Potentials for a Cylindrical Electromagnetic Cavity

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1 Problem

Deduce scalar and vector potentials relevant to the lowest electromagnetic mode in a right circular cylindrical cavity of radius R and axial extend $2D$, assuming the walls to be perfect conductors, and the interior of the cavity to be vacuum.

2 Solution

For the case of a rectangular cavity, see [1].

2.1 E and B Fields of the Cavity Modes

In cylindrical coordinates (r, ϕ, z) with the z-axis being that of the cavity, only E_z and B_ϕ are nonzero, such that for time dependence $e^{-i\omega t}$ we have (in Gaussian units; see, for example, sec. 8.7 of [2]),

$$E_z = E_0\, J_0(kr)\, e^{-i\omega t}, \tag{1}$$

$$B_\phi = -iE_0\, J_1(kr)\, e^{-i\omega t}, \tag{2}$$

where the resonant frequency is,

$$\omega = kc = \frac{2.405c}{R}, \tag{3}$$

such that $J_0(kR) = 0$ so the tangential electric field is zero at $r = R$, and c is the speed of light in vacuum. The magnetic field is related to the electric field by Faraday's law,

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial ct} = ik\mathbf{B}. \tag{4}$$

2.2 Potentials

The electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ can be related to scalar and vector potentials V and $\mathbf{A}$ according to,

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\boldsymbol{\nabla} V + ik\mathbf{A}, \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{5}$$

In sec. 2.1 we deduced the electromagnetic fields inside the cavity, but did not comment on their values outside it. While it seems most reasonable to consider the case that **E** and **B** are zero outside the cavity, we can also imagine the case of periodic boundary conditions, in which the space at $r < R$ space is filled with an infinite collection of cavities similar to the specified one.

2.2.1 Hamiltonian Gauge

A simple option for the potentials is to adopt the so-called Hamiltonian gauge in which the scalar potential is everywhere zero (see, for example, sec. 8 of [3]),[1,2]

$$V^{(\mathrm{H})} = 0, \qquad \mathbf{A}^{(\mathrm{H})} = -\frac{i\mathbf{E}}{k} = \begin{cases} -\frac{iE_0}{k}\, J_0(kr)\, e^{-i\omega t}\, \hat{\mathbf{z}} & \text{(inside)}, \\ 0 & \text{(outside)}. \end{cases} \tag{6}$$

Then, $\boldsymbol{\nabla} \times \mathbf{A}^{(\mathrm{H})} = \mathbf{B}$ is confirmed by use of Faraday's law, eq. (4), and also by noting that $dJ_0(kr)/dr = -kJ_1(kr)$. Clearly, the form $\mathbf{A}^{(\mathrm{H})} = -i\mathbf{E}/k$ holds for all other electromagnetic modes of the cavity.

This vector potential is not continuous on the planar faces of the cavity. However, this is not a formal problem in that the computation $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ next to the surface does not involve derivatives normal to that surface.[3]

2.2.2 Poincaré Gauge

In cases where the **E** and **B** fields are known, we can compute the potentials in the so-called Poincaré gauge (see sec. 9A of [3] and [6, 7]),[4] in which,

$$V^{(\mathrm{P})}(\mathbf{x}, t) = -\mathbf{x} \cdot \int_0^{u_0=1} du\, \mathbf{E}(u\mathbf{x}, t), \qquad \mathbf{A}^{(\mathrm{P})}(\mathbf{x}, t) = -\mathbf{x} \times \mathbf{I}(\mathbf{B}), \tag{7}$$

where,

$$\mathbf{I}(\mathbf{B}) = \int_0^{u_0=1} u\, du\, \mathbf{B}(u\mathbf{x}, t). \tag{8}$$

[1] This gauge appears to have been first used by Gibbs in 1896 [4].

[2] For a static electric field the Hamiltonian-gauge vector potential is $\mathbf{A}^{(\mathrm{H})} = -c(t - t_0)\mathbf{E}$, while for a static magnetic field the vector potential is the same as that in the Coulomb gauge (and also in the Lorenz gauge).

[3] In Hamiltonian dynamics of a particle with charge q the normal (z) component of the canonical momentum $\mathbf{p} = \mathbf{p}_{\mathrm{mech}} + q\mathbf{A}/c$ takes a discontinuous step when a particle enters or exits the rf cavity through the planar faces. This undesirable feature can be mitigated by switching from coordinates (x, y, z) with independent variable t to coordinates (x, y, t) with independent variable z, in which case the canonical momentum of the t-coordinate is $p_t = -E_{\mathrm{mech}} - qV$ (see, for example, sec. 1.6 of [5]), which is just $-E_{\mathrm{mech}}$ in the Hamiltonian gauge. Then, if the faces of the cavity traversed by particles are at constant z, all three canonical momenta p_x, p_y and p_t are continuous. *Only if the particles are muons would it be considered practical to used closed rf cavities.*

[4] The Poincaré gauge is also called the multipolar gauge [8].

These forms are remarkable in that they depend on the instantaneous value of the fields only along a line between the origin and the point of observation.

For points $\mathbf{x}$ outside the cavity (whose center is at the origin) such that the vector $\mathbf{x}$ passes through the cavity wall at $r = R$, then $E_z(ur)$ and $B_\phi(ur)$ are nonzero only for $u < u_0 = R/r$, whereas if the vector $\mathbf{x}$ passes through the cavity wall at $|z| = D$, then $E_z(ur)$ and $B_\phi(ur)$ are nonzero only for $u < u_0 = D/|z|$. Thus, using 8.402 and 5.51 of [9] the scalar potential is,

$$
\begin{aligned}
V^{(\mathrm{P})} &= -zE_0\, e^{-i\omega t} \int_0^{u_0} du\, J_0(kur) = -\frac{zE_0}{kr}\, e^{-i\omega t} \int_0^{ku_0 r} dv\, J_0(v) \\
&= -\frac{zE_0}{kr}\, e^{-i\omega t} \int_0^{ku_0 r} dv\, \left[1 - \frac{v^2}{2^2} + \frac{v^4}{2^6} - \frac{v^6}{2^6 \cdot 3^2} + \cdots\right] \\
&= -zE_0 u_0\, e^{-i\omega t} \left[1 - \frac{(ku_0 r)^2}{2^2 \cdot 3} + \frac{(ku_0 r)^4}{2^6 \cdot 5} - \frac{(ku_0 r)^6}{2^6 \cdot 3^2 \cdot 7} + \cdots\right] \\
&= -\frac{2zE_0}{kr}\, e^{-i\omega t} \sum_{j=0}^{\infty} J_{2j+1}(ku_0 r). \qquad (9)
\end{aligned}
$$

Since $-\boldsymbol{\nabla} V$ has a nonzero radial component, and its axial component is not equal to the electric field (1), we anticipate that the vector potential will have both radial and axial components.

For the vector potential we have, using 8.402 of [9], that,

$$
\begin{aligned}
I_\phi(\mathbf{B}) &= -iE_0\, e^{-i\omega t} \int_0^{u_0} u\, du\, J_1(kur) = -\frac{iE_0}{k^2 r^2}\, e^{-i\omega t} \int_0^{ku_0 r} v\, dv\, J_1(v) \\
&= -\frac{iE_0}{k^2 r^2}\, e^{-i\omega t} \int_0^{ku_0 r} dv\, \left[\frac{v^2}{2} - \frac{v^4}{2^4} + \frac{v^6}{2^7 \cdot 3} - \cdots\right] \\
&= -iE_0 kr u_0^3\, e^{-i\omega t} \left[\frac{1}{2 \cdot 3} - \frac{(kru_0)^2}{2^4 \cdot 5} + \frac{(kru_0)^4}{2^7 \cdot 3 \cdot 7} - \cdots\right] \\
&= \frac{iE_0}{k^2 r^2}\, e^{-i\omega t} \int_0^{ku_0 r} v\, dv\, \frac{dJ_0(v)}{dv} = \frac{iE_0}{k^2 r^2}\, e^{-i\omega t} \int_0^{ku_0 r} dv\, \left[\frac{d(vJ_0(v))}{dv} - J_0(v)\right] \\
&= \frac{iE_0}{k^2 r^2}\, e^{-i\omega t} \left[ku_0 r J_0(ku_0 r) - 2\sum_{j=0}^{\infty} J_{2j+1}(ku_0 r)\right]. \qquad (10)
\end{aligned}
$$

The vector potential $\mathbf{A}^{(\mathrm{P})} = -\mathbf{x} \times \mathbf{I}(\mathbf{B})$ has components,

$$
A_r^{(\mathrm{P})} = zI_\phi(\mathbf{B}) = \frac{izE_0}{k^2 r^2}\, e^{-i\omega t} \left[ku_0 r J_0(ku_0 r) - 2\sum_{j=0}^{\infty} J_{2j+1}(ku_0 r)\right], \qquad (11)
$$

$$
A_z^{(\mathrm{P})} = -rI_\phi(\mathbf{B}) = -\frac{iE_0}{k^2 r}\, e^{-i\omega t} \left[ku_0 r J_0(ku_0 r) - 2\sum_{j=0}^{\infty} J_{2j+1}(ku_0 r)\right]. \qquad (12)
$$

On the z-axis, u_0 is either 1 or $D/|z|$, so $I_\phi(\mathbf{B})$ (and the vector potential) vanishes there. In the central region outside the cavity, where $r/|z| > R/D$, we have that $u_0 = R/r$ so $I_\phi(\mathbf{B})$

falls off as $1/r^2$ and hence $\mathbf{A}^{(\mathrm{P})}$ falls off as $1/r$ at large r. In the forward and backward regions outside the cavity, where $r/|z| < R/D$, we have that $u_0 = D/|z|$ so $I_\phi(\mathbf{B})$ falls off as $r/|z|^3$ and hence $A_r^{(\mathrm{P})}$ falls off as $r/|z|^2 < R/D\,|z|$, and $A_z^{(\mathrm{P})}$ falls off as $r^2/|z|^3 < R^2/D^2\,|z|$ at large r.

In sum, the Poincaré potentials fall off as $1/|\mathbf{x}|$ at large $|\mathbf{x}|$.

Further Discussion of the Exterior Potentials

The electric field is related by eq. (5), so outside the cavity where $\mathbf{E} = 0 = \mathbf{B}$ we have that,

$$\mathbf{A}^{(\mathrm{P})} = -\frac{i}{k}\boldsymbol{\nabla}V, \qquad A_r^{(\mathrm{P})} = -\frac{i}{k}\frac{\partial V}{\partial r}, \qquad A_\phi^{(\mathrm{P})} = 0, \qquad A_z^{(\mathrm{P})} = -\frac{i}{k}\frac{\partial V}{\partial z}. \tag{13}$$

Then $\mathbf{B} = (i/k)\boldsymbol{\nabla}\times\boldsymbol{\nabla}V^{(\mathrm{P})} = 0$ as expected. In the central exterior region, $r > R$, $r/|z| > R/D$ and $u_0 = R/r$, we have that,

$$V^{(\mathrm{P})} = = -\frac{2zE_0}{kr}\, e^{-i\omega t}\sum_{j=0}^{\infty} J_{2j+1}(kR), \tag{14}$$

$$A_r^{(\mathrm{P})} = -\frac{2izE_0}{k^2r^2}\, e^{-i\omega t}\sum_{j=0}^{\infty} J_{2j+1}(kR), \tag{15}$$

$$A_z^{(\mathrm{P})} = \frac{2iE_0}{k^2r}\, e^{-i\omega t}\sum_{j=0}^{\infty} J_{2j+1}(kR), \tag{16}$$

in agreement with eqs. (11)-(12).

2.2.3 Lorenz Gauge

The potentials in the Lorenz gauge include the well-known retarded potentials.[5] For a cavity with perfectly conducting walls, and time dependence $e^{-i\omega t}$, the only charge and current densities reside on these walls, so the retarded potentials (in the frequency domain) have the form,

$$V^{(\mathrm{L,ret})}(\mathbf{x}) = \int \frac{\sigma(\mathbf{x}')\, e^{ikr}}{r}\, d\mathrm{Area}' = \int \frac{\mathbf{E}(\mathbf{x}')\cdot\hat{\mathbf{n}}'\, e^{ikr}}{4\pi r}\, d\mathrm{Area}', \tag{17}$$

$$\mathbf{A}^{(\mathrm{L,ret})}(\mathbf{x}) = \int \frac{\mathbf{K}(\mathbf{x}')\, e^{ikr}}{cr}\, d\mathrm{Area}' = \int \frac{\hat{\mathbf{n}}'\times\mathbf{B}(\mathbf{x}')\, e^{ikr}}{4\pi r}\, d\mathrm{Area}', \tag{18}$$

where σ and $\mathbf{K}$ are the surface charge and current densities, and $\hat{\mathbf{n}}'$ is the inward unit vector normal to the bounding surface. These potentials are nonzero both inside and outside of the cavity.

It does not seem possible to give analytic expressions for these potentials in the present example.

Following a comment in prob. 14.2 of [11], we note that the Hamiltonian-gauge potentials (6) satisfy the Lorenz-gauge condition $\boldsymbol{\nabla}\cdot\mathbf{A}^{(\mathrm{L})} = -\partial V^{(\mathrm{L})}/\partial ct$ (and the vector potential

[5]Other forms are possible as well, as Lorenz-gauge potentials are subject to a class of restricted gauge transformations, as discussed, for example, in sec. 2.3.1 of [10].

satisfies the Coulomb-gauge condition $\boldsymbol{\nabla}\cdot\mathbf{A}^{(\mathrm{L})}=0$) inside the cavity, although the spatial derivatives are not defined on the planar cavity walls. Can we infer that at least in the interior of the cavity the vector potential (18) is both the Lorenz-gauge and Coulomb-gauge potential, and that the Lorenz-gauge scalar potential is zero inside the cavity? For potentials to be in some gauge, they must satisfy the gauge condition everywhere, and not just in some restricted region as considered in [11].

The surface charge density σ has opposite signs on the planar walls of the cavity, and is independent of the cavity length, such that the retarded potential (17) is nonzero. For example, consider a cavity that extends from $z=0$ to d where $kd\gg 1$. Then, for a point inside the cavity far from both ends, $z\gg 1/k$ and $d-z\gg 1/k$, the retarded potential is,

$$V^{(\mathrm{L,ret})}(\mathbf{x})\approx Q\left(\frac{e^{ikz}}{z}-\frac{e^{ik(d-z)}}{d-z}\right)\neq 0, \tag{19}$$

where Q is the peak charge density on the planar ends. This potential also holds for points outside the cavity such that $\sqrt{x^2+y^2}$ is small compared to $z\gg 1/k$ and $d-z\gg 1/k$. Only on the plane $z=d/2$ does $V^{(\mathrm{L,ret})}=0$. Then, the retarded vector potential in this region follows as,

$$\begin{aligned}A_z^{(\mathrm{L,ret})}(\mathbf{x})=-\frac{i}{k}\left(\frac{\partial V}{\partial z}+E_z\right)\ &\approx\ -\frac{iQ}{k}\left[\left(ik-\frac{1}{z}\right)\frac{e^{ikz}}{z}+\left(ik-\frac{1}{d-z}\right)\frac{e^{ik(d-z)}}{d-z}\right]\\ &\quad+\begin{cases}-\frac{iE_0}{k}\,J_0(kr) & \text{(inside)},\\ 0 & \text{(outside)}.\end{cases}\end{aligned} \tag{20}$$

Similar approximations can be given for the (nonzero) Lorenz-gauge retarded potentials outside the cavity for $z\ll 1/k$ and $z-d\gg 1/k$.

2.2.4 Coulomb Gauge

The Coulomb gauge (favored by Maxwell) is defined by the condition that $\boldsymbol{\nabla}\cdot\mathbf{A}^{(\mathrm{C})}=0$. The Coulomb gauge (like the Lorenz gauge) is a velocity gauge, with the speed v of propagation of the scalar potential $V^{(\mathrm{C})}$ being infinite. Coulomb-gauge potentials are not unique, but different variants can be related by a gauge transformations whose gauge function χ obeys $\nabla^2\chi=0$.

Coulomb-gauge potentials for cavities and waveguides are extensively discussed in chap. 13 of [12], to which the reader is referred.[6]

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