A Driven Disk with Chaotic Motion

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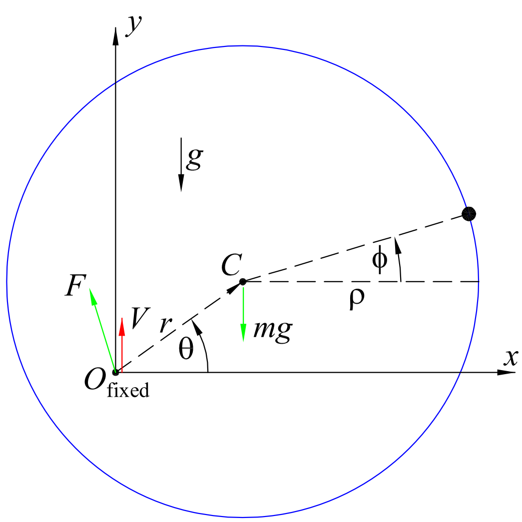
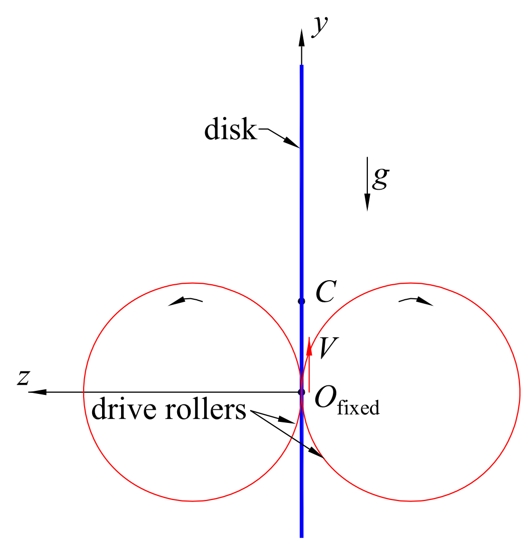
1 Problem

A uniform, thin disk of mass m and radius ρ is constrained to lie in the (vertical) x-y plane, where the x-axis is horizontal, the y-axis is vertical, subject to gravitational acceleration $\mathbf{g} = -g\,\hat{\mathbf{y}}$. The center C of the disk is not fixed, but rather some point on the disk, instantaneously at fixed point O in the lab frame, is constrained to have constant, vertical velocity $\mathbf{V} = V\,\hat{\mathbf{y}}$ with respect to a fixed origin O in the lab frame. The constraint could be applied by a pair of rollers, driven with constant angular velocity, that lie in the (vertical) y-z plane, where where $\hat{\mathbf{z}} = \hat{\mathbf{x}} \times \hat{\mathbf{y}}$ points out of the page in the left figure below; the rollers make contact with the disk at the fixed point O.

Deduce the equations of motion of this system, supposing that the drive rollers exert both a force $\mathbf{F}$ and a torque $\boldsymbol{\tau}_O$ on the disk, where the latter can be approximated at $-k\dot{\theta}\,\hat{\mathbf{z}}$ with $k > 0$.





This problem is based on [1].

2 Solution

The center of mass of the disk, at point $C = (x, y)$ in the x-y plane, is at distance $\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}$ from origin O that is fixed in the lab frame. We define θ as the angle between $\mathbf{r}$ (line OC) and the x-axis. Then, $\boldsymbol{\Omega} = d\theta/dt\,\hat{\mathbf{z}} = \dot{\theta}\,\hat{\mathbf{z}}$ can be called the “orbital” angular velocity of the disk about fixed point O.

The disk has “spin” angular velocity $\boldsymbol{\omega} = \dot{\phi}\,\mathbf{z}$ about its center, C, where angle ϕ is measured with respect to a horizontal axis through the center C of the disk, for some reference point fixed on, say, the rim of the moving disk.

This problem can be described by three independent coordinates, x, y, or r, θ, of the center of the disk and and angle ϕ of a point on the disk to the x-axis.

2.1 Constraint Relations

A key first step in analyses of examples of constrained motion is to identify the constraint. The (nonholonomic) constraint on the velocity $\mathbf{V}$ of the point on the disk instantaneously at the origin O can be written as

$$\mathbf{V} = V\,\hat{\mathbf{y}} = \dot{\mathbf{r}} - \boldsymbol{\omega} \times \mathbf{r}, \tag{1}$$

which is the sum of the velocity $\dot{\mathbf{r}} = \dot{x}\,\hat{\mathbf{x}} + \dot{y}\,\hat{\mathbf{y}}$ of the center of the disk and the velocity $\boldsymbol{\omega} \times (-\mathbf{r}) = y\dot{\phi}\,\hat{\mathbf{x}} - x\dot{\phi}\,\hat{\mathbf{y}}$, relative to the center of the disk, of the point on the disk at instantaneously at O.

The x- and y-components of the constraint relation (1) are then,

$$\dot{x} = -y\dot{\phi}, \qquad \dot{y} = V + x\dot{\phi}. \tag{2}$$

While our analysis will emphasize (x, y) coordinates, it is useful to display some results in terms of (r, θ) coordinates. First, we note that

$$\dot{\mathbf{r}} = \dot{r}\,\hat{\mathbf{r}} + r\,\dot{\theta}\,\hat{\boldsymbol{\theta}}, \tag{3}$$

in cylindrical coordinates (r, θ). Then, we find from eqs. (1) and (3) that

$$\mathbf{V} \cdot \mathbf{r} = Vr\sin\theta = Vy = \mathbf{r} \cdot \dot{\mathbf{r}} = r\dot{r}, \tag{4}$$

$$V\sin\theta = \dot{r}. \tag{5}$$

Also, we can relate $\dot{\theta}$ to $\dot{\phi}$, x and y by noting that

$$[\mathbf{V} \times \mathbf{r}]_z = -Vr\cos\theta = [(\dot{\mathbf{r}} - \boldsymbol{\omega} \times \mathbf{r}) \times \mathbf{r}]_z = [(\dot{r}\,\hat{\mathbf{r}} + r\,\dot{\theta}\,\hat{\boldsymbol{\theta}} - \dot{\phi}\,\hat{\mathbf{z}} \times \mathbf{r}) \times \mathbf{r}]_z = -r^2\dot{\theta} + r^2\dot{\phi}, \tag{6}$$

$$V\cos\theta = r\left(\dot{\theta} - \dot{\phi}\right), \tag{7}$$

$$Vr\cos\theta = Vx = r^2\left(\dot{\theta} - \dot{\phi}\right), \tag{8}$$

$$\dot{\theta} = \dot{\phi} + \frac{Vx}{x^2 + y^2}, \tag{9}$$

since $r^2 = x^2 + y^2$.

2.2 Equations of Motion

The forces on the disk are $-mg\,\hat{\mathbf{y}}$ due to gravity, acting at the center of the disk, and a frictional force $\mathbf{F}$ (in the x-y plane) acting at the fixed origin O to maintain the constant velocity $\mathbf{V}$ of the point on the disk instantaneously at point O. In addition, there exists a frictional torque $\boldsymbol{\tau}_O$ on the disk about point O, which can be approximated as $\boldsymbol{\tau}_O = -k\boldsymbol{\Omega} = -k\dot{\theta}\,\hat{\mathbf{z}}$ with $k > 0$.

The equations of motion of the disc are then

$$m\ddot{\mathbf{r}} = \mathbf{F} - mg\,\hat{\mathbf{y}}, \tag{10}$$

and

$$\frac{d\mathbf{J}}{dt} = \boldsymbol{\tau}_O + \mathbf{r} \times (-mg\,\hat{\mathbf{y}}) = -k\,\dot{\theta}\,\hat{\mathbf{z}} - mgx\,\hat{\mathbf{z}}. \tag{11}$$

Here, $\mathbf{J}$ is the angular momentum of the disk about the fixed origin O,

$$\mathbf{J} = \mathbf{r} \times m\dot{\mathbf{r}} + I_C\,\boldsymbol{\omega} = (mx\dot{y} - my\,\dot{x} + I_C\dot{\phi})\,\hat{\mathbf{z}}, \tag{12}$$

being the sum of the "orbital" angular momentum $\mathbf{r} \times m\dot{\mathbf{r}}$ of the center of mass about point O and the "spin" angular momentum $I_C\,\boldsymbol{\omega}$ of the disk about its center of mass, where $I_C = m\rho^2/2$ is the moment of inertia of a uniform disk of radius ρ about its center. Recalling eq. (2), we can rewrite eq. (12) as

$$\mathbf{J} = J\,\hat{\mathbf{z}} = \left(mxV + mx^2\dot{\phi} + my^2\dot{\phi} + I_C\,\dot{\phi}\right)\,\hat{\mathbf{z}} = \left(mxV + mr^2\dot{\phi} + I_C\,\dot{\phi}\right)\,\hat{\mathbf{z}}. \tag{13}$$

Then, recalling eqs. (2), and (4), the angular equation of motion (11) reduces to

$$\frac{dJ}{dt} = -myV\dot{\phi} + 2mr\dot{r}\dot{\phi} + mr^2\ddot{\phi} + I_C\,\ddot{\phi} = myV\dot{\phi} + \left(mr^2 + I_C\right)\ddot{\phi} = -k\,\dot{\theta} - mgx, \tag{14}$$

$$I_O\,\ddot{\phi} = -k\,\dot{\phi} - \frac{kVx}{x^2 + y^2} - mgx - mVy\,\dot{\phi}. \tag{15}$$

where we have introduced $I_O = mr^2 + I_C$ as the (time-dependent) moment of inertia of the disk about point O, and used eq. (9) to obtain eq. (15) from (14).

We do not need to use the force equation (10) to solve for the motion as we already have the results of eq. (2),

$$\dot{x} = -y\dot{\phi}, \qquad \dot{y} = V + x\dot{\phi}, \tag{16}$$

from the constraint equation (1), which are in effect first integrals of second-order differential equations of motion.[1] Once the motion is known, eq. (10) could be used to compute the constraint force $\mathbf{F}$.

The equations of motion (15)-(16) are not simple, and in general the motion is complex/chaotic, bearing some relation to the cases of a double pendulum and a driven, inverted pendulum. See [2]-[31] for discussion of these themes.

We could rewrite eqs. (15)-(16) in terms of the "spin" angular velocity $\omega = \dot{\phi}$ as

$$\dot{x} = -y\omega, \qquad \dot{y} = V + x\omega, \qquad I_O\,\dot{\omega} = -k\omega - \frac{kVx}{x^2 + y^2} - mgx - mVy\,\omega, \tag{19}$$

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[1]We can also display the equations of motion in terms of the coordinates r, θ and ϕ. The vector constraint, eq. (1), provides two scalar relations among these three coordinates, eqs. (4) and (7),

$$\dot{r} = v\sin\theta, \qquad r\left(\dot{\theta} - \dot{\phi}\right) = V\cos\theta. \tag{17}$$

The needed third equation of motion is obtained from eq. (14), rewriting it in terms of r, θ and ϕ as

$$I_O\,\ddot{\phi} = -k\,\dot{\theta} - mgr\cos\theta - mVr\sin\theta\,\dot{\phi}. \tag{18}$$

such that the three first-order differential equations (19) are closely related to eqs. (25)-(27) in the paper [32] by Lorenz that stimulated modern "chaos theory".[2] See also [34, 35]. See [36]-[41] for a different "tabletop" experiment that illustrates Lorenz' equations.[3] A somewhat related paper is [42].

The only solution with steady motion for $V > 0$ is with $x = 0$, $\theta = 90°$, $\phi =$ constant,[4] in which case $y = r = -\rho + Vt$. This steady, vertical motion of the disk, without rotation, holds only for $0 < t < 2\rho/V$, and is unstable against small perturbations.

However, there exists a motion for large V that can be slowly varying, with the center of the disk at rest at $y = 0$ and $x = -V/\dot{\phi}$ according to eq. (16). Then, $\dot{x} \approx 0 = \dot{y}$, and the equation of motion (15) reduces to $I_O\ddot{\phi} \approx -mgx$, such that the "spin" angular velocity $\omega = \dot{\phi}$ changes slowly for small x (and hence large $\dot{\phi}$). Here, the disk simply spins with its center near the point of contact with the drive wheels.[5]

For smaller V there exists a quasisteady motion very similar to that of a simple pendulum with the center of the disk moving in a quasicircular arc at a nearly constant distance from the point of contact with the drive wheels.[6]

A Appendix: The Plane of the Disk is Horizontal

If the disk is constrained to lie in a horizontal plane (rather than in a vertical plane as above),[7] then the preceding analysis holds supposing the y axis is horizontal and the z-axis in vertical, and g is set to zero (as gravity has no effect on the motion in this case).

Then, the equation of motion (15) simplifies slightly to

$$I_O\,\ddot{\phi} = -k\,\dot{\phi} - \frac{kVx}{x^2 + y^2} - mVy\,\dot{\phi}, \tag{20}$$

while the constraint relations (16) hold as before.

Now, the motion with center of the disk at rest with $y = 0$ and $x = -V/\dot{\phi}$ is steady, with constant $\dot{\phi} = \omega_0$. However, this motion is unstable in that if y is perturbed to a negative value, then $I_O\ddot{\phi} \approx -mVy\,\dot{\phi}$, so $\dot{\phi}$ moves away from ω_0 and does not return to this condition.

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[2] Lorenz is perhaps best known for coining the term "butterfly effect" [33] as a description of mathematical chaos.

[3] See also https://sciencedemonstrations.fas.harvard.edu/presentations/chaotic-waterwheel

[4] For $V < 0$, the steady motion has $x = 0$, $\theta = -90°$, $\phi =$ constant and $y = r = \rho + Vt$ for $0 < t < -2\rho/V$.

[5] See times 2:47-2:58 of the video in [1].

[6] See times 2:27-2:38 of the video in [1].

[7] Additional support rollers are required to maintain the disk in a horizontal plane.

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