Displacement Current of a Uniformly Moving Charge

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1 Problem

It is well known that the magnetic field $\mathbf{B}$ of a (point) charge q that moves with uniform velocity $\mathbf{v}$ in vacuum can be related to its electric field $\mathbf{E}$ (in SI units) by,

$$\mathbf{B} = \frac{\mathbf{v}}{c^2} \times \mathbf{E}. \tag{1}$$

Deduce a relation between the magnetic field and the "displacement-current density" $\epsilon_0\, \partial\mathbf{E}/\partial t$.

2 Solution

The solution follows [1], which contains a good discussion of the displacement current.[1]

The electric field $\mathbf{E}$ at an observation point $\mathbf{r}$ at the moment when the moving charge is at the origin can be deduced by a Lorentz transformation of the field $\mathbf{E}^\star = q\,\mathbf{r}^\star/4\pi\epsilon_0 r^{\star 3}$ in the rest frame of the charge,

$$\mathbf{E} = \mathbf{E}^\star_\parallel + \gamma\mathbf{E}^\star_\perp = (\mathbf{E}^\star\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}} + \gamma[\mathbf{E}^\star - (\mathbf{E}^\star\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}}] = \gamma\mathbf{E}^\star + (1-\gamma)(\mathbf{E}^\star\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}}, \tag{2}$$

where $\gamma = 1/\sqrt{1-\beta^2}$, $\beta = v/c$, and the distance $\mathbf{r}^\star$ is related by,

$$\mathbf{r}^\star = \mathbf{r}^\star_\parallel + \mathbf{r}^\star_\perp = \gamma\mathbf{r}_\parallel + \mathbf{r}_\perp = \gamma(\mathbf{r}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}} + \mathbf{r} - (\mathbf{r}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}} = \mathbf{r} + (\gamma-1)(\mathbf{r}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}}, \tag{3}$$

since $\mathbf{r}_\parallel = \mathbf{r}^\star_\parallel/\gamma$ according to the Lorentz contraction. Then,

$$r^{\star 2} = \gamma^2 r_\parallel^2 + r_\perp^2 = \gamma^2[(\mathbf{r}\cdot\hat{\mathbf{v}})^2 + (1-\beta^2)(r^2 - (\mathbf{r}\cdot\hat{\mathbf{v}})^2) = \gamma^2 r^2(1-\beta^2\sin^2\theta), \tag{4}$$

where θ is the angle between $\mathbf{r}$ and $\mathbf{v}$. Combining eqs. (2)-(4) we obtain,

$$\mathbf{E} = \gamma\frac{q\,\mathbf{r}}{4\pi\epsilon_0 r^{\star 3}} = \frac{q\,\mathbf{r}}{4\pi\epsilon_0\gamma^2 r^3(1-\beta^2\sin^2\theta)^{3/2}}, \tag{5}$$

such that the electric field of the uniformly moving charge points away from the present position of the charge, and varies with angle for fixed r, being largest at $\theta = 90^\circ$.

The magnetic field $\mathbf{B}$ can be obtained from the electric field in the charge's rest frame according to the transformation,

$$\mathbf{B} = \gamma\frac{\mathbf{v}}{c^2}\times\mathbf{E}^\star = \frac{\mathbf{v}}{c^2}\times\mathbf{E} = \gamma\frac{q}{4\pi\epsilon_0 c^2 r^{\star 3}}\mathbf{v}\times\mathbf{r} = \gamma\frac{\mu_0 q}{4\pi r^{\star 3}}\mathbf{v}\times\mathbf{r}, \tag{6}$$

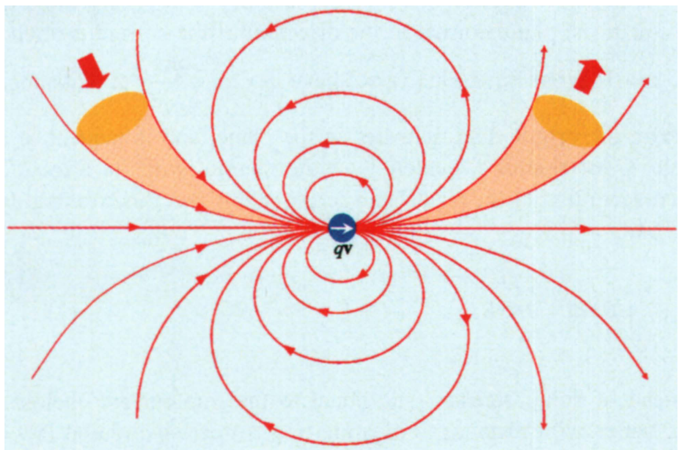
[1]In what may be the earliest paper on the fields of a uniformly moving charge [2], J.J. Thomson implied that he would relate these to the displacement current, his eq. (1), but he did not do this in the sense of the present note. See also pp. 149-150 of [3], and [4].

recalling eq. (2).[2]

The displacement-current density is,

$$\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \epsilon_0 \frac{\mathbf{E}(q \text{ at } v\,dt) - \mathbf{E}(q \text{ at origin})}{dt} = \epsilon_0 \mathbf{E}_{\text{dipole}} = 3\gamma^2 \frac{(\gamma q\,\mathbf{v}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}}{4\pi r^{\star 3}} - \frac{\gamma q\,\mathbf{v}}{4\pi r^{\star 3}}\,, \tag{7}$$

where $\mathbf{E}_{\text{dipole}}$ is the field of an electric dipole that has velocity $\mathbf{v}$ and separation of charges $\pm q$ by distance v in the lab frame, and hence by distance γv in the rest frame of the dipole.[3] That is, the dipole moment in its rest frame is $\mathbf{p}^\star = \gamma q\,\mathbf{v}$. Lines of the displacement-current density (7) form closed loops passing through the charge q, as shown in the figure below (from [1]). For large velocity, the field pattern is flattened perpendicular to $\mathbf{v}$.



Comparing with eq. (6), we see that the magnetic field can be expressed in terms of the displacement-current density (7) as,

$$\mathbf{B} = \mu_0 \mathbf{r} \times \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\,. \tag{10}$$

The forms (6) for the magnetic field at the observer relate it to the distant, moving charge, while the form (10) relates it to the time-dependent electric field at the observer. The former relations have the character of action at a distance, while the latter relation illustrates the spirit of Maxwell's vision of a dynamical field theory of electromagnetism [8]. One can interpret eq. (10) as implying that the "local" cause of the magnetic field at the observer is the nearby displacement-current density, while eq. (6) indicates that the "ultimate" cause of the magnetic field is the distant, moving charge.[4]

[2]The results (5)-(6) were first obtained in 1888 by Heaviside [5] and by Thomson (1889) [6] via different methods. For commentary on this, see the end of [7].

[3]We can confirm eq. (7) by differentiation of eq. (5), noting that $r_\perp = r_\perp^\star$ is constant in time while $d\mathbf{r}/dt \equiv \dot{\mathbf{r}} = -\mathbf{v}$ since vector $\mathbf{r}$ points from the moving charge to the observer. Thus, $\dot{r}_\parallel = -v$, and from eq. (4),

$$\dot{r}^\star = \gamma^2 \frac{r_\parallel \dot{r}_\parallel}{r^\star} = -\gamma^2 \frac{\mathbf{r}\cdot\mathbf{v}}{r^\star}\,, \tag{8}$$

whence,

$$\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \gamma \frac{q\,\dot{\mathbf{r}}}{4\pi r^{\star 3}} - 3\gamma \frac{q\,\dot{r}^\star \mathbf{r}}{4\pi r^{\star 4}} = -\gamma \frac{q\,\mathbf{v}}{4\pi r^{\star 3}} + 3\gamma^3 \frac{q\,(\mathbf{r}\cdot\mathbf{v})\,\mathbf{r}}{4\pi r^{\star 5}} = \epsilon_0 \mathbf{E}_{\text{dipole}}, \tag{9}$$

where $\mathbf{E}_{\text{dipole}}$ is the field of an electric dipole of moment $\mathbf{p}^\star = \gamma q\,\mathbf{v}$ that has velocity $\mathbf{v}$.

[4]For related discussion, see Appendix A.5 of [9] and the Appendix of [10].

As usual, the presence of the displacement-current term in Maxwell's fourth equation,

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right), \tag{11}$$

(in vacuum) provides consistency for Ampère's law in the form,

$$\oint_{\rm loop} \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\rm thru} + \mu_0 \int_{\rm loop} \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{Area}, \tag{12}$$

since, in general, the current of the moving charge cannot be said to pass through the area of the loop considered in the integration.[5,6,7]

[5]For example, consider a circular loop centered on the **v**-axis that passes through the observer at distance **r** from the moving charge, and take the area of loop to consist of a conical surface that extends to "infinity" with vector **r** as a generator, with the surface completed by a spherical cap also at "infinity" (on which the fields are negligible). Then, eq. (12), together with eq. (7), tells us that $2\pi r_\perp B_\phi$,

$$\begin{aligned} 2\pi r \sin\theta B_\phi &= \mu_0 \int_{\rm loop} \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{Area} = \mu_0 \int_r^\infty \frac{\gamma q v}{4\pi r^{\star 3}} 2\pi r \sin\theta \, dr \sin\theta = \frac{2\pi\mu_0 q v \sin^2\theta}{4\pi\gamma^2(1-\beta^2\sin^2\theta)^{3/2}} \int_r^\infty \frac{dr}{r^2} \\ &= -\frac{2\pi\mu_0 q v \sin^2\theta}{4\pi\gamma^2 r(1-\beta^2\sin^2\theta)^{3/2}} = -2\pi r \sin\theta \frac{qv}{4\pi\gamma^2 r^3(1-\beta^2\sin^2\theta)^{3/2}} \mu_0 r \sin\theta = -2\pi r \sin\theta \left| \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \times \mu_0 \mathbf{r} \right| \end{aligned} \tag{13}$$

recalling that the observer is at angle θ to the **v**-axis, which leads again to eq. (10) for the magnetic field of the uniformly moving charge.

[6]A calculation was made in 1939 by Cullwick, pp. 149-150 of [3], taking the surface of the loop to be a spherical cap of radius r. Here,

$$\mu_0 \int_{\rm loop} \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{Area} = \mu_0 \int_{\cos\theta}^1 \frac{qv(3\gamma^2-1)\cos\theta}{4\pi\gamma^2 r^3(1-\beta^2\sin^2\theta)^{3/2}} 2\pi r^2 \, d\cos\theta. \tag{14}$$

This integral is difficult, so (like Cullwick) we consider only low velocities, where $\beta^2 \approx 0$ and $\gamma \approx 1$. Then,

$$\begin{aligned} \mu_0 \int_{\rm loop} \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{Area} &= \mu_0 \int_{\cos\theta}^1 \frac{2qv\cos\theta}{4\pi r^3} 2\pi r^2 \, d\cos\theta = \frac{2\pi\mu_0 q v(1-\cos^2\theta)}{4\pi r} = 2\pi r \sin\theta \frac{\mu_0 q v \sin\theta}{4\pi r^2} \\ &= 2\pi r \sin\theta B_\phi, \end{aligned} \tag{15}$$

as expected.

[7]Taking the surface of the loop to be a disk of radius $R = r\sin\theta$ at distance $z = r\cos\theta$ from the moving charge, and again considering only low velocities, we have, noting that $r = \sqrt{R^2+z^2}$,

$$\begin{aligned} \mu_0 \int_{\rm loop} \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{Area} &= \mu_0 \int_0^R \left(\frac{3qvz^2}{4\pi r^5} - \frac{qv}{4\pi r^3} \right) 2\pi R \, dR = \frac{2\pi\mu_0 q v}{4\pi} \left(-\frac{3z^2}{3r^3} + \frac{3z^2}{3z^3} + \frac{1}{r} - \frac{1}{z} \right) \\ &= \frac{2\pi\mu_0 q v \sin^2\theta}{4\pi r} = 2\pi r \sin\theta \frac{\mu_0 q v \sin\theta}{4\pi r^2} = 2\pi r \sin\theta B_\phi, \end{aligned} \tag{16}$$

using Dwight 201.03 and 201.05 [11].

References

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