Diamagnetic Levitation

Robert H. Austin and Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(November 15, 2001; updated January 19, 2017)

1 Problem

Discuss the stability of levitation of a small diamagnetic sphere of radius b, mass m and (relative) permeability $\mu < 1$ in the magnetic field due to a horizontal disk of radius a and height $h \ll a$ that has uniform magnetization $\mathbf{M} = M\hat{\mathbf{z}}$ in the vertical direction. "Small" means that $b \ll a$.

Find the range of equilibrium heights z_0 above the plane of the magnetized disc for which the motion is stable against small perturbations.

2 Solution

The possibility of diamagnetic levitation was mentioned in the first theoretical paper on diamagnetism [1] (by W. Thomson in 1847 at age 23). See also [2].

Diamagnetic levitation was first observed in 1939 [3]. See also [4, 5]. A superconductor can be regarded as an extreme form of diamagnetism, with relative permeability $\mu = 0$; levitation of a superconductor was first observed in 1945 [6, 7].

This problem is closely related to that of the Levitron™, a well-known science toy [8, 9, 10, 11, 12].[1] The 2000 IgNobel Prize in Physics was awarded to Berry and Geim for their study of diamagnetic levitation [13]. See also [14, 15].

Other forms of levitation are possible, such as via eddy currents in liquid and solid metals induced by AC currents [16]. For general reviews, see [17, 18]. A recently discovered variant on diamagnetic levitation [19] is discussed in sec. 2.5.

See the Appendices for general remarks on stability of test objects in a static electric or magnetic field. In particular, Earnshaw's theorem [21] does not preclude stability of a diamagnetic object in a static magnetic field.

2.1 The Magnetic Field On the Axis of the Disk

We recall that the uniform magnetization $\mathbf{M}$ of the disk can be thought of a due to internal current loops whose net current density is zero everywhere inside the disk, but which leads to a surface current density $c\mathbf{M} \times d\mathbf{S}$ on surface area element $d\mathbf{S}$ (in Gaussian units). If $\mathbf{M}$ is vertical, the equivalent surface currents exist only on the vertical sides of the disk, and the total current circulating around the sides, of height h is,

$$I = cMh. \tag{1}$$

[1]A variant in which a magnetized disk is levitated between a pair of diamagnetic sheets is marketed at http://www.kjmagnetics.com/proddetail.asp?prod=LEV2

When height h is much less than the radius a of the disk, this current can be thought of as a line current when calculating the magnetic field.

The magnetic field on the axis of the disk is readily calculated using the Biot-Savart law,

$$B_z(0,z) = \frac{I}{c}\oint \frac{(d\mathbf{l}\times\mathbf{r})|_z}{r^3} = \frac{2\pi a I}{c}\frac{a}{r^3} = \frac{2\pi a^2 h M}{(a^2+z^2)^{3/2}}\,. \tag{2}$$

2.2 The Induced Magnetic Moment of the Diamagnetic Sphere

In the presence of an external magnetic field the diamagnetic sphere takes on a magnetic moment $\mathbf{m}$. In the present problem the radius of the sphere b is small compared to that of the characteristic spatial extent a of the magnetic field, so we take the external field to be uniform over the sphere.

We recall that the magnetization density $\mathbf{M}$ of a permeable sphere is uniform when the sphere is placed in a uniform external magnetic field. The magnetic moment $\mathbf{m}$ is related to the magnetization density by,

$$\mathbf{m} = \frac{4\pi}{3}b^3\mathbf{M}. \tag{3}$$

We also recall that the self magnetic field inside a uniform magnetized sphere is uniform, and that the self magnetic field outside the sphere is just that of the dipole $\boldsymbol{\mu}$.

One way to relate these quantities is to imagine the uniform magnetization $\mathbf{m}$ as causing (or being caused by) a surface density $\sigma = \mathbf{M}\cdot\hat{\mathbf{n}}$ of magnetic poles. Then, if we consider a Gaussian pillbox that encloses a small "polar cap" on the magnetized sphere, we learn that the self field $\mathbf{H}$ obeys,

$$H_{\rm out}(\mathrm{pole}) - H_{\rm in} = 4\pi\sigma = 4\pi M = \frac{3}{b^3}\left|\mathbf{m}\right|. \tag{4}$$

The self field outside the sphere is the dipole form,

$$\mathbf{H}_{\rm out} = \frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3}\,, \tag{5}$$

so the self field just above the pole is,

$$H_{\rm out}(\mathrm{pole}) = \frac{2\left|\mathbf{m}\right|}{b^3}\,. \tag{6}$$

Hence, eq. (4) tells us that,

$$\mathbf{H}_{\rm in} = -\frac{\mathbf{m}}{b^3}\,. \tag{7}$$

The internal self magnetic field $\mathbf{B}$ is also uniform, and is related by (in Gaussian units),

$$\mathbf{B}_{\rm in} = \mathbf{H}_{\rm in} + 4\pi\mathbf{M} = \mathbf{H}_{\rm in} + \frac{3\mathbf{m}}{b^3} = \frac{2\mathbf{m}}{b^3}\,. \tag{8}$$

The total fields inside the sphere are the sum of the internal fields and the external fields,

$$\mathbf{B} = \mathbf{B}_{\rm ext} + \mathbf{B}_{\rm in} = \mathbf{B}_{\rm ext} + \frac{2\mathbf{m}}{b^3}\,, \qquad \mathbf{H} = \mathbf{H}_{\rm ext} + \mathbf{H}_{\rm in} = \mathbf{B}_{\rm ext} - \frac{\mathbf{m}}{b^3}\,, \tag{9}$$

since the external field obeys $\mathbf{B}_{\rm ext} = \mathbf{H}_{\rm ext}$. Inside the permeable sphere we have $\mathbf{B} = \mu\mathbf{H}$, where μ is the (relative) magnetic permeability, so,

$$\mathbf{B}_{\rm ext} + \frac{2\mathbf{m}}{b^3} = \mu\left(\mathbf{B}_{\rm ext} - \frac{\mathbf{m}}{b^3}\right) . \tag{10}$$

Thus, the magnetic moment of the sphere is,

$$\mathbf{m} = \frac{\mu - 1}{\mu + 2} b^3 \mathbf{B}_{\rm ext}. \tag{11}$$

For a diamagnetic sphere, $\mu < 1$, so the induced magnetic moment is opposite in direction to that of the external field, as is to be expected from Lenz' law.

2.3 Conditions for Stability

To discuss the center of mass motion, we construct a potential energy $U(r, z)$. Then at an equilibrium point $\mathbf{F} = -\boldsymbol{\nabla} U = 0$, and the equilibrium is stable only if the energy increases for any small departure from equilibrium, *i.e.*, only if the second spatial derivatives are positive at the equilibrium.

The gravitational potential energy is just mgz, taking the z-axis as vertically upwards. The potential energy of a permeable object with magnetic dipole moment $\mathbf{m}$ (and no remnant magnetization) at rest in an external magnetic field $\mathbf{B}$ is $-\mathbf{m}\cdot\mathbf{B}/2$ (rather than $-\mathbf{m}\cdot\mathbf{B}$ as for a "permanent" magnetic moment). See, for example, sec. B.6 of [22]. In the present case, the moment is induced by the field according to eq. (11), so,

$$U(r, z) = mgz - \frac{\mathbf{m}\cdot\mathbf{B}(r, z)}{2} = mgz - \frac{1}{2}\frac{\mu - 1}{\mu + 2} B^2(r, z) . \tag{12}$$

For a circularly symmetric field, $\mathbf{B}(r, z)$, the equilibrium points will be on the z-axis of symmetry, and the condition that $(0, z_0)$ be an equilibrium point is,[2]

$$F_z = -\frac{\partial U(0, z_0)}{\partial z} = 0 = -mg + \frac{1}{2}\frac{\mu - 1}{\mu + 2}\frac{\partial B^2(0, z_0)}{\partial z} , \tag{13}$$

$$F_r = -\frac{\partial U(0, z_0)}{\partial r} = 0 = \frac{1}{2}\frac{\mu - 1}{\mu + 2}\frac{\partial B^2(0, z_0)}{\partial r} . \tag{14}$$

Conditions that this equilibrium be stable are,[3]

$$\frac{\partial^2 U(0, z_0)}{\partial z^2} = -\frac{1}{2}\frac{\mu - 1}{\mu + 2}\frac{\partial^2 B^2(0, z_0)}{\partial z^2} > 0 , \tag{15}$$

$$\frac{\partial^2 U(0, z_0)}{\partial r^2} = -\frac{1}{2}\frac{\mu - 1}{\mu + 2}\frac{\partial^2 B^2(0, z_0)}{\partial r^2} > 0 . \tag{16}$$

[2]Equations (13)-(14) indicate that the magnetic force on a diamagnetic object ($\mu < 1$) is towards regions of weaker magnetic field, as noted by Faraday in Art. 2269 of [23], in which he first identified diamagnetism. An implication is that a diamagnetic object could be levitated above a suitable magnet, as considered here, while a paramagnetic/ferromagnetic object (with $\mu > 1$) might be levitated/suspended below a magnet. The latter effect was first observed in 1937 [24]; but, as noted in Appendix B, such a suspension of a ferromagnetic needle is not fully stable, and a supplementary mechanism for horizontal stability is required.

[3]These conditions are necessary, but would not be sufficient if $\partial^2 U(0, z_0)/\partial x \partial z$ were nonzero, as mentioned in Appendix C. However, as indicated by eq. (30) below, this derivative is zero for the present example.

Since $\mu < 1$ for a diamagnetic sphere, the stability conditions are ,

$$\frac{\partial^2 B^2(0,z_0)}{\partial z^2} > 0\,, \tag{17}$$

$$\frac{\partial^2 B^2(0,z_0)}{\partial r^2} > 0\,. \tag{18}$$

2.4 Evaluation of the Field Derivatives

To complete the problem, we express the magnitude of the field B in terms of only its z-component, $B_z(0,z)$ from eq. (2). The approach is to use Maxwell's equations, $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ and $\boldsymbol{\nabla}\times\mathbf{B} = 0$, to relate B_r to B_z. From the above, we see that we will use only the first and second derivatives of B, so it suffices to use a series expansion to second order in r and z. Say,

$$B_z(r,z) = B_0 + B_1(z-z_0) + B_2(z-z_0)^2 + B_3 r + B_4 r^2 + B_5 r(z-z_0)\,, \tag{19}$$

and,

$$B_r(r,z) = C_0 + C_1(z-z_0) + C_2(z-z_0)^2 + C_3 r + C_4 r^2 + C_5 r(z-z_0)\,. \tag{20}$$

In cylindrical coordinates we have,

$$\boldsymbol{\nabla}\cdot\mathbf{B} = \frac{1}{r}\frac{\partial r B_r}{\partial r} + \frac{\partial B_z}{\partial z} = 0\,, \tag{21}$$

and,

$$(\boldsymbol{\nabla}\times\mathbf{B})_\phi = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = 0\,. \tag{22}$$

From eq. (21),

$$\begin{gathered}\left[C_0 + C_1(z-z_0) + C_2(z-z_0)^2\right]/r + \\ 2C_3 + 3C_4 r + 2C_5(z-z_0) + \\ B_1 + 2B_2(z-z_0) + B_5 r = 0\,,\end{gathered} \tag{23}$$

and so,

$$C_0 = C_1 = C_2 = 0, \quad C_3 = -\frac{B_1}{2}, \quad C_4 = -\frac{B_5}{3}, \quad C_5 = -B_2\,. \tag{24}$$

That is,

$$B_r(r,z) = -\frac{B_1 r}{2} - \frac{B_5 r^2}{3} - B_2 r(z-z_0)\,. \tag{25}$$

Then, from eq. (22),

$$-B_2 r - B_3 - 2B_4 r - B_5(z-z_0) = 0\,, \tag{26}$$

and hence,

$$B_3 = B_5 = 0, \quad B_4 = -\frac{B_2}{2}\,. \tag{27}$$

Altogether,

$$B_z(r,z) = B_0 + B_1(z-z_0) + B_2(z-z_0)^2 - \frac{B_2 r^2}{2}\,, \tag{28}$$

and,

$$B_r(r,z) = -\frac{B_1 r}{2} - B_2 r(z - z_0)\,, \tag{29}$$

accurate to second order in r and z.

To evaluate the equilibrium conditions (13)-(14) and the stability conditions (17)-(18) we need B^2 accurate to second order,

$$B^2 = B_z^2 + B_r^2 = B_0^2 + 2B_0B_1(z - z_0) + (B_1^2 + 2B_0B_2)(z - z_0)^2 + \left(\frac{B_1^2}{4} - B_0B_2\right) r^2. \tag{30}$$

The axial magnetic field is given by eq. (2),

$$B_z(0,z) = \frac{2\pi a^2 hM}{(a^2 + z^2)^{3/2}}\,. \tag{31}$$

Thus,

$$B_0 = B_z(0,z_0) = \frac{2\pi a^2 hM}{(a^2 + z_0^2)^{3/2}}\,, \tag{32}$$

$$B_1 = \frac{\partial B_z(0,z_0)}{\partial z} = -\frac{6\pi a^2 hMz_0}{(a^2 + z_0^2)^{5/2}}\,, \tag{33}$$

and,

$$B_2 = \frac{1}{2}\frac{\partial^2 B_z(0,z_0)}{\partial z^2} = \frac{3\pi a^2 hM(4z_0^2 - a^2)}{(a^2 + z_0^2)^{7/2}}\,. \tag{34}$$

The condition (13) for vertical equilibrium is that,

$$-2mg\frac{\mu + 2}{\mu - 1} = -\frac{\partial B^2(0,z_0)}{\partial z} = -2B_0B_1 = \frac{24\pi^2 a^4 h^2 M^2 z_0}{(a^2 + z_0^2)^4}\,, \tag{35}$$

The righthand side of eq. (35) reaches a maximum for $z_0 = a/\sqrt{7}$, so vertical equilibrium exists only if,

$$-mg\frac{\mu + 2}{\mu - 1} < \frac{1029\sqrt{7}\pi^2 h^2 M^2}{1024 a^3}\,. \tag{36}$$

When the equilibrium exists, there are always two solutions, one with $z_0 < a/\sqrt{7}$ and the other with $z_0 > a/\sqrt{7}$. We find below that only the solution with $z_0 > a/\sqrt{7}$ is stable.

The condition (14) for radial equilibrium is trivially satisfied at $r = 0$.

The condition (18) that the radial equilibrium be stable is that,

$$2\frac{\partial^2 B^2(0,z_0)}{\partial z^2} = B_1^2 - 4B_0B_2 = \frac{12\pi^2 a^4 h^2 M^2(2a^2 - 5z_0^2)}{(a^2 + z_0^2)^5} > 0, \tag{37}$$

which requires that $z_0 < a\sqrt{2/5}$.

The condition (17) that the vertical equilibrium be stable is that,

$$\frac{1}{2}\frac{\partial^2 B^2(0,z_0)}{\partial z^2} = B_1^2 + 2B_0B_2 = \frac{12\pi^2 a^4 h^2 M^2(7z_0^2 - a^2)}{(a^2 + z_0^2)^5} > 0, \tag{38}$$

which requires that $z_0 > a/\sqrt{7}$.

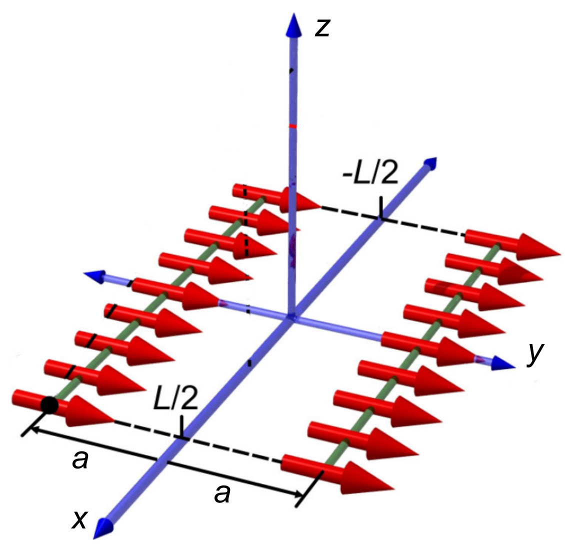
In sum, diamagnetic levitation is possible provided condition (36) is satisfied.[4] The equilibrium radius is, of course, zero, and the equilibrium height z_0 is given by eq. (35). The equilibrium is stable against both radial and vertical perturbations provided $a/\sqrt{7} = 0.38a < z_0 < \sqrt{2/5}a = 0.63a$. Surprisingly, in many cases there are two stable equilibria, one above $a/\sqrt{3} = 0.58a$ and one below.

We note that as the radius a of the magnetized disk goes to zero, the condition (37) for radial stability can no longer be satisfied. That is, the field of the magnetization directly under the diamagnetic sphere forces the sphere away from the axis of that magnetization; the destabilizing effect can be overcome by the field of magnetization at nonzero radius, which forces the sphere back towards the axis of the magnetized disk.

2.5 A Variant

In the present example, the external magnetic field was provided by a permanently magnetized disk which gives an upward force on a diamagnetic sphere above the disk; gravity provides a downward force, and levitation/trapping is possible at the point where the total force is zero. For large enough radius of the disk the horizontal force restores the diamagnetic sphere to the field axis.

A trapping configuration in which the permanent magnetization lies in a plane has recently been demonstrated in [19]. As shown below, two linear arrays of permanent magnetic dipoles in the horizontal plane provide the external magnetic field.[5]



Trapping in the horizontal (x-y) plane is provided by the magnetic field, but the field alone does not provide trapping in the vertical (z) direction, where a combination of gravity and magnetism can give trapping. The latter is a bit subtle, in that the vertical equilibrium point is far enough from the origin that a quadratic approximation to the energy U does not

[4]This condition cannot be satisfied for paramagnetic materials ($\mu > 1$), in agreement with the general conclusion of Appendix B.

[5]The magnetic field due to a long rong with transverse magnetization is deduced in [20].

suffice. See the Supplementary Material, particularly sec. D, of [19] for details.

Appendix A: Earnshaw's Theorem

In 1839, Earnshaw [21, 25, 26, 27, 28, 29, 30] noted the a stable equilibrium is impossible for an electric charge q in an external electrostatic field $\mathbf{E}_{\rm ext}$ in an otherwise charge-free region. Stability requires a restoring force for any small displacement of the charge from equilibrium (where $\mathbf{E}_{\rm ext} = 0$), *i.e.*, $\boldsymbol{\nabla} \cdot \mathbf{F} = \boldsymbol{\nabla} \cdot q\mathbf{E}_{\rm ext} < 0$, but in the otherwise charge-free region $\boldsymbol{\nabla} \cdot \mathbf{E}_{\rm ext} = 0$.

This argument appears to hold whether or not the external electric field is static, but a time-dependent electric field would be associated with a magnetic field that exerts additional force on the charge during small displacements from equilibrium, which might result in a stable equilibrium. Hence, Earnshaw's theorem is considered to apply only to static electric fields.

This theorem is often expressed in terms of the electric scalar potential $V_{\rm ext}$, where the static external electric field is related by $\mathbf{E}_{\rm ext} = -\boldsymbol{\nabla} V_{\rm ext}$, where in a charge-free region $\nabla^2 V_{\rm ext} = 0$. Then, a requirement for a stable equilibrium is that $\boldsymbol{\nabla} \cdot \mathbf{F} = -q\nabla^2 V_{\rm ext} < 0$, which is not satisfied by $V_{\rm ext}$.

Clearly, Earnshaw's argument also forbids a stable equilibrium for a magnetic charge (monopole) p in an external, static magnetic field $\mathbf{B}_{\rm ext}$ in an otherwise magnetic-charge-free region where $\boldsymbol{\nabla} \cdot \mathbf{B}_{\rm ext} = 0$.

Appendix B: Braunbek's Extension of Earnshaw's Theorem

In 1939, Braunbek [31] extended Earnshaw's theorem to include electric and magnetic dipoles, $\mathbf{p}$ and $\mathbf{m}$ respectively, as well as dielectric and diamagnetic objects.[6]

The torque $\mathbf{p} \times \mathbf{E}_{\rm ext}$ (or $\mathbf{m} \times \mathbf{B}_{\rm ext}$) on an electric (or magnetic) dipole in an external electric (or magnetic) field must be zero for equilibrium, so again the external field is zero at a possible equilibrium point. The force on an electric dipole in a static electric field is $\mathbf{F} = (\mathbf{p} \cdot \boldsymbol{\nabla})\mathbf{E}_{\rm ext} = \boldsymbol{\nabla}(\mathbf{p} \cdot \mathbf{E}_{\rm ext})$, so for small displacements of a dipole of fixed magnitude in which no rotation occurs, $\boldsymbol{\nabla} \cdot \mathbf{F} = \nabla^2(\mathbf{p} \cdot \mathbf{E}_{\rm ext}) = \mathbf{p} \cdot \nabla^2 \mathbf{E}_{\rm ext} = 0$, and the equilibrium is not stable.

We also consider a small object made of a linear dielectric material with (constant) relative permittivity ϵ, for which its electric dipole momentum is $\mathbf{p} = (\epsilon - 1)\mathbf{E}_{\rm ext}/4\pi$ in Gaussian units. Then, $\boldsymbol{\nabla} \cdot \mathbf{F} = \nabla^2(\mathbf{p} \cdot \mathbf{E}_{\rm ext}) = (\epsilon - 1)\nabla^2 E_{\rm ext}^2/4\pi$. At a point where $\nabla^2 V_{\rm ext} = 0$, $\nabla^2 E_{\rm ext}^2 = 2(\partial_i E_j)(\partial_i E_j) - 2(\mathbf{E}_{\rm ext} \cdot \boldsymbol{\nabla})\nabla^2 V_{\rm ext} \ge 0$, and stable equilibrium ($\boldsymbol{\nabla} \cdot \mathbf{F} < 0$) could be possible if $\epsilon < 1$. This is not the case for ordinary materials (and for static fields), so it is generally considered that there is no stable equilibrium for a dielectric object in a static electric field.[7]

---

[6] See also sec. 2 of [17].

[7] Artifical dielectrics [32] (now called metamaterials) exist with $\epsilon(\omega) < 1$, and more particularly with $\epsilon(\omega) < 0$, but only for wave fields (angular frequency $\omega > 0$). Such artificial dielectrics do not constitute an

However, when we turn to magnetic materials of relative permeability μ, a requirement for stability of a small object at a point where $\mathbf{B}_{\rm ext} = 0$ is that $\boldsymbol{\nabla} \cdot \mathbf{F} = (\mu - 1)\nabla^2 B_{\rm ext}^2/4\pi < 0$, which can be satisfied for magnetic fields with appropriate spatial variation and diamagnetic materials ($\mu < 1$), although not for paramagnetic/ferromagnetic materials ($\mu > 1$).[8]

Appendix C: Stability Requirement Based on Energy

When a system can be described by a (potential) "energy" $U(\mathbf{x})$, with "force" $\mathbf{F} = -\boldsymbol{\nabla} U$, an equilibrium point $\mathbf{x}_e$ where $\mathbf{F} = 0$ is stable against small displacements from equilibrium if the energy U is a local minimum at $\mathbf{x}_e$.[9]

A requirement for stability against small displacements in coordinate x_i is that $\partial^2 U(\mathbf{x}_e)/\partial x_i^2 > 0$. If $\nabla^2 U = \sum_i \partial^2 U/\partial x_i^2 = 0$, then stability is not possible, which is Earnshaw's theorem.

The requirement for stability against small displacements in any direction from an equilibrium point is that the eigenvalues of the Jacobian matrix, $J_{ij} = \partial^2 U(\mathbf{x}_e)/\partial x_i \partial x_j$, at the equilibrium point all have positive real parts.

If the Jacobian matrix is diagonal (as in the present problem), its eigenvalues are the diagonal elements $\partial^2 U/\partial x_i^2$, and the general criterion for stability reduces to the simpler case that $\partial^2 U(\mathbf{x}_e)/\partial x_i^2 > 0$ for each coordinate x_i.

If the (symmetric) Jacobian matrix is not diagonal, there exists a coordinate rotation about the point $\mathbf{x}_e$ to a set of coordinates $\mathbf{x}'$ for which the Jacobian matrix is diagonal at the equilibrium point, with eigenvalues $\partial^2 U(\mathbf{x}'_e)/\partial x'^2_i$ that must all be positive for stability. And, the eigenvalues of the Jacobian matrix are invariant under this rotation.

References

[1] W. Thomson, *On the Forces Experienced by Small Spheres under Magnetic Influence; and on Some of the Phenomena Presented by Diamagnetic Substances*, Camb. Dublin Math. J. **2**, 230 (1847), http://kirkmcd.princeton.edu/examples/EM/thomson_cdmj_2_230_47.pdf

[2] W. Thomson, *Remarks on the Forces experienced by inductively Magnetized Ferromagnetic or Diamagnetic Non-crystalline Substances*, Phil. Mag. **37**, 241 (1850), http://kirkmcd.princeton.edu/examples/EM/thomson_pm_37_241_50.pdf

[3] W. Braunbek, *Freies Schweben diamagnetischer Körper im Magnetfeld*, Z. Phys. **112**, 764 (1939), http://kirkmcd.princeton.edu/examples/EM/braunbek_zp_112_764_39.pdf

[4] A.H. Boerdijk, *Technical Aspects of Levitation*, Philips Res. Rep. **11**, 45 (1956), http://kirkmcd.princeton.edu/examples/EM/boerdijk_prr_11_45_56.pdf

[5] A.H. Boerdijk, *Levitation by static magnetic fields*, Philips Tech. Rev. **18**, 125 (1957), http://kirkmcd.princeton.edu/examples/EM/boerdijk_ptr_18_125_57.pdf

exception to the Braunbek/Earnshaw's theorem for small homogeneous objects in static fields.

[8] Ferromagnetic materials can be levitated in a time-dependent magnetic field [24, 33, 34].

[9] This criterion for stability was given by Lagrange, and notably elaborated upon by Lyapunov [35].

[6] V. Arkadjev, *Hovering of a Magnet over a Superconductor*, J. Phys. (USSR) **9**, 148 (1945), http://kirkmcd.princeton.edu/examples/EM/arkadjev_jpussr_9_148_45.pdf

[7] V. Arkadiev, *A Floating Magnet*, Nature **160**, 30 (1947), http://kirkmcd.princeton.edu/examples/EM/arkadiev_nature_160_30_47.pdf

[8] The Levitron^{TM} is marketed by Fascinations Inc, 19224 Des Moines Way South, Suite 100, Seattle, WA 98148, http://www.levitron.com

[9] M. and K. Sherlock, *The Hidden History of the Levitron*, http://www.amasci.com/maglev/lev/expose.html

[10] M.V. Berry, *The Levitron^{TM}: an adiabatic trap for spins*, Proc. Roy. Soc. London A **452**, 1207 (1996), http://kirkmcd.princeton.edu/examples/EM/berry_prsla_452_1207_96.pdf

[11] M.D. Simon, L.O. Heflinger and S.L. Ridgway, *Spin stabilized magnetic levitation*, Am. J. Phys. **65**, 286 (1997), http://www.physics.ucla.edu/marty/levitron/ http://kirkmcd.princeton.edu/examples/EM/simon_ajp_65_286_97.pdf

[12] K.T. McDonald, *The Levitron^{TM}* (April 4, 1997), http://kirkmcd.princeton.edu/examples/levitron.pdf

[13] M.V. Berry and A.K. Geim, *Of flying frogs and levitrons*, Eur. J. Phys. **18**, 307 (1997), http://kirkmcd.princeton.edu/examples/EM/berry_ejp_18_307_97.pdf
See also, http://www.phy.bris.ac.uk/people/Berry_mv/igberry.html

[14] A.K. Geim *et al.*, *Magnet levitation at your fingertips*, Nature **400**, 323 (1999), http://kirkmcd.princeton.edu/examples/EM/geim_nature_400_323_99.pdf

[15] M.D. Simon and A.K. Geim, *Diamagnetic levitation: Flying frogs and floating magnets*, J. Appl. Phys. **87**, 6200 (2000), http://kirkmcd.princeton.edu/examples/EM/simon_jap_87_6200_00.pdf

[16] E.C. Okress *et al.*, *Electromagnetic Levitation of Solid and Molten Metals* , J. Appl. Phys. **23**, 545 (1952), http://kirkmcd.princeton.edu/examples/EM/okress_jap_23_545_52.pdf

[17] B.V. Jayawant, *Electromagnetic suspension and levitation*, Rep. Prog. Phys. **44**, 411 (1981), http://kirkmcd.princeton.edu/examples/EM/jayawant_rpp_44_411_81.pdf

[18] E.H. Brandt, *Levitation in Physics*, Science **243**, 349 (1989), http://kirkmcd.princeton.edu/examples/EM/brandt_science_243_349_89.pdf

[19] O. Gunawan, Y. Virgus and K.-F. Tai, *A parallel dipole line system*, Appl. Phys. Lett. **106**, 062407 (2015), http://kirkmcd.princeton.edu/examples/EM/gunawan_apl_106_062407_15.pdf

[20] K.T. McDonald, *Long Rod with Uniform Magnetization Transverse to its Axis* (Nov. 3, 1999), http://kirkmcd.princeton.edu/examples/magrod.pdf

[21] S. Earnshaw, *On the Nature of the Molecular Forces which regulate the Constitution of the Luminiferous Ether*, Trans. Camb. Phil. Soc. **7**, 97 (1842), http://kirkmcd.princeton.edu/examples/EM/earnshaw_tcps_7_97_39.pdf

[22] F.J. Castro Paredes and K.T. McDonald, *A Paradox Concerning the Energy of a Dipole in a Uniform External Field* (May 3, 2004), http://kirkmcd.princeton.edu/examples/dipoleparadox.pdf

[23] M. Faraday, *Experimental Researches in Electricity.—Twentieth Series*, Phil. Trans. Roy. Soc. London **136**, 21 (1846), http://kirkmcd.princeton.edu/examples/EM/faraday_ptrsl_136_21_46.pdf

[24] F.T. Holmes, *Axial Magnetic Suspensions*, Rev. Sci. Instr. **8**, 444 (1937), http://kirkmcd.princeton.edu/examples/EM/holmes_rsi_8_444_37.pdf

[25] W. Thomson, *Demonstration of a Fundamental Proposition in the Mechanical Theory of Electricity*, Camb. Dublin Math. J. **4**, 223 (1845), http://kirkmcd.princeton.edu/examples/EM/thomson_cmj_4_223_45.pdf

[26] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, 3rd ed., vol. I, Art. 116 (Clarendon Press, 1904), http://kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v1_04.pdf

[27] Lord Kelvin and P.G. Tait, *Treatise on Natural Philosophy*, Part II, p. 50 (Cambridge U. Press, 1903), http://kirkmcd.princeton.edu/examples/mechanics/thomson_tait_treatise_v2.pdf

[28] L. Tonks, *Notes on Earnshaw's Theorem*, Elec. Eng. **69**, 118 (1940), http://kirkmcd.princeton.edu/examples/EM/tonks_ee_59_118_40.pdf

[29] W.T. Scott, *Who was Earnshaw?* Am. J. Phys. **27**, 418 (1959), http://kirkmcd.princeton.edu/examples/EM/scott_ajp_27_418_59.pdf

[30] R. Bassani, *Earnshaw (1805-1888) and Passive Magnetic Levitation*, Meccanica **41**, 375 (2006), http://kirkmcd.princeton.edu/examples/EM/bassani_meccanica_41_375_06.pdf

[31] W. Braunbek, *Freischwebende Körper im elektrischen und magnetischen Feld*, Z. Phys. **112**, 753 (1939), http://kirkmcd.princeton.edu/examples/EM/braunbek_zp_112_753_39.pdf

[32] J. Brown, *Artifical Dielectrics Having Refractive Indices Less Than Unity*, Proc. IEE **100**, 51 (1953), http://kirkmcd.princeton.edu/examples/EM/brown_piee_100_51_53.pdf

[33] W. Kemper, *Schwebende Aufhängung durch elektromagnetische Kräfte: eine Möglichkeit für eine grundsätzlich neue Fortbewegungsart*, Elek. Z. **59**, 391 (1938), http://kirkmcd.princeton.edu/examples/EM/kemper_ez_59_391_38.pdf

[34] B.D. Bedford, L.H.B. Peer and L. Tonks, *The Electromagnetic Levitator*, Gen. Elec. Rev. **42**, 246 (1939), http://kirkmcd.princeton.edu/examples/EM/bedford_ger_42_246_39.pdf

[35] A.M. Lyapunov, *The General Problem of the Stability of Motion* (in Russian), Doctoral dissertation (U. Kharkov, 1892). English translations: *Stability of Motion* (Academic Press, 1966), *The General Problem of the Stability of Motion*, A.T. Fuller, trans. (Taylor and Francis, 1992).