Classical "Dressing" of a Free Electron in a Plane Electromagnetic Wave

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Abstract

The energy and momentum densities of the fields of a free electron in a plane electromagnetic wave include interference terms that are the classical version of the "dressing" of the electron the arises in a quantum analysis. The transverse mechanical momentum of the oscillating electron is balanced by the field momentum resulting from the interference between the driving wave and the static part of the electron's field. The interference between the wave and the oscillating part of the electron's field leads to a longitudinal field momentum and a negative field energy that compensate for the longitudinal momentum and kinetic energy of the electron. The interference terms are dominated by the near zone, so that as the wave passes the electron by the latter reverts to its energy and momentum prior to the arrival of the wave.

1 Introduction

The behavior of a free electron in a electromagnetic wave is one of the most commonly discussed topics in classical electromagnetism. Yet, several basic issues remain to be clarified. These relate to the question: to what extent can net energy be transferred from an electromagnetic pulse (such as that of a laser) in vacuum to a free electron?

These issues are made more complex by quantum considerations, including the role of the "quasimomentum" and the "effective mass" of an electron that is "dressed" by an electromagnetic wave [1].

As a small step towards understanding of the larger issues, we consider a simpler question here. The response of a free electron to a plane electromagnetic wave is oscillatory motion in the plane perpendicular to the direction of the wave, in the first approximation. Thus, the electron has momentum transverse to the direction of the wave. However, the wave contains momentum only in its direction, and the radiated wave contains no net momentum (in the nonrelativistic limit). How is momentum conserved in this process?

The general sense of the answer has been given by Poynting [2], who noted that an electromagnetic field can be said to contain a flux of energy (energy per unit area per unit time) given by

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B}, \tag{1}$$

in Gaussian units, where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field (taken to be in vacuum throughout this paper) and c is the speed of light.

J.J. Thomson [3, 4, 5] (and later Poincaré [6]) noted that this flow of energy can also be associated with a momentum density given by

$$\mathbf{P}_{\rm field} = \frac{\mathbf{S}}{c^2} = \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\,, \tag{2}$$

Hence, in the case of a free electron in a plane electromagnetic wave we are led to seek an electromagnetic field momentum that is equal and opposite to the mechanical momentum of the electron.

In this paper we demonstrate that indeed the mechanical momentum of the oscillating electron is balanced by the field momentum in the interference term between the incident wave and the static field of the electron. We are left with some subtleties when we consider the interference between the incident wave and the oscillating field of the electron.

2 Generalities

2.1 Motion of an Electron in a Plane Wave

We consider a plane electromagnetic wave that propagates in the $+z$ direction of a rectangular coordinate system. A fairly general form of this wave is

$$\begin{aligned}\mathbf{E}_{\rm wave} &= E_x\cos(kz-\omega t)\,\hat{\mathbf{x}} - E_y\sin(kz-\omega t)\,\hat{\mathbf{y}},\\ \mathbf{B}_{\rm wave} &= E_y\sin(kz-\omega t)\,\hat{\mathbf{x}} + E_x\cos(kz-\omega t)\,\hat{\mathbf{y}},\end{aligned} \tag{3}$$

where $\omega = kc$ is the angular frequency of the wave, $k = 2\pi/\lambda$ is the wave number and $\hat{\mathbf{x}}$ is a unit vector in the x direction, *etc.* When either E_x or E_y is zero we have a linearly polarized wave, while for $E_x = \pm E_y$ we have circular polarization.

A free electron of mass m oscillates in this field such that its average position is at the origin. This simple statement hides the subtlety that our frame of reference is the average rest frame of the electron when inside the wave, and is not the lab frame of an electron that is initially at rest, but which is overtaken by a wave[7, 8, 9]. If the velocity of the oscillating electron is small, we can ignore the $\mathbf{v}/c\times\mathbf{B}$ force and take the motion to be entirely in the plane $z = 0$. Then, (also ignoring radiation damping) the equation of motion of the electron is

$$m\,\ddot{\mathbf{x}} = e\mathbf{E}_{\rm wave}(0,t) = e(E_x\cos\omega t\,\hat{\mathbf{x}} + E_y\sin\omega t)\,\hat{\mathbf{y}}. \tag{4}$$

Using eq. (3) we find the position of the electron to be

$$\mathbf{x} = -\frac{e}{m\omega^2}(E_x\cos\omega t\,\hat{\mathbf{x}} + E_y\sin\omega t)\,\hat{\mathbf{y}}, \tag{5}$$

and the mechanical momentum of the electron is

$$\mathbf{p}_{\rm mech} = m\,\dot{\mathbf{x}} = \frac{e}{\omega}(E_x\sin\omega t\,\hat{\mathbf{x}} - E_y\cos\omega t)\,\hat{\mathbf{y}}. \tag{6}$$

The root-mean-square (rms) velocity of the electron is

$$v_{\rm rms} = \sqrt{\langle\dot{x}^2+\dot{y}^2\rangle} = \frac{e}{m\omega}\sqrt{\frac{E_x^2+E_y^2}{2}} = \frac{eE_{\rm rms}}{m\omega c}c. \tag{7}$$

The condition that the $\mathbf{v}/c \times \mathbf{B}$ force be small is then

$$\eta \equiv \frac{eE_{\rm rms}}{m\omega c} \ll 1, \tag{8}$$

where the dimensionless measure of field strength, η, is a Lorentz invariant. Similarly, the rms departure of the electron from the origin is

$$x_{\rm rms} = \frac{eE_{\rm rms}}{m\omega^2} = \frac{\eta\lambda}{2\pi}. \tag{9}$$

Thus, condition (8) also insures that the extent of the motion of the electron is small compared to a wavelength, and so we may use the dipole approximation when considering the fields of the oscillating electron.

In the weak-field approximation, we can now use (6) for the velocity to evaluate the second term of the Lorentz force:

$$e\frac{\mathbf{v}}{c} \times \mathbf{B} = \frac{e^2(E_x^2 - E_y^2)}{2m\omega c} \sin 2\omega t\, \hat{\mathbf{z}}. \tag{10}$$

This term vanishes for circular polarization, in which case the motion is wholly in the transverse plane. However, for linear polarization the $\mathbf{v}/c \times \mathbf{B}$ force leads to oscillations along the z axis at frequency 2ω, as first analyzed in general by Landau [10]. For polarization along the x-axis, the x-z motion has the form of a "figure 8", which for weak fields ($\eta \ll 1$) is described by

$$x = -\frac{eE_x}{m\omega^2} \cos\omega t, \qquad z = -\frac{e^2E_x^2}{8m^2\omega^3 c} \sin 2\omega t. \tag{11}$$

If the electron had been at rest before the arrival of the plane wave, then inside the wave it would move with an average drift velocity given by

$$v_z = \frac{\eta^2/2}{1+\eta^2/2} c, \tag{12}$$

along the direction of the wave vector, as first deduced by McMillan [11]. In the present paper we work in the frame in which the electron has no average velocity along the z axis. Therefore, prior to its encounter with the plane wave the electron had been moving in the negative z direction with speed given by (12).

2.2 Field Momentum

The fields associated with the electron can be regarded as the superposition of those of an electron at rest at the origin plus those of a dipole consisting of the actual oscillating electron and a positron at rest at the origin. Thus, we can write the electric field of the electron as $\mathbf{E}_{\rm static} + \mathbf{E}_{\rm osc}$ and the magnetic field as $\mathbf{B}_{\rm osc}$, where the oscillating fields have the pure frequency ω in the low-velocity limit.

The entire electromagnetic momentum density can then be written

$$\mathbf{P}_{\rm field} = \frac{(\mathbf{E}_{\rm wave} + \mathbf{E}_{\rm static} + \mathbf{E}_{\rm osc}) \times (\mathbf{B}_{\rm wave} + \mathbf{B}_{\rm osc})}{4\pi c}. \tag{13}$$

However, in seeking the field momentum that opposes the mechanical momentum of the electron, we should not include either of the self-momenta $\mathbf{E}_{\rm wave} \times \mathbf{B}_{\rm wave}$ or $(\mathbf{E}_{\rm static} + \mathbf{E}_{\rm osc}) \times \mathbf{B}_{\rm osc}$. The former is independent of the electron, while the latter can be considered as a part of the mechanical momentum of the electron according to the concept of "renormalization".

We therefore restrict our attention to the interaction field momentum

$$\mathbf{P}_{\rm int} = \mathbf{P}_{\rm wave,static} + \mathbf{P}_{\rm wave,osc}, \tag{14}$$

where

$$\mathbf{P}_{\rm wave,static} = \frac{\mathbf{E}_{\rm static} \times \mathbf{B}_{\rm wave}}{4\pi c}. \tag{15}$$

and

$$\mathbf{P}_{\rm wave,osc} = \frac{\mathbf{E}_{\rm wave} \times \mathbf{B}_{\rm osc} + \mathbf{E}_{\rm osc} \times \mathbf{B}_{\rm wave}}{4\pi c}. \tag{16}$$

We recall from eqs. (6) and (11) that the transverse mechanical momentum of the oscillating electron has pure frequency ω. Since the wave and the oscillating part of the electron's field each have frequency ω, the term $\mathbf{P}_{\rm wave,osc}$ contains harmonic functions of ω^2, which can be resolved into a static term plus ones in frequency 2ω. Hence, we should not expect this term to cancel the mechanical momentum. Rather, we look to the term $\mathbf{P}_{\rm wave,static}$, since this has pure frequency ω.

3 The Momentum $\mathbf{P}_{\rm wave,static}$

The static field of the electron at the origin is, in rectangular coordinates,

$$\mathbf{E}_{\rm static} = \frac{e}{r^3}(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}), \tag{17}$$

where r is the distance from the origin to the point of observation. Combing this with eq. (3) we have

$$\begin{aligned} \mathbf{P}_{\rm wave,static} &= \frac{e}{4\pi c r^3}\{-zE_x \cos(kz-\omega t)\,\hat{\mathbf{x}} + zE_y \sin(kz-\omega t)\,\hat{\mathbf{y}} \\ &\qquad + [xE_x \cos(kz-\omega t) - yE_y \cos(kz-\omega t)]\,\hat{\mathbf{z}}\}. \end{aligned} \tag{18}$$

When we integrate this over all space to find the total field momentum, the terms in $\hat{\mathbf{z}}$ vanish as they are odd in either x or y. Likewise, after expanding the cosine and sine of $kz - \omega t$, the terms proportional to $z\cos kz$ vanish on integration. The remaining terms are thus

$$\begin{aligned} \mathbf{p}_{\rm wave,static} &= \int \mathbf{P}_{\rm wave,static} \\ &= \frac{e}{4\pi c}(-E_x \sin\omega t\,\hat{\mathbf{x}} + E_y \cos\omega t\,\hat{\mathbf{y}}) \int \frac{z\sin kz}{r^3}\, d\mathrm{Vol} \\ &= \frac{e}{\omega}(-E_x \sin\omega t\,\hat{\mathbf{x}} + E_y \cos\omega t\,\hat{\mathbf{y}}) = -\mathbf{p}_{\rm mech}, \end{aligned} \tag{19}$$

after an elementary volume integration.

It is noteworthy that the integration is independent of any hypothesis as to the size of a classical electron. Indeed, the integrand of (19) can be expressed as $\cos\theta \sin(kr\cos\theta)/r^2$ via the substitution $z = r\cos\theta$. Hence, the integral over a spherical shell is independent of r for $kr \ll 1$, and significant contributions to the integral occur for radii up to one wavelength of the electromagnetic wave. This contrasts with the self-momentum density of the electron which is formally divergent; if the integration is cut off at a minimum radius (the classical electron radius), the dominant contribution occurs within twice that radius.

Thus, we have demonstrated the principal result of this paper.

4 The Momentum $\mathbf{P}_{\text{wave,osc}}$

Several subtleties in the argument appear when we consider the other interference term in the momentum density (14). For this, we must first display the electromagnetic fields of an oscillating electron.

4.1 The Fields $\mathbf{E}_{\text{osc}}$ and $\mathbf{B}_{\text{osc}}$

Since we restrict our attention to an electron that oscillates with amplitude much less than a wavelength of the driving wave, and the electron attains velocities that are much less than the speed of light, it is sufficient to use the dipole approximation to the fields of the electron. While these fields are well known, they are typically presented in complex notation, of which only the real part has physical significance. This notation is very useful for discussions in which only time-averaged behavior is of interest. However, we wish to consider the details of momentum balance at an arbitrary moment, and it is preferable to use purely real notation.

We begin by noting that the retarded vector potential of the oscillating electron at a point $\mathbf{r}$ at time t can be written

$$\begin{aligned} \mathbf{A}_{\text{osc}}(\mathbf{r},t) &= \frac{e}{c}\frac{\dot{\mathbf{x}}(t'=t-r/c)}{r} \\ &= -\frac{e^2}{m\omega c r}[E_x \sin(kr-\omega t)\,\hat{\mathbf{x}} + E_y \cos(kr-\omega t)\,\hat{\mathbf{y}}], \end{aligned} \tag{20}$$

using eq. (5) for the motion $\mathbf{x}$ of the electron. The oscillating part of the scalar potential is obtained by integration of the Lorenz gauge condition:

$$\nabla\cdot\mathbf{A}_{\text{osc}} + \frac{1}{c}\frac{\partial\phi_{\text{osc}}}{\partial t} = 0. \tag{21}$$

We find

$$\begin{aligned} \phi_{\text{osc}} &= -\frac{e^2}{m\omega^2}\left\{E_x\left[\frac{kx}{r^2}\sin(kr-\omega t) + \frac{x}{r^3}\cos(kr-\omega t)\right]\right. \\ &\quad \left. + E_y\left[\frac{ky}{r^2}\cos(kr-\omega t) - \frac{y}{r^3}\sin(kr-\omega t)\right]\right\}. \end{aligned} \tag{22}$$

The constant static potential is omitted in the above.

The scalar potential could also be deduced from the retarded potential of a moving charge. Equation (22) results on expanding the retarded distance to first order in the field strength of the plane wave.

The electric and magnetic fields are, of course, found from the potentials via

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \text{and} \qquad \mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}. \tag{23}$$

The lengthy expressions for the rectangular components of the fields are

$$\begin{aligned}
B_{\mathrm{osc},x} &= -\frac{e^2 E_y}{m\omega^2}\left[\frac{k^2 z}{r^2}\sin(kr-\omega t) + \frac{kz}{r^3}\cos(kr-\omega t)\right], \\
B_{\mathrm{osc},y} &= -\frac{e^2 E_x}{m\omega^2}\left[\frac{k^2 z}{r^2}\cos(kr-\omega t) - \frac{kz}{r^3}\sin(kr-\omega t)\right], \\
B_{\mathrm{osc},z} &= \frac{e^2 E_x}{m\omega^2}\left[\frac{k^2 y}{r^2}\cos(kr-\omega t) - \frac{ky}{r^3}\sin(kr-\omega t)\right] \\
&+ \frac{e^2 E_y}{m\omega^2}\left[\frac{k^2 x}{r^2}\sin(kr-\omega t) + \frac{kx}{r^3}\cos(kr-\omega t)\right],
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
E_{\mathrm{osc},x} &= -\frac{e^2 E_x}{m\omega^2}\left[\left(\frac{3kx^2}{r^4} - \frac{k}{r^2}\right)\sin(kr-\omega t) + \left(\frac{k^2}{r} - \frac{k^2x^2}{r^3} + \frac{3x^2}{r^5} - \frac{1}{r^3}\right)\cos(kr-\omega t)\right] \\
&\quad -\frac{e^2 E_y}{m\omega^2}\left[\frac{3kxy}{r^4}\cos(kr-\omega t) + \left(\frac{k^2xy}{r^3} - \frac{3xy}{r^5}\right)\sin(kr-\omega t)\right], \\
E_{\mathrm{osc,y}} &= -\frac{e^2 E_x}{m\omega^2}\left[\frac{3kxy}{r^4}\sin(kr-\omega t) - \left(\frac{k^2xy}{r^3} - \frac{3xy}{r^5}\right)\cos(kr-\omega t)\right] \\
&\quad -\frac{e^2 E_y}{m\omega^2}\left[\left(\frac{3ky^2}{r^4} - \frac{k}{r^2}\right)\cos(kr-\omega t) - \left(\frac{k^2}{r} - \frac{k^2y^2}{r^3} + \frac{3y^2}{r^5} - \frac{1}{r^3}\right)\sin(kr-\omega t)\right], \\
E_{\mathrm{osc,z}} &= -\frac{e^2 E_x}{m\omega^2}\left[\frac{3kxz}{r^4}\sin(kr-\omega t) - \left(\frac{k^2xz}{r^3} - \frac{3xz}{r^5}\right)\cos(kr-\omega t)\right] \\
&\quad -\frac{e^2 E_y}{m\omega^2}\left[\frac{3kyz}{r^4}\cos(kr-\omega t) + \left(\frac{k^2yz}{r^3} - \frac{3yz}{r^5}\right)\sin(kr-\omega t)\right].
\end{aligned} \tag{25}$$

These expressions can also be deduced from the Liénard-Wiechert forms for the fields of an accelerated charge, keeping terms only to first order in the strength of the plane wave.

4.2 Components of $\mathbf{P}_{\mathrm{wave,osc}}$

Since the wave fields have no z component, the x component of $\mathbf{P}_{\mathrm{wave,osc}}$ is given by

$$P_{\mathrm{wave,osc},x} = \frac{E_{\mathrm{wave},y}B_{\mathrm{osc},z} - E_{\mathrm{osc},z}B_{\mathrm{wave},y}}{4\pi c}. \tag{26}$$

From eqs. (24) and (25) we see that both $B_{\mathrm{osc},z}$ and $E_{\mathrm{osc},z}$ are odd in either x or y. Therefore, the volume integral of $P_{\mathrm{wave,osc},x}$ vanishes, and we do not consider it further. Likewise,

$P_{\mathrm{wave,osc},y}$ vanishes on integration. This confirms the claim made at the end of sec. II that the interference term $\mathbf{P}_{\mathrm{wave,osc}}$ is not relevant to the balance of transverse momentum between the electron and the fields.

However, the z component of $\mathbf{P}_{\mathrm{wave,osc}}$ does not vanish on integration, and requires further discussion. As the details include some surprises (to the authors), we present them at length.

$$
\begin{aligned}
& P_{\mathrm{wave,osc},z} = \\
& \frac{E_{\mathrm{w},x}B_{\mathrm{o},y} - E_{\mathrm{w},y}B_{\mathrm{o},x} + E_{\mathrm{o},x}B_{\mathrm{w},y} - E_{\mathrm{o},y}B_{\mathrm{w},x}}{4\pi c} = \\
- \quad & \frac{e^2E_x^2\cos(kz-wt)}{4\pi m\omega^2 c}\Big[\frac{k^2z}{r^2}\cos(kr-\omega t) - \frac{kz}{r^3}\sin(kr-\omega t)\Big] \\
- \quad & \frac{e^2E_y^2\sin(kz-wt)}{4\pi m\omega^2 c}\Big[\frac{k^2z}{r^2}\sin(kr-\omega t) + \frac{kz}{r^3}\cos(kr-\omega t)\Big] \\
- \quad & \frac{e^2E_x^2\cos(kz-wt)}{4\pi m\omega^2 c}\Big[\left(\frac{3kx^2}{r^4} - \frac{k}{r^2}\right)\sin(kr-\omega t) + \left(\frac{k^2}{r} - \frac{k^2x^2}{r^3} + \frac{3x^2}{r^5} - \frac{1}{r^3}\right)\cos(kr-\omega t)\Big] \\
- \quad & \frac{e^2E_xE_y\cos(kz-wt)}{4\pi m\omega^2 c}\Big[\frac{3kxy}{r^4}\cos(kr-\omega t) + \left(\frac{k^2xy}{r^3} - \frac{3xy}{r^5}\right)\sin(kr-\omega t)\Big] \qquad (27) \\
- \quad & \frac{e^2E_xE_y\sin(kz-wt)}{4\pi m\omega^2 c}\Big[-\frac{3kxy}{r^4}\sin(kr-\omega t) + \left(\frac{k^2xy}{r^3} - \frac{3xy}{r^5}\right)\cos(kr-\omega t)\Big] \\
+ \quad & \frac{e^2E_y^2\sin(kz-wt)}{4\pi m\omega^2 c}\Big[\left(\frac{3ky^2}{r^4} - \frac{k}{r^2}\right)\cos(kr-\omega t) - \left(\frac{k^2}{r} - \frac{k^2y^2}{r^3} + \frac{3y^2}{r^5} - \frac{1}{r^3}\right)\sin(kr-\omega t)\Big].
\end{aligned}
$$

The terms of $P_{\mathrm{wave,osc},z}$ that are proportional to E_xE_y are odd on both x and y, and so will vanish on integration.

We now consider the implications of eq. (27) separately for waves of circular and linear polarization.

4.3 Circular Polarization

For a circularly polarized wave, we have $E_x^2 = E_y^2$. Consequently, the dimensionless measure of field strength is $\eta = eE_x/m\omega c = eE_y/m\omega c$, according to (8). The prefactors $e^2E_x^2/4\pi m\omega^2 c$ and $e^2E_y^2/4\pi m\omega^2 c$ can therefore both be written $\eta^2 mc/4\pi$, and have dimensions of momentum.

The terms of eq. (27) in E_x^2 and E_y^2 can be combined in pairs via the identities

$$
\begin{aligned}
& \cos(kz-\omega t)\cos(kr-\omega t) + \sin(kz-\omega t)\sin(kr-\omega t) \\
& = \cos kz\cos kr + \sin kz\sin kr, \qquad (28)
\end{aligned}
$$

and

$$
\begin{aligned}
& \sin(kz-\omega t)\cos(kr-\omega t) - \cos(kz-\omega t)\sin(kr-\omega t) \\
& = \sin kz\cos kr - \cos kz\sin kr. \qquad (29)
\end{aligned}
$$

A detail: the second term of eq. (27) in E_x^2 contains factors of x^2, while second term of in E_y^2 contains factors of y^2. But during integration, we can replace y^2 by x^2, after which the terms can be combined via (28-29).

We see already that the volume integral of $P_{\mathrm{wave,osc},z}$ will contain no time dependence!

On integration, terms such as $f(x,r)\sin kz$ and $g(x,r)z\cos kz$ that are odd in z will vanish. The integrated field momentum is thus,

$$p_{\mathrm{wave,osc},z} = \int_V P_{\mathrm{wave,osc},z} = -\frac{\eta^2 mc}{4\pi} I_1 = -\frac{4}{3}\eta^2 mc, \tag{30}$$

where I_1 is the volume integral whose integrand is

$$\frac{k^2 z}{r^2}\sin kz\ \sin kr + \frac{kz}{r^3}\sin kz\ \cos kr + \left(\frac{3kx^2}{r^4} - \frac{k}{r^2}\right)\cos kz\ \sin kr$$
$$+\left(\frac{k^2}{r} - \frac{k^2 y^2}{r^3} + \frac{3y^2}{r^5} - \frac{1}{r^3}\right)\cos kz\ \cos kr. \tag{31}$$

We return to the significance of eq. (30) after describing the evaluation of integral I_1.

As seen from eq. (30), the integral I_1 must be dimensionless, although it is apparently a function of the wave number k. However, the form of (31) indicates that I_1 is actually independent of the length scale, so we can set $k = 1$ during integration.

To perform the integration we consider a volume element $r^2 dr\ d\cos\theta\ d\phi$ in a spherical coordinate system with angle θ defined relative to the z axis. It is more convenient to keep $z = r\cos\theta$ as a variable of integration, using $dz = r d\cos\theta$. Then, the volume integration has the form

$$\int_V = \int_0^\infty r\, dr \int_{-r}^{r} dz \int_0^{2\pi} d\phi. \tag{32}$$

Most terms of (31) are independent of ϕ, so their ϕ integral is just 2π. For the terms in x^2, we have

$$\int_0^{2\pi} x^2\ d\phi = \int r^2 \sin^2\theta \cos^2\phi\ d\phi = \pi(r^2 - z^2). \tag{33}$$

While each of the four main terms of (31) diverges on integration, it turns out that the two terms in $\cos z$ taken together are finite (and likewise for the two terms in $\sin z$). We find that

$$I_1 = I_A + I_B = \frac{16\pi}{3}, \tag{34}$$

where

$$I_A = 2\pi \int_0^\infty dr\ \frac{\sin r}{r} \int_{-r}^{r} dz\ z\sin z + 2\pi \int dr\ \frac{\cos r}{r^2} \int dz\ z\sin z = 4\pi, \tag{35}$$

and

$$I_B = \pi \int dr\ \frac{\sin r}{r} \int dz\ \cos z - 3\pi \int dr\ \frac{\sin r}{r^3} \int dz\ z^2 \cos z \tag{36}$$
$$+\pi \int dr\ \cos r \left(1 + \frac{1}{r^2}\right) \int dz\ \cos z + \pi \int dr\ \frac{\cos r}{r^2} \left(1 - \frac{3}{r^2}\right) \int dz\ z^2 \cos z = \frac{4\pi}{3}\,.$$

From detailed evaluation of the radial integral, we find that the integrand approaches a constant value as r goes to zero, and that the contribution to the integral at large r diminishes as $1/r$. That is, the principal contribution is from the region $kr \approx 1$.

We are left with the result (30) that the integral of the interference term in the field momentum density has a constant longitudinal term for an electron oscillating in a circularly polarized wave.

Recall that we have performed the analysis in a frame in which the electron has no longitudinal momentum. However, as remarked in sec. IIA, prior to its encounter with the wave, the electron had velocity $v_z = -\eta^2 c/2$ (assuming $\eta^2 \ll 1$), and therefore had initial mechanical momentum $p_{\mathrm{mech},z} = -\eta^2 mc/2$. So, we would expect that this initial mechanical momentum had been converted to field momentum, if momentum is to be conserved.

We continue to be puzzled as to why the result (30) is 8/3 times larger than that required to satisfy momentum conservation.

4.4 Linear Polarization

Consider now the case of a linearly polarized wave with electric field along the x axis. Then, $E_{\mathrm{rms}} = E_x/\sqrt{2}$, and the prefactors in (27) can be written as $\eta^2 mc/2\pi$.

The remaining terms in the momentum density $P_{\mathrm{wave,osc},z}$ have time dependences that can be expressed as sums of pure frequencies via the identities

$$\begin{aligned} & 2\cos(kz-\omega t)\cos(kr-\omega t) \\ = \;& \cos kz\cos kr + \sin kz\sin kr + (\cos kz\cos kr - \sin kz\sin kr)\cos 2\omega t \\ & +(\cos kz\sin kr + \sin kz\sin kr)\sin 2\omega t, \end{aligned} \tag{37}$$

and

$$\begin{aligned} & 2\cos(kz-\omega t)\sin(kr-\omega t) \\ = \;& \cos kz\sin kr - \sin kz\cos kr + (\cos kz\sin kr + \sin kz\cos kr)\cos 2\omega t \\ & +(\sin kz\sin kr - \cos kz\cos kr)\sin 2\omega t, \end{aligned} \tag{38}$$

Inserting these into eq. (27) and keeping only those terms that are even in z, we find the integrated field momentum to be

$$p_{\mathrm{wave,osc},z} = \int_V P_{\mathrm{wave,osc},z} = -\frac{\eta^2 mc}{4\pi}(I_1 + I_2\cos 2\omega t + I_3 \sin 2\omega t), \tag{39}$$

where integral $I_1 = 16\pi/3$ has been discussed in eqs. (31-36),

$$I_2 = -I_A + I_B = -\frac{8\pi}{3}, \tag{40}$$

and integral I_3 has the integrand,

$$\begin{aligned} & \frac{k^2 z}{r^2}\sin kz\;\sin kr - \frac{kz}{r^3}\sin kz\;\cos kr - \left(\frac{3kx^2}{r^4} - \frac{k}{r^2}\right)\cos kz\;\sin kr \\ & + \left(\frac{k^2}{r} - \frac{k^2 y^2}{r^3} + \frac{3y^2}{r^5} - \frac{1}{r^3}\right)\cos kz\;\cos kr. \end{aligned} \tag{41}$$

On evaluation, $I_3 = 0$.

Hence, the longitudinal component of the interference field momentum of a free electron in a linearly polarized wave is

$$p_{\mathrm{wave,osc},z} = -\frac{4}{3}\eta^2 mc + \frac{2}{3}\eta mc \cos 2\omega t. \tag{42}$$

The constant term is the same as that found in eq. (30) for circular polarization, and represents the initial mechanical momentum of the electron that became stored in the electromagnetic field once the electron became immersed in the wave.

As for the second term of (42), recall from eq. (11) that for linear polarization the electron oscillates along the z axis at frequency 2ω. Hence the z component of the mechanical momentum of the electron is

$$p_{\mathrm{mech},z} = m\dot{z} = -\frac{\eta^2 mc}{2}\cos 2\omega t. \tag{43}$$

The term in $p_{\mathrm{wave,osc},z}$ at frequency 2ω is $-4/3$ of the longitudinal component of the mechanical momentum associated with the "figure 8" motion of the electron. Thus, we have not been completely successful in accounting for momentum conservation when the small, oscillatory longitudinal momentum is considered.

The factors of 4/3 and 8/3 are presumably not the same as the famous factor of 4/3 that arise in analyses of the electromagnetic energy and momentum of the self fields of an electron [12, 13]. A further appearance of a factor of 8/3 in the present example occurs when we consider the field energy of the interference terms.

5 The Interference Field Energy

It is also interesting to examine the electromagnetic field energy of an electron in a plane wave. As for the momentum density (13), we can write

$$U_{\mathrm{total}} = \frac{(\mathbf{E}_{\mathrm{wave}} + \mathbf{E}_{\mathrm{static}} + \mathbf{E}_{\mathrm{osc}})^2 + (\mathbf{B}_{\mathrm{wave}} + \mathbf{B}_{\mathrm{osc}})^2}{8\pi}, \tag{44}$$

for the field energy density. Again, we no not consider the divergent energies of the self fields, but only the interference terms,

$$U_{\mathrm{int}} = U_{\mathrm{wave,static}} + U_{\mathrm{wave,osc}}, \tag{45}$$

where

$$U_{\mathrm{wave,static}} = \frac{\mathbf{E}_{\mathrm{wave}} \cdot \mathbf{E}_{\mathrm{static}}}{4\pi}. \tag{46}$$

and

$$U_{\mathrm{wave,osc}} = \frac{\mathbf{E}_{\mathrm{wave}} \cdot \mathbf{E}_{\mathrm{osc}} + \mathbf{B}_{\mathrm{wave}} \cdot \mathbf{B}_{\mathrm{osc}}}{4\pi}. \tag{47}$$

In general, the interference field energy density is oscillating. Here, we look for terms that are nonzero after averaging over time. We see at once that

$$\langle U_{\mathrm{wave,static}} \rangle = 0, \tag{48}$$

since all terms have time dependence of $\cos\omega t$ or $\sin\omega t$. In contrast, $\langle U_{\rm wave,osc}\rangle$ will be nonzero as its terms are products of sines and cosines:

$$
\begin{aligned}
& U_{\rm wave,osc} = \\
- \;& \frac{e^2E_x^2\cos(kz-wt)}{4\pi m\omega^2}\Big[\left(\frac{3kx^2}{r^4}-\frac{k}{r^2}\right)\sin(kr-\omega t)+\left(\frac{k^2}{r}-\frac{k^2x^2}{r^3}+\frac{3x^2}{r^5}-\frac{1}{r^3}\right)\cos(kr-\omega t)\Big] \\
- \;& \frac{e^2E_xE_y\cos(kz-wt)}{4\pi m\omega^2}\Big[\frac{3kxy}{r^4}\cos(kr-\omega t)+\left(\frac{k^2xy}{r^3}-\frac{3xy}{r^5}\right)\sin(kr-\omega t)\Big], \\
+ \;& \frac{e^2E_xE_y\sin(kz-wt)}{4\pi m\omega^2}\Big[\frac{3kxy}{r^4}\sin(kr-\omega t)-\left(\frac{k^2xy}{r^3}-\frac{3xy}{r^5}\right)\cos(kr-\omega t)\Big] \\
+ \;& \frac{e^2E_y^2\sin(kz-wt)}{4\pi m\omega^2}\Big[\left(\frac{3ky^2}{r^4}-\frac{k}{r^2}\right)\cos(kr-\omega t)-\left(\frac{k^2}{r}-\frac{k^2y^2}{r^3}+\frac{3y^2}{r^5}-\frac{1}{r^3}\right)\sin(kr-\omega t)\Big] \\
- \;& \frac{e^2E_y^2\sin(kz-wt)}{4\pi m\omega^2}\Big[\frac{k^2z}{r^2}\sin(kr-\omega t)+\frac{kz}{r^3}\cos(kr-\omega t)\Big] \\
- \;& \frac{e^2E_x^2\cos(kz-wt)}{4\pi m\omega^2}\Big[\frac{k^2z}{r^2}\cos(kr-\omega t)-\frac{kz}{r^3}\sin(kr-\omega t)\Big].
\end{aligned}
\tag{49}
$$

The terms in E_xE_y will vanish on integration over volume. The various time averages are

$$
\begin{aligned}
\langle 2\cos(kz-\omega t)\cos(kr-\omega t)\rangle &= \cos kz\cos kr+\sin kz\sin kr, \\
\langle 2\sin(kz-\omega t)\cos(kr-\omega t)\rangle &= \sin kz\cos kr-\cos kz\sin kr, \\
\langle 2\cos(kz-\omega t)\sin(kr-\omega t)\rangle &= \cos kz\sin kr-\sin kz\cos kr, \\
\langle 2\sin(kz-\omega t)\sin(kr-\omega t)\rangle &= \cos kz\cos kr+\sin kz\sin kr.
\end{aligned}
\tag{50}
$$

After performing the time average on eq. (49), we keep only terms that are even in z. These terms have the form (31), and so we find that

$$
u_{\rm int}=\int_V\langle U_{\rm wave,osc}\rangle=-\frac{e^2(E_x^2+E_y^2)}{8\pi m\omega^2}I_1=-\frac{4}{3}\eta^2mc^2, \tag{51}
$$

for waves of either linear or circular polarization. As with the case of the interference field momentum, this interference field energy is distributed over a volume of order a cubic wavelength around the electron. Being an interference term, its sign can be negative.

We can interpret the quantity,

$$
\frac{u_{\rm int}}{c^2}=-\frac{4}{3}\eta^2m, \tag{52}
$$

as compensation for the relativistic mass increase of the oscillating electron, which scales as $v_{\rm rms}^2/c^2$ and hence as η^2 (for small η, recall eq. (7)). Indeed, a general result for the motion of an electron in a plane wave of arbitrary strength η is that its rms relativistic mass, often called its effective mass, is [7, 10]

$$
m_{\rm eff}=m\sqrt{1+\eta^2}. \tag{53}
$$

For small η, the increase in mass is

$$\Delta m \approx \frac{1}{2}\eta^2 m. \tag{54}$$

Thus, the decrease in field energy due to the interference terms between the electromagnetic fields of the wave and electron is $-8/3$ times the mass increase it should compensate.

6 Discussion

6.1 Temporary Acceleration

We remarked in sec. IIA that the preceding analysis holds in the average rest frame of the electron. If instead the electron had been at rest prior to the arrival of the plane wave, the velocity of the average rest frame would be $v_z = (\eta^2/2)/(1+\eta^2/2)$. For this, the amplitude of the plane wave is presumed to have a slow rise from zero to a long plateau at strength η, followed by a slow decline back to zero.

Once the wave has passed by the electron, the interference field energy, (51), goes to zero since the integral is dominated by the contribution at distances of order a wavelength from the electron. Hence, the electron's kinetic energy must return to zero (or to its initial value if that was nonzero). A plane wave, or more precisely, a long pulse that is very nearly a plane wave, cannot transfer net energy to an electron. The acceleration of the electron from zero velocity to v_z is only temporary, *i.e.,* for the duration of the plane wave pulse. This result was first deduced by di Francia [14] and by Kibble [7] by different arguments.[1]

6.2 The Radiation Reaction

Our analysis of the energy balance of an electron in a plane wave is not quite complete. We have neglected the energy radiated by the electron. Since the rate of radiation is constant (once the electron is inside the plane wave), the total radiated energy grows linearly with time, and eventually becomes large. The interference energy, (51), is constant in time, and hence cannot account for the radiated energy.

More to follow.....

7 Appendix: Liénard-Wiechert Fields

As an alternative to the dipole approximation, we consider the use of the Liénard-Wiechert potentials and fields of a moving electron. We have limited our analysis to the case of a weak plane wave ($\eta \ll 1$), for which the velocity of the electron is always small ($\beta = v/c \ll 1$). In this case we may approximate the time-dependent part of the fields of the electron as proportional to the strength of the field of the plane wave (proportional to η). Then we find that the Liénard-Wiechert fields of the electron are the same as the fields in the dipole approximation.

[1] The present consensus is that a focused laser beam can transfer some net energy to a "free" electron in "vacuum".

We can show this in two ways. First, we verify that the Liénard-Wiechert potentials reduce to eqs. (20) and (22). Second, we can verify directly that the Liénard-Wiechert fields are the same as eqs. (24) and (25).

The Liénard-Wiechert potentials are

$$\phi = \left[\frac{e}{R(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})}\right], \qquad \mathbf{A} = \left[\frac{e\boldsymbol{\beta}}{R(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})}\right], \tag{55}$$

where the electron is at position $\mathbf{x}$, the observer is at $\mathbf{r}$, their separation is $\mathbf{R} = \mathbf{r} - \mathbf{x}$, the unit vector $\hat{\mathbf{n}}$ is $\mathbf{R}/R$, and the brackets, [ ], indicate that quantities within are to be evaluated at the retarded time, $t' = t - R/c$.

We work in the average rest frame of the electron. In the weak-field approximation we ignore the longitudinal motion of the electron, (11), which is quadratic in the strength of the plane wave. Then the velocity vector of the electron is

$$\boldsymbol{\beta}(t) = \frac{e}{m\omega c}\left(E_x \sin\omega t\,\hat{\mathbf{x}} - E_y \cos\omega t\,\hat{\mathbf{y}}\right), \tag{56}$$

from eq. (6). The retarded velocity is thus,

$$[\boldsymbol{\beta}] = \boldsymbol{\beta}(t' = t - R/c) = -\frac{e}{m\omega c}\left(E_x \sin(kR-\omega t)\,\hat{\mathbf{x}} + E_y \cos(kr-\omega t)\,\hat{\mathbf{y}}\right). \tag{57}$$

Distance R differs from r because the electron's oscillatory motion takes it away from the origin. However, the amplitude of the motion is proportional to strength of the plane wave. Hence, we may replace R by r in eq. (57) with error only in the second order of field strength.

Since the vector potential includes a factor $\boldsymbol{\beta}$ in the numerator, we can replace R by r and $1 - \boldsymbol{\beta}\cdot\hat{\mathbf{n}}$ by 1 in the first order in the field strength of the plane wave. Thus,

$$\mathbf{A} = -\frac{e^2}{m\omega c r}\left(E_x \sin(kr-\omega t)\,\hat{\mathbf{x}} + E_y \cos(kr-\omega t)\right)\,\hat{\mathbf{y}}, \tag{58}$$

in agreement with eq. (23).

In the scaler potential, we first bring $\boldsymbol{\beta}$ to the numerator:

$$\phi \approx \frac{e[1+\boldsymbol{\beta}\cdot\hat{\mathbf{n}}]}{[R]}. \tag{59}$$

Unit vector $[\hat{\mathbf{n}}]$ differs from unit vector $\hat{\mathbf{r}}$ due to the oscillation of the electron, which is proportional to the field strength of the plane wave. For the scalar potential, however, we must expand the factor $1/[R]$ to first order in the field strength. Now,

$$[R] = |\mathbf{r} - \mathbf{x}(t')| = \sqrt{r^2 - 2\mathbf{r}\cdot\mathbf{x}(t') + \mathbf{x}^2(t')}, \tag{60}$$

with

$$\mathbf{x}(t') \approx -\frac{e}{m\omega^2}\left(E_x \cos\omega t'\,\hat{\mathbf{x}} + E_y \cos\omega t'\,\hat{\mathbf{y}}\right) \approx -\frac{e}{m\omega^2}\left(E_x \cos\omega(kr-\omega t)\,\hat{\mathbf{x}} - E_y \cos(kr-\omega t)\,\hat{\mathbf{y}}\right), \tag{61}$$

again approximating R by r in the arguments of the cosine and sine, accurate to first order in the field strength. Hence,

$$\frac{1}{[R]} \approx \frac{1}{r}(1 + \mathbf{r}\cdot\mathbf{x}(\mathbf{t}')) \approx \frac{1}{r}\left\{1 - e\frac{(xE_x\cos(kr-\omega t) - yE_y \sin(kr-\omega t))}{m\omega^2 r^2}\right\}. \tag{62}$$

Altogether,

$$\phi \approx \frac{e}{r} - \frac{e^2}{m\omega^2}\left\{E_x\left(\frac{kx}{r^2}\sin(kr-\omega t)+\frac{x}{r^3}\cos(kr-\omega t)\right)\right.$$
$$\left.+E_y\left(\frac{ky}{r^2}\cos(kr-\omega t)-\frac{y}{r^3}\sin(kr-\omega t)\right)\right\}, \quad (63)$$

in agreement with eq. (22).

Similarly, we could proceed from the Liénard-Wiechert fields,

$$\mathbf{E} = e\left[\frac{\hat{\mathbf{n}}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R^2}\right]+\frac{e}{c}\left[\frac{\hat{\mathbf{n}}\times\left\{(\hat{\mathbf{n}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right\}}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R}\right],$$
$$\mathbf{B} = [\hat{\mathbf{n}}\times\mathbf{E}]. \quad (64)$$

After some work, we find that these fields are the same as eqs. (24-25), to first order in the strength of the plane wave.

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