

Does Destructive Interference Destroy Energy?

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1 Problem

In principle, a pair of counterpropagating waves (with separate sources) whose waveforms are the negative of each other can completely cancel at some moment in time. Does this destructive interference also destroy the energy of the waves at this moment?

Comment also on the case of waves from a single source.

2 Solution

While the answer is well-known to be **NO**, and energy is conserved in the superposition of waves, discussion of this is sparse in textbooks.^{1,2,3}

A key here is that the energy associated with a wave has two forms, generically called “kinetic” and “potential”, that are equal for a wave propagating in a single direction, while in the case of destructive (or constructive) interference of counterpropagating waves one form of energy decreases and the other increases such that the total energy remains constant. The character of energy conservation in the case of two sources is well illustrated by a pair of counterpropagating waves in one dimension, as discussed in sec. 2.1.

¹Thanks to Hans Schantz for pointing this out. Pedagogic discussions of this issue include R.C. Levine, *False paradoxes in superposition of electric and acoustic waves*, Am. J. Phys. **48**, 28 (1980), http://kirkmcd.princeton.edu/examples/EM/levine_ajp_48_28_80.pdf

W.N. Mathews, Jr, *Superposition and energy conservation for small amplitude mechanical waves*, Am. J. Phys. **54**, 233 (1986), http://kirkmcd.princeton.edu/examples/mechanics/mathews_ajp_54_233_86.pdf

N. Gauthier, *What happens to energy and momentum when two oppositely-moving wave pulses overlap?* Am. J. Phys. **71**, 787 (2003), http://kirkmcd.princeton.edu/examples/EM/gauthier_ajp_71_787_03.pdf

R. Drosd, L. Minkin and A.S. Shapovalov, *Interference and the Law of Energy Conservation*, Phys. Teach. **52**, 428 (2014), http://kirkmcd.princeton.edu/examples/mechanics/drosd_pt_52_428_14.pdf

²A one-dimensional wave moving in one direction can have only one source (in that a one-dimensional wave from one source cannot pass through another source), and there can be only one such wave at a given point, such that wave interference is not a relevant concept here. We can write $0 = \sin(kx - \omega t) - \sin(kx - \omega t)$, but this mathematical identity does not have the physical implication that two distinct waves are present, each with nonzero energy.

A one-dimensional wave could have a source at $x = a$ and a sink at $x = b > a$, such that energy is transmitted from a to b , where fraction $\alpha < 1$ of it is absorbed. For example, if the wave for $a < x < b$ has the form $\sin(kx - \omega t)$, we could say that the wave $\sqrt{1 - \alpha} \sin(kx - \omega t)$ for $x > b$ is the result of interference, say, $\sin(kx - \omega t) + (1 - \sqrt{1 - \alpha}) \sin(kx - \omega t + \pi)$. While one might say that this is an example where destructive interference “destroyed” energy, it seems better to say that the energy was absorbed at $x = b$, which reduced the amplitude of the wave for $x > b$. *This footnote is based on comments by Carlo Mantovani, May 8, 2018.*

³It appears that the trivial case of a null wave (with zero energy) is sometimes mistakenly described as an example of destructive interference of two waves moving with opposite amplitudes (but the same energies) in the same direction, which has led to the misimpression that destructive interference can destroy energy.

Interference in a wave from a single source can only occur for propagation in two or more dimensions where the system includes entities (such as “slits” in a screen) that “scatter” one portion of the wave onto another portion. Then, energy which would appear in one region in the absence of “scattering” appears elsewhere in its presence. The character of energy conservation in the case of a single source is illustrated by double-slit interference in sec. 2.2.

2.1 Counterpropagating Waves from Two Sources in One Dimension

Here, we present three related arguments for counterpropagating one-dimensional waves.

2.1.1 Transverse Waves on a Stretched String

A string of linear mass density ρ under tension T has wave speed,

$$c = \sqrt{\frac{T}{\rho}}. \quad (1)$$

Writing the transverse displacement as $y(x, t)$, the kinetic energy associated with this waveform is,

$$\text{KE} = \int \frac{\rho \dot{y}^2}{2} dx, \quad (2)$$

where $\dot{y} = dy/dt$. The potential energy can be taken the work done in stretching the string,

$$\text{PE} = \int T dl = \int T \left(\sqrt{1 + y'^2} - 1 \right) dx \approx \int \frac{T y'^2}{2} dx, \quad (3)$$

where $y' = dy/dx$. The first derivatives for traveling waves $y(x \pm ct)$ are related by,

$$\dot{y} = \pm c y', \quad \dot{y}^2 = c^2 y'^2 = \frac{T}{\rho} y'^2, \quad (4)$$

which implies that $\text{KE} = \text{PE}$ for a wave traveling in a single direction, and that the total energy U of such a wave is given by,

$$U = \text{KE} + \text{PE} = 2\text{KE} = 2\text{PE}. \quad (5)$$

For two waves propagating in opposite directions with similar waveforms,

$$y_1(x, t) = y(x - ct), \quad y_2(x, t) = \pm y_1(x, -t) = \pm y(x + ct). \quad (6)$$

The total energy of the waves when they don't overlap is,

$$U_{\text{total}} = \text{KE}_1 + \text{PE}_1 + \text{KE}_2 + \text{PE}_2 + 2\text{KE} + 2\text{PE} = 4\text{KE} = 4\text{PE}, \quad (7)$$

where the energies without subscripts are the common values associated with the individual waves. The first derivatives for the superposition $y_{\text{tot}} = y_1 + y_2 = y(x - ct) \pm y(x + ct)$ of the two waves are,

$$\dot{y}_{\text{tot}} = \dot{y}_1 + \dot{y}_2, = -c [y'(x - ct) \mp y'(x + ct)] \quad y'_{\text{tot}} = y'_1 + y'_2 = y'(x - ct) \pm y'(x + ct), \quad (8)$$

Destructive Interference at $t = 0$

Destructive interference in the wave y_{tot} at time $t = 0$ corresponds to the lower signs in eq. (8), in which case we have at time $t = 0$,⁴

$$\dot{y}_{\text{tot}}(t = 0) = 2\dot{y}(x), \quad y'_{\text{tot}}(t = 0) = 0, \quad (9)$$

$$\text{KE}_{\text{tot}}(t = 0) = \int \frac{\rho \dot{y}_{\text{tot}}^2}{2} dx = 4 \int \frac{\rho \dot{y}^2}{2} dx = 4\text{KE}, \quad (10)$$

$$\text{PE}_{\text{tot}}(t = 0) \approx \int \frac{T y_{\text{tot}}'^2}{2} dx = 0, \quad (11)$$

$$U_{\text{tot}}(t = 0) = \text{KE}_{\text{tot}}(t = 0) + \text{PE}_{\text{tot}}(t = 0) = 4\text{KE} = U_{\text{total}}, \quad (12)$$

such that energy is conserved. This destructive interference destroys the potential energy, but doubles the kinetic energy, at time $t = 0$.

Constructive Interference at $t = 0$

Similarly, constructive interference in y_{tot} at time $t = 0$ corresponds to the upper signs in eq. (8), in which case we have at time $t = 0$,

$$\dot{y}_{\text{tot}}(t = 0) = 0, \quad y'_{\text{tot}}(t = 0) = 2y'(x), \quad (13)$$

$$\text{KE}_{\text{tot}}(t = 0) = \int \frac{\rho \dot{y}_{\text{tot}}^2}{2} dx = 0, \quad (14)$$

$$\text{PE}_{\text{tot}}(t = 0) \approx \int \frac{T y_{\text{tot}}'^2}{2} dx = 4 \int \frac{T y'^2}{2} dx = 4\text{PE}, \quad (15)$$

$$U_{\text{tot}}(t = 0) = \text{KE}_{\text{tot}}(t = 0) + \text{PE}_{\text{tot}}(t = 0) = 4\text{PE} = U_{\text{total}}. \quad (16)$$

and again energy is conserved. This constructive interference doubles the potential energy, but destroys the kinetic energy, at time $t = 0$.

2.1.2 Transmission Line

A two-conductor transmission line is characterized by capacitance C and inductance L per unit length (ignoring the electrical resistance of the conductors). We consider the voltage difference $V(x, t)$ between the two conductors, which carry equal and opposite currents $\pm I(x, t)$. Then, the speed of waves along the line is,

$$c = \frac{1}{\sqrt{LC}}, \quad (17)$$

⁴This case also corresponds to destructive interference at time $t = 0$ in y'_{tot} but constructive interference (at this time) in \dot{y}_{tot} .

and the voltage and currents are related by,

$$V = \pm IZ, \quad Z = \sqrt{\frac{L}{C}}, \quad (18)$$

where the upper(lower) sign holds for waves moving in the $+x(-x)$ direction, and Z is the (real) transmission-line impedance.⁵

The energy of a wave has both capacitive and inductive terms,

$$U_C = \int \frac{CV(x)^2}{2} dx, \quad U_L = \int \frac{LI(x)^2}{2} dx, \quad (19)$$

For a wave that moves only in a single direction,

$$U_C = \int \frac{CV(x)^2}{2} dx = \int \frac{CZ^2I(x)^2}{2} dx = \int \frac{LI(x)^2}{2} dx = U_L, \quad (20)$$

and the total energy of such a wave is,

$$U = U_C + U_L = 2U_C = 2U_L. \quad (21)$$

For two waves propagating in opposite directions with similar waveforms,

$$V_1(x, t) = V(x - ct), \quad V_2(x, t) = \pm V_1(x, -t) = \pm V(x + ct), \quad (22)$$

$$I_1(x, t) = \frac{V_1}{Z} = \frac{V(x - ct)}{Z} = I(x - ct), \quad I_2(x, t) = -\frac{V_2}{Z} = \mp I(x + vt). \quad (23)$$

The total energy of the waves when they don't overlap is,

$$U_{\text{total}} = U_{C,1} + U_{L,1} + U_{C,2} + U_{L,2} = 2U_C + 2U_L = 4U_C = 4U_L, \quad (24)$$

where the energies without subscripts are the common values associated with the individual waves. The total voltage and current for the superposition of the two waves are,

$$V_{\text{tot}} = V_1 + V_2 = V(x - ct) \pm V(x + ct), \quad I_{\text{tot}} = I_1 + I_2 = I(x - ct) \mp I(x + ct). \quad (25)$$

Destructive Interference at $t = 0$

Destructive interference of the voltage V_{tot} at time $t = 0$ corresponds to the lower signs in eq. (25), in which case we have at time $t = 0$,⁶

$$V_{\text{tot}}(t = 0) = 0, \quad I_{\text{tot}}(t = 0) = 2I(x), \quad (26)$$

$$U_{C,\text{tot}}(t = 0) = \int \frac{CV_{\text{tot}}^2}{2} dx = 0, \quad (27)$$

$$U_{L,\text{tot}}(t = 0) = \int \frac{LI_{\text{tot}}^2}{2} dx = 4 \int \frac{LI^2}{2} dx = 4U_L, \quad (28)$$

$$U_{\text{tot}}(t = 0) = U_{C,\text{tot}}(t = 0) + U_{L,\text{tot}}(t = 0) = 4U_L = U_{\text{total}}, \quad (29)$$

⁵See, for example, K.T. McDonald, *Distortionless Transmission Line* (Nov. 11, 1996), <http://kirkmcd.princeton.edu/examples/distortionless.pdf>.

⁶This case also corresponds to constructive interference at time $t = 0$ in the current I_{tot} .

such that energy is conserved. This destructive interference doubles the inductive energy, but destroys the capacitive energy, at time $t = 0$.

Constructive Interference at $t = 0$

Constructive of the voltage V_{tot} at time $t = 0$ corresponds to the upper signs in eq. (25), in which case we have at time $t = 0$,

$$V_{\text{tot}}(t = 0) = 2V(x), \quad I_{\text{tot}}(t = 0) = 0, \quad (30)$$

$$U_{C,\text{tot}}(t = 0) = \int \frac{CV_{\text{tot}}^2}{2} dx = 4 \int \frac{CV^2}{2} dx = 4U_C, \quad (31)$$

$$U_{L,\text{tot}}(t = 0) = \int \frac{LI_{\text{tot}}^2}{2} dx = 0, \quad (32)$$

$$U_{\text{tot}}(t = 0) = U_{C,\text{tot}}(t = 0) + U_{L,\text{tot}}(t = 0) = 4U_C = U_{\text{total}}, \quad (33)$$

and again energy is conserved. This constructive interference destroys the inductive energy, but doubles the capacitive energy, at time $t = 0$.

2.1.3 Plane Electromagnetic Waves

A plane electromagnetic wave propagating in vacuum in the x -direction with, say, y polarization had electric and magnetic fields (in Gaussian units),

$$\mathbf{E} = \mathbf{E}(x - ct) = E(x - ct) \hat{\mathbf{y}}, \quad \mathbf{B} = \mathbf{B}(x - ct) = E(x - ct) \hat{\mathbf{z}}, \quad (34)$$

where c is the speed of light in vacuum.

The energy of a wave has both electric and magnetic terms,

$$U_E = \int \frac{E^2}{8\pi} d\text{Vol}, \quad U_M = \int \frac{B^2}{8\pi} d\text{Vol} = U_E, \quad (35)$$

and the total energy of such a wave is,

$$U = U_E + U_M = 2U_E = 2U_M. \quad (36)$$

For two waves propagating in opposite directions with similar waveforms,

$$\mathbf{E}_1(x, t) = \mathbf{E}(x - ct), \quad \mathbf{B}_1(x, t) = \mathbf{B}(x - ct), \quad (37)$$

$$\mathbf{E}_2(x, t) = \pm \mathbf{E}(x + ct), \quad \mathbf{B}_2(x, t) = \mp \mathbf{B}(x + ct). \quad (38)$$

The total energy of the waves when they don't overlap is,

$$U_{\text{total}} = U_{E,1} + U_{M,1} + U_{E,2} + U_{M,2} = 2U_E + 2U_M = 4U_E = 4U_M, \quad (39)$$

where the energies without subscripts are the common values associated with the individual waves. The total electric and magnetic fields for the superposition of the two waves are,

$$\mathbf{E}_{\text{tot}} = \mathbf{E}_1 + \mathbf{E}_2 = \mathbf{E}(x - ct) \pm \mathbf{E}(x + ct), \quad \mathbf{B}_{\text{tot}} = \mathbf{B}_1 + \mathbf{B}_2 = \mathbf{B}(x - ct) \mp \mathbf{B}(x + ct). \quad (40)$$

Destructive Interference at $t = 0$

Destructive interference in the magnetic field \mathbf{B}_{tot} at time $t = 0$ corresponds to the lower signs in eq. (40), in which case we have at time $t = 0$,⁷

$$\mathbf{E}_{\text{tot}}(t = 0) = 2\mathbf{E}(x), \quad \mathbf{B}_{\text{tot}}(t = 0) = 0, \quad (41)$$

$$U_{E,\text{tot}}(t = 0) = \int \frac{E_{\text{tot}}^2}{8\pi} d\text{Vol} = 4 \int \frac{E^2}{8\pi} d\text{Vol} = 4U_E, \quad (42)$$

$$U_{M,\text{tot}}(t = 0) = \int \frac{B_{\text{tot}}^2}{8\pi} d\text{Vol} = 0, \quad (43)$$

$$U_{\text{tot}}(t = 0) = U_{E,\text{tot}}(t = 0) + U_{B,\text{tot}}(t = 0) = 4U_E = U_{\text{total}}, \quad (44)$$

such that energy is conserved. This destructive interference destroys the magnetic energy at time $t = 0$, but doubles the electric energy, at time $t = 0$.

Constructive Interference at $t = 0$

Similarly, constructive interference in the magnetic field \mathbf{B}_{tot} at time $t = 0$ corresponds to the upper signs in eq. (40), in which case we have at time $t = 0$,

$$\mathbf{E}_{\text{tot}}(t = 0) = 0, \quad \mathbf{B}_{\text{tot}}(t = 0) = 2\mathbf{B}(x), \quad (45)$$

$$U_{E,\text{tot}}(t = 0) = \int \frac{E_{\text{tot}}^2}{8\pi} d\text{Vol} = 0, \quad (46)$$

$$U_{M,\text{tot}}(t = 0) = \int \frac{B_{\text{tot}}^2}{8\pi} d\text{Vol} = 4 \int \frac{B^2}{2} d\text{Vol} = 4U_M, \quad (47)$$

$$U_{\text{tot}}(t = 0) = U_{E,\text{tot}}(t = 0) + U_{M,\text{tot}}(t = 0) = 4U_M = U_{\text{total}}, \quad (48)$$

and again energy is conserved. This constructive interference doubles the magnetic energy, but destroys the electric energy, at time $t = 0$.

2.2 The Double Slit Experiment with a Single Source

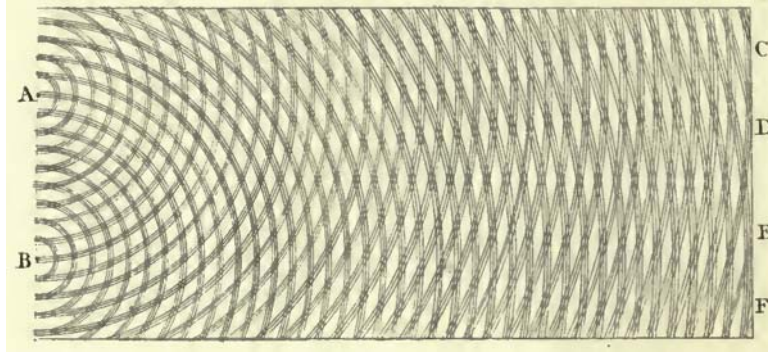
Following Young,⁸ we consider a line source of cylindrical light waves of wavelength λ that impinge on a planar screen that is parallel to the line source at closest distance large compared to λ , as shown in the figure on the next page. The screen has two narrow line slits, parallel to the line source and separated by distance D , located symmetrically about the line on the screen closest to the source. Beyond the planar screen is a cylindrical screen of radius $R \gg \lambda$ on which the intensity of the light is observed.

We first to consider scalar waves (*i.e.*, scalar diffraction theory), with amplitude $A e^{-i\omega t}$ at each of the two slits, where A is a complex number, $\omega = kc = 2\pi c/\lambda$ and c is the speed of light in vacuum. The intensity of a wave at some point is proportional to the absolute square of its amplitude there. We suppose the units of A are such that the (time-average) power per unit length passing through each slit is A^2 .

⁷This case also corresponds to constructive interference at time $t = 0$ in the electric field \mathbf{E}_{tot} .

⁸T. Young, *On the Nature of Light and Colours*, Lecture 39, *Course of Lectures on Natural Philosophy and Mechanical Arts* (London, 1897),

http://kirkmcd.princeton.edu/examples/optics/young_lecture_39.pdf.



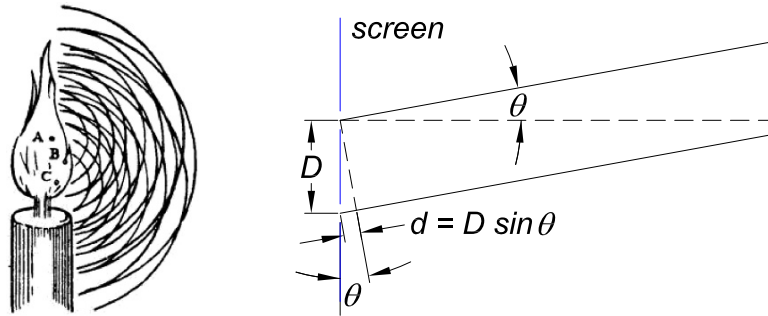
The intensity on the cylindrical screen is uniform, with total power $P_0 = A^2$ per unit length, and hence the angular distribution of (time-average) power per unit length on the half-cylinder screen from the light that passes through a single slit is,

$$\frac{dP}{d\theta} = \frac{P_0}{\pi} = \frac{A^2}{\pi}, \quad P = \int_{-\pi/2}^{\pi/2} \frac{dP}{d\theta} d\theta = P_0. \quad (49)$$

In the absence of interference, the power incident on the distance cylindrical screen would be $2P_0$.

The interference pattern of the waves beyond the slits can be computed according to the method of Huygens.⁹ The path length for light traveling at angle θ from the two slits to the distant screen has path difference $d = D \sin \theta$, and hence phase difference,

$$\Delta\phi = \frac{2\pi d}{\lambda} = \frac{2\pi D \sin \theta}{\lambda} = kD \sin \theta. \quad (50)$$



The (time-average) power incident on the distant cylindrical screen at angle θ depends on the absolute square of the sum of the amplitudes of the light that passes through the two slits, and has angular distribution $dP/d\theta$ and total power P given by,

$$\frac{dP}{d\theta} = \frac{P_0 |1 + e^{i\Delta\phi}|^2}{\pi} = \frac{2P_0}{\pi} (1 + \cos \Delta\phi), \quad (51)$$

$$P = \int_{-\pi/2}^{\pi/2} \frac{dP}{d\theta} d\theta = \frac{2P_0}{\pi} \int_{-\pi/2}^{\pi/2} [1 + \cos(kD \sin \theta)] d\theta = 2P_0 [1 + J_0(kD)], \quad (52)$$

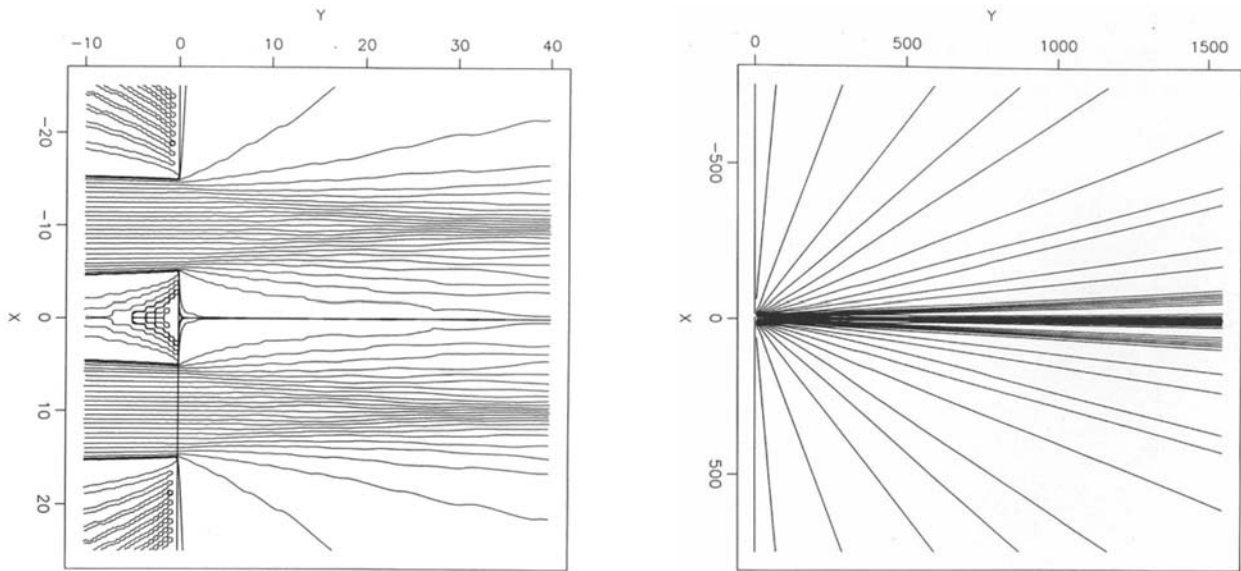
⁹C. Huygens, *Treatise on Light* (1678, English translation, Macmillan, 1912), p. 21, http://kirkmcd.princeton.edu/examples/optics/huygens_treatise_on_light.pdf.

noting that¹⁰

$$J_0(x) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos(x \sin \theta) d\theta. \quad (53)$$

Since the Bessel function $J_0(x)$ oscillates about zero, energy is conserved only “on average” in an analysis of the double-slit experiment via scalar diffraction theory.¹¹

A more accurate theory of electromagnetic waves characterizes the flow of energy via the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$.^{12,13} An analytic solution for the double-slit experiment apparently does not exist, but numerical computations are reported by Jeffers *et al.*,¹⁴ with lines of (energy-conserving) Poynting flux as shown below.¹⁵ The electromagnetic analysis matches that of scalar diffraction theory for small angles to the direction of incidence, but differs at large angles such that energy is conserved in detail for any slit separation.



¹⁰See, for example, eq. 9.1.18 of M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions* (NBS, 1964), http://kirkmcd.princeton.edu/examples/EM/abramowitz_and_stegun.pdf.

¹¹This is noted in sec. 5 of A. Lundin, *Interference and Energy Conservation in Phased Antenna Arrays and Young’s Double Slit Experiment* (Dec. 12, 2012), <http://www.diva-portal.org/smash/get/diva2:576304/FULLTEXT01.pdf>.

¹²J.H. Poynting, *On the Transfer of Energy in the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **175**, 343 (1884), http://kirkmcd.princeton.edu/examples/EM/Poynting_ptrsl_175_343_84.pdf.

¹³In so-called Bohmian quantum mechanics a photon is a point particle whose “Bohmian trajectory” is essentially a line of the Poynting vector field. See, for example, C. Philippidis, C. Dewdney and B.J. Hiley, *Quantum Interference and the Quantum Potential*, Nuovo Cim. **52B**, 15 (1979), http://kirkmcd.princeton.edu/examples/QM/philippidis_nc_52b_15_79.pdf.

¹⁴S. Jeffers *et al.*, *Classical Electromagnetic Theory of Diffraction and Interference: Edge, Single-Slit and Double-Slit Solutions*, in *Waves and Particles in Light and Matter*, A. van der Merwe and A. Garuccio, eds. (Plenum Press, New York, 1994), p. 309, http://kirkmcd.princeton.edu/examples/EM/jeffers_wplm_309_94.pdf.

¹⁵See also S. Kocsis *et al.*, *Observing the Average Trajectories of Single Photons in a Two-Slit Interferometer*, Science **332**, 1170 (2011), http://kirkmcd.princeton.edu/examples/QM/kocsis_science_332_1170_11.pdf and K.Y. Bliokh *et al.*, *Photon trajectories, anomalous velocities and weak measurements: a classical interpretation*, New J. Phys. **15**, 073022 (2013), http://kirkmcd.princeton.edu/examples/QM/bliokh_njp_15_073022_13.pdf

The essence of energy conservation in waves from a single source is that regions of higher energy density are compensated by regions of lower energy density.¹⁶

2.3 Equal and Opposite Electric Charges (Aug. 19, 2021)

We consider the idealized case where an electron and positron approach one another with equal and opposite, constant velocities, until they meet at the origin at time $t = 0$, and either annihilate one another or pass through each other.¹⁷

If the positive charge has position $\mathbf{x}_+(t) = \mathbf{v}t$ and the negative charge has position $\mathbf{x}_-(t) = -\mathbf{v}t$ for constant velocity \mathbf{v} , then their electromagnetic fields are,¹⁸

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}_+ + \mathbf{E}_- = \frac{q}{\gamma^2 r_+^2 (1 - \beta^2 \sin^2 \phi)^{3/2}} \hat{\mathbf{r}}_+ - \frac{q}{\gamma^2 r_-^2 (1 - \beta^2 \sin^2 \phi)^{3/2}} \hat{\mathbf{r}}_-, \quad (55)$$

$$\mathbf{B}(\mathbf{x}, t) = \mathbf{B}_+ + \mathbf{B}_- = \boldsymbol{\beta} \times (\mathbf{E}_+ - \mathbf{E}_-), \quad (56)$$

where $\mathbf{r}_\pm = \mathbf{x} - \mathbf{x}_\pm$.

At time $t = 0$, the electric field (55) is zero everywhere, which is a kind of destructive interference. However, the magnetic field (56) is nonzero everywhere, $\mathbf{B}(t = 0) = 2\mathbf{B}_+(t = 0)$. Hence, the energy $U_{\text{EM}} = \int (E^2 + B^2) d\text{Vol}/8\pi c$ of the electromagnetic field does not vanish at time $t = 0$, but remains spread out over all space.

If the charges somehow pass through one another at the origin and continued thereafter with uniform velocities, eqs. (55)-(56) would hold for all times.

If the charges annihilate one another at time $t = 0$, then for time $t > 0$, both \mathbf{E} and \mathbf{B} are zero within a sphere of radius ct about the origin, while outside this sphere eqs. (55)-(56) still hold.

¹⁶Comments on energy conservation in the closely related case of 3-dimensional waves from two sources are given, for example, in Y.-S. Hoh, *On the electromagnetic wave omnidirectional interference phenomena*, Am. J. Phys. **55**, 570 (1987), http://kirkmc.d.princeton.edu/examples/EM/hoh_ajp_55_570_87.pdf.

¹⁷For more details of this example, and references to papers that found it “paradoxical”, see <http://kirkmc.d.princeton.edu/examples/annihilate.pdf>

¹⁸The electromagnetic fields of a charge q with uniform velocity \mathbf{v} were first deduced in 1888 by Heaviside, http://kirkmc.d.princeton.edu/examples/EM/heaviside_electrician_22_147_88.pdf (and perhaps more accessibly by Thomson in 1889, http://kirkmc.d.princeton.edu/examples/EM/thomson_pm_28_1_89.pdf.)

$$\mathbf{E}(\mathbf{x}, t) = \frac{q}{\gamma^2 r^2 (1 - \beta^2 \sin^2 \phi)^{3/2}} \hat{\mathbf{r}}, \quad \mathbf{B} = \boldsymbol{\beta} \times \mathbf{E}, \quad \boldsymbol{\beta} = \frac{\mathbf{v}}{c}, \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad (54)$$

in Gaussian units, where $r = |\mathbf{r}|$ is the distance from the present position $\mathbf{x}_q(t)$ of the charge to that of the observer at \mathbf{x} , ϕ is the angle between \mathbf{r} and \mathbf{v} , and c is the speed of light in vacuum.