Momentum of a Charged Particle Orbiting a Magnetic Dipole

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1 Problem

Discuss the momentum in a system consisting of an electric charge e and mass m that moves with speed $v \ll c$ in a circular orbit in the symmetry plane of the magnetic field of an Ampèrian magnetic dipole moment $\boldsymbol{\mu}$, where c is the speed of light in vacuum.

2 Solution

This problem was suggested by Mike Romalis.

The magnetic field of the magnetic moment $\boldsymbol{\mu}$ in a frame with the moment at the origin is (in Gaussian units),

$$\mathbf{B} = \frac{3(\boldsymbol{\mu} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \boldsymbol{\mu}}{r^3} . \tag{1}$$

The equation of motion of the electric charge at $r\,\hat{\mathbf{r}}$ that moves with low velocity v in a circular orbit of radius r is,

$$\mathbf{F} = -\frac{mv^2}{r}\,\hat{\mathbf{r}} = e\frac{\mathbf{v}}{c} \times \mathbf{B} = e\frac{\mathbf{v}}{c} \times -\frac{\boldsymbol{\mu}}{r^3} = \frac{ev\mu}{cr^3}\,\hat{\boldsymbol{\mu}} \times \hat{\mathbf{v}}. \tag{2}$$

Hence, the mechanical momentum of the orbiting charge is,

$$\mathbf{P}_{\rm mech} = m\mathbf{v} = \frac{e\mu}{cr^2}\,\hat{\mathbf{v}} = \frac{e\mu}{cr^2}\,\hat{\boldsymbol{\mu}} \times \hat{\mathbf{r}}. \tag{3}$$

Assuming that the magnetic dipole $\boldsymbol{\mu}$ is Ampèrian, meaning that it is due to electric currents rather than to hypothetical magnetic "poles", the electromagnetic field momentum of the magnetic moment plus charge can be written in the quasistatic approximation as,[1]

$$\mathbf{P}_{\rm EM} = \frac{\mathbf{E}(r=0) \times \boldsymbol{\mu}}{c} = -\frac{e\,\hat{\mathbf{r}}}{r^2} \times \frac{\boldsymbol{\mu}}{c} = \frac{e\mu}{cr^2}\,\hat{\boldsymbol{\mu}} \times \hat{\mathbf{r}} = \frac{e\mu}{cr^2}\,\hat{\mathbf{v}} = \mathbf{P}_{\rm mech}. \tag{4}$$

Thus, the total momentum is twice the mechanical momentum of the orbiting charge in the frame in which the magnetic moment $\boldsymbol{\mu}$ is at rest.

[1]See eq. (37) of [1], eq. (71) of [2] and eq. (37) of [3].

3 Betatron with Magnetic Dipole at the Origin

A variant of the preceding is to suppose that the electron orbits in the plane $z = 0$ inside a betatron[2] whose magnetic field is zero at the origin, where a magnetic dipole $\boldsymbol{\mu} = \mu\,\hat{\mathbf{z}}$ is located. In this case the energy delivered by the betatron to the system of (relativistic) electron plus magnetic dipole includes the increase in the energy $\gamma m_0 c^2$ of the electron (which can be regarded as the sum of mechanical plus electromagnetic energies), where $\gamma = 1/\sqrt{1 - v^2/c^2}$ and m_0 is the rest mass of the electron, as well as the interaction energy,

$$U_{\rm int} = -\boldsymbol{\mu} \cdot \mathbf{B}_{\rm electron} \approx -\frac{\gamma e \mu}{r^2}\,, \tag{5}$$

where $-e$ is the charge of the electron, and we suppose that the acceleration of the electron is weak. The minus sign in eq. (5) follows by supposing that the direction of the velocity $\mathbf{v}$ of the electron is such that it would be in a circular orbit about the magnetic dipole in the absence of the betatron magnetic field.

The energy of the electron plus magnetic dipole can be written as,

$$U = \gamma \left(m_0 c^2 - \frac{e\mu}{r^2} \right) = \gamma m_{\rm eff} c^2, \qquad m_{\rm eff} \equiv m_0 - \frac{e\mu}{r^2 c^2}\,. \tag{6}$$

That is, the system of electron plus magnetic dipole can be regarded as a kind of quasiparticle with effective mass, as in eq. (6), that is less that the mass of a "free" electron.[3]

References

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[3] K.T. McDonald, *Mansuripur's Paradox* (May 2, 2012),
http://kirkmcd.princeton.edu/examples/mansuripur.pdf

[4] K.T. McDonald, *Stability of Transverse Oscillations in a Betatron* (Nov. 26, 2012),
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[5] K.T. McDonald and K. Shmakov, *Temporary Acceleration of Electrons While Inside an Intense Electromagnetic Pulse* (Nov. 20, 1997), Phys. Rev. STAB **2**, 121301 (1999),
http://kirkmcd.princeton.edu/accel/acceleration2.pdf

[2]See, for example, [4].

[3]This contrasts with the case of an electron inside an intense laser pulse, where the effective mass is larger than the electron rest mass. See, for example, [5].