with the help of comoving inertial frames chosen so that the points of observation on the circumferences of the rotating disks are instantaneously at rest in the comoving frame.

In the comoving rest frame, the local charge density on the two disks is $\pm\rho_0$, and the local current density is zero.

In the rest frame of the centers of the disks, charges $\pm q$ have velocities $\pm\mathbf{u}$, so the charge and current densities follow from a Lorentz transformation as,

$$\rho^{\star\pm} = \pm\gamma_u\rho_0 \approx \pm\rho_0 \qquad \mathbf{J}^{\star\pm} = \gamma_u(\pm\rho_0)(\pm\mathbf{u}) = \gamma_u\rho_0\mathbf{u} \approx \rho_0\mathbf{u}, \tag{2}$$

where $\gamma_u = 1/\sqrt{1 - u^2/c^2} \approx 1$ and the approximations hold for $u \ll c$ as is assumed in this problem.[1]

The total charge and current density in the inertial rest frame of the centers of the disks is,

$$\rho^\star_{\rm total} = \rho^{\star +} + \rho^{\star -} = 0, \qquad \mathbf{J}^\star_{\rm total} = \mathbf{J}^{\star +} + \mathbf{J}^{\star -} = 2\gamma_u \rho_0 \mathbf{u} \approx 2\rho_0 \mathbf{u} \equiv J_0\, \hat{\mathbf{u}} \equiv \mathbf{J}_0. \tag{3}$$

The total current associated with the counter-rotating disks is $I^\star = J^\star_{\rm total} A^\star = J_0 A$, and their combined magnetic moment has magnitude $\mu^\star = \pi a^{\star 2} I^\star/c = \pi a^2 J_0 A/c$. In a medium composed of a uniform distribution of pairs of counter-rotating disks (with axes of rotation parallel to the z-axis), where each pair occupies a volume $V^\star$, the average bulk magnetization is,

$$\mathbf{M}^\star_{\rm ave} = \frac{\mu^\star}{V^\star}\,\hat{\mathbf{z}} = \frac{\pi a^2 J_0 A}{c V^\star}\,\hat{\mathbf{z}}. \tag{4}$$

2.2 The Centers of the Disks Have Uniform Velocity v with Respect to the Lab Frame

In this case it is simplest to transform the charge and current densities from the inertial * frame (in which the centers of the rotating disks are at rest) to the inertial lab frame via a Lorentz transformation,

$$\rho = \gamma_v \left(\rho^\star + \frac{\mathbf{v}\cdot\mathbf{J}^\star_{\rm total}}{c^2}\right) \approx \frac{\mathbf{v}\cdot\mathbf{J}_0}{c^2} \qquad \mathbf{J} = \gamma_v(\mathbf{J}^\star_{\rm total} + \rho^\star \mathbf{v}) \approx \mathbf{J}_0, \tag{5}$$

where for $v \ll c$ we make the approximation $\gamma_v \approx 1$. The total charge density ρ is nonzero in the lab frame, and varies with azimuth around the disks.

We can confirm eq. (5) by transformations between the lab frame and the comoving inertial frames of points on the disks, noting that a positive charge has velocity $\mathbf{v} + \mathbf{u}$ *with respect to the lab frame, while a negative charge has velocity* $\mathbf{v} - \mathbf{u}$. *Then,*

$$\rho^+ = \gamma_{\mathbf{v}+\mathbf{u}}\rho_0, \qquad \rho^- = -\gamma_{\mathbf{v}-\mathbf{u}}\rho_0, \qquad \rho = \rho_0(\gamma_{\mathbf{v}+\mathbf{u}} - \gamma_{\mathbf{v}-\mathbf{u}}) \approx 2\rho_0 \frac{\mathbf{u}\cdot\mathbf{v}}{c^2} = \frac{\mathbf{v}\cdot\mathbf{J}_0}{c^2}\,, \tag{6}$$

where we must expand $\gamma_{\mathbf{v}\pm\mathbf{u}}$ *to second order to maintain sufficient accuracy.*

If the velocity $\mathbf{v}$ has a component in the plane of the counter-rotating disks they appear to have an electric dipole moment in the lab frame (while remaining neutral overall). For the case that $\mathbf{v} = v\,\hat{\mathbf{x}}$ (and $v \ll c$) we write the position vector of a point on the rim of a disk in the lab frame as $\mathbf{x} \approx (vt + a\cos\phi)\,\hat{\mathbf{x}} + a\sin\phi\,\hat{\mathbf{y}}$ where ϕ is the angle between the x-axis and radius vector $\mathbf{a}$ on the disks. Then, the electric dipole moment is,

$$\begin{aligned} \mathbf{p} &= \int \rho\,\mathbf{x}\,d{\rm Vol} \approx \int_0^{2\pi} \frac{-J_0 v\sin\phi}{c^2}[(vt + a\cos\phi)\,\hat{\mathbf{x}} + a\sin\phi\,\hat{\mathbf{y}}]\,aA\,d\phi \\ &= -\frac{\pi a^2 A J_0 v}{c^2}\,\hat{\mathbf{y}} = \frac{\mathbf{v}}{c}\times \mathbf{M}^\star_{\rm ave} V^\star. \end{aligned} \tag{7}$$

[1]We tacitly assume that the rotating disks are rigid, which assumption is consistent only for $u \ll c$ [2]. Then, we avoid Ehrenfest's paradox that the total charge on a disk might appear to be different in the lab frame and in the rotating frame of the disk.

For a medium consisting of many pairs of counter-rotating disks, the average bulk electric polarization in the rotating frame is,

$$\mathbf{P}_{\rm mag} = \frac{\mathbf{p}}{V} \approx \frac{\mathbf{v}}{c} \times \mathbf{M}^\star_{\rm ave}, \tag{8}$$

noting that to order v/c the volumes are related by $V^\star \approx V =$. The polarization density (8) is not related to the (relative) permittivity ϵ of the medium, which has been taken to be unity. It is related to the magnetization of the medium (whether or not the medium can be characterized by a permeability μ), as indicated by our use of the subscript $_{\rm mag}$.

The relation (7) is the low-velocity limit of the well-known result,

$$\mathbf{P} = \gamma_v \left(\mathbf{P}^\star + \frac{\mathbf{v}}{c} \times \mathbf{M}^\star \right), \tag{9}$$

of special relativity (see, for example, chap. E III of [3] or sec. 18.6 of [4]), where $\mathbf{P}^\star$ is the electric polarization density in the rest frame of the medium.[2]

2.3 The Centers of the Disks Rotate with Angular Velocity $\boldsymbol{\omega} = \omega\,\hat{\mathbf{z}}$ with Respect to the Lab Frame

Suppose the centers of the counter-rotating disks are at $\mathbf{x}' = (r', 0, 0)$ in the rotating frame. The velocity of the centers of the disks with respect to the lab frame is then $\mathbf{v}_{\rm cen} = \boldsymbol{\omega} \times \mathbf{x}' = (0, \omega r', 0)$.

We can use the instantaneous comoving frame with velocity $\mathbf{v}_{\rm cen}$ and the results of sec. 2.2 to learn that the local charge and current densities in the lab frame are,

$$\rho \approx \frac{\mathbf{J}_0 \cdot \mathbf{v}_{\rm cen}}{c^2}, \qquad \mathbf{J} \approx \mathbf{J}_0, \tag{10}$$

where $\mathbf{J}_0 = 2\rho_0 \mathbf{u}$ is the current density associated with the counter-rotating disks in an inertial frame where the centers of the disks are at rest (the * frame of sec. 2.1).

According to the transformations from the lab to the rotating frame [1], the local charge density and current densities in the rotating frame are,

$$\rho' = \rho \approx \frac{\mathbf{J}_0 \cdot \mathbf{v}_{\rm cen}}{c^2}, \qquad \mathbf{J}' = \mathbf{J}_0 - \rho\mathbf{v} = \mathbf{J}_0 - \frac{\mathbf{J}_0 \cdot \mathbf{v}_{\rm cen}}{c^2}\,\mathbf{v} \approx \mathbf{J}_0, \tag{11}$$

where $\mathbf{v}$ is the velocity with respect to the lab frame of the point of observation in the rotating frame, so that $v \approx v_{\rm cen}$ and we neglect the term in $v\,v_{\rm cen}/c^2$ as being second order in v/c.

Thus, the counter-rotating disks appear to have an electric dipole moment in the rotating frame as well as in the lab frame (while remaining neutral overall). Writing the position

[2] The fact that a moving magnetization is associated with an electric polarization is often stated to be a result of special relativity, with the implication that this result was not anticipated by prerelativistic electrodynamics. However, Maxwell's early vision of the nature of electromagnetic media placed emphasis on "molecular vortices" from which his notions of electric charge and polarization were derived as secondary concepts. For a historical review, see [5].

vector of a point on the rim of a disk as $\mathbf{x}' = (r' + a\cos\phi)\,\hat{\mathbf{x}}' + a\sin\phi\,\hat{\mathbf{y}}'$ where ϕ is the angle between the x'-axis and radius vector $\mathbf{a}$ on the disks, the electric dipole moment is,

$$\begin{aligned}\mathbf{p}' &= \int \rho'\,\mathbf{x}'\,d\mathrm{Vol}' \approx \int_0^{2\pi} \frac{J_0\omega r'\cos\phi}{c^2}[(r' + a\cos\phi)\,\hat{\mathbf{x}}' + a\sin\phi\,\hat{\mathbf{y}}']\,aA\,d\phi \\ &= \frac{\pi a^2 A J_0 \omega r'}{c^2}\,\hat{\mathbf{x}}' = \frac{\boldsymbol{\omega}\cdot\mathbf{M}^\star_{\mathrm{ave}}}{c} r' V^\star\,\hat{\mathbf{x}}'. \end{aligned} \tag{12}$$

For a medium consisting of many pairs of counter-rotating disks, the average bulk electric polarization in the rotating frame is,

$$\mathbf{P}'_{\mathrm{mag}} = \frac{\mathbf{p}'}{V'} \approx \frac{\boldsymbol{\omega}\cdot\mathbf{M}^\star_{\mathrm{ave}}}{c}\,r'\,\hat{\mathbf{r}}' = \frac{\boldsymbol{\omega}\cdot\mathbf{M}'_{\mathrm{ave}}}{c}\,r'\,\hat{\mathbf{r}}', \tag{13}$$

noting that the electric dipole moment of a pair of counter-rotating disks is in the radial direction with respect to the center of the rotating frame, that the volumes are related by $V^\star \approx V = V'$, and that the magnetizations are related by $\mathbf{M}^\star \approx \mathbf{M} = \mathbf{M}'$ [1]. The polarization density (13) is not related to the permittivity ϵ of the medium, which has been taken to be unity. It is related to the magnetization of the medium (whether or not the medium can be characterized by a permeability μ), as indicated by our use of the subscript $_{\mathrm{mag}}$.

Associated with the bulk polarization density $\mathbf{P}'_{\mathrm{mag}}$ is an average bulk charge density,

$$\rho'_{\mathrm{ave}} = -\boldsymbol{\nabla}'\cdot\mathbf{P}'_{\mathrm{mag}} = -\frac{1}{r'}\frac{\partial(r' P'_{\mathrm{mag},r})}{\partial r'} = -\frac{2\boldsymbol{\omega}\cdot\mathbf{M}'_{\mathrm{ave}}}{c}\,. \tag{14}$$

This is exactly the form of the bound charge density in the rotating frame given in eq. (1) in case of a uniform magnetization $\mathbf{M}'$.

The present derivation of eq. (14) was based on the use of comoving inertial frames to deduce the charge density ρ in the lab frame, followed by the transformation $\rho' = \rho$ from the lab frame to the rotating frame. In contrast, eq. (1) was deduced via the principle of general covariance [1]. The latter technique is very powerful, but leads to forms in noninertial frames that may be "new" or surprising from the perspective of inertial observers. Here, we have shown how one of these possibly surprising noninertial effects is consistent with a sequence of arguments that may be appealing to inertial observers. That is, the charge density $-2\boldsymbol{\omega}\cdot\mathbf{M}'/c$ that appears in a rotating magnetized medium is in effect the density $-\boldsymbol{\nabla}'\cdot\mathbf{P}'_{\mathrm{mag}}$ associated with an electric polarization density $\mathbf{P}'_{\mathrm{mag}}$ that arises when electric currents circulate in the rotating frame. In addition, there may be an "ordinary" electric polarization $\mathbf{P}'_{\mathrm{el}}$ associated with deformations of molecules induced by the electric field in the rotating frame, with which there is associated a bulk charge density $-\boldsymbol{\nabla}'\cdot\mathbf{P}'_{\mathrm{el}}$.

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