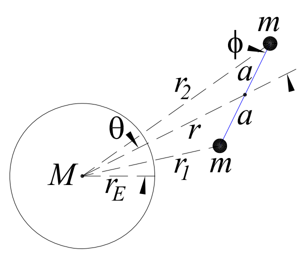
Dumbell in Orbit

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(April 9, 2021)

1 Problem

Discuss the stability of the orbital motion of a dumbell about a massive force center M (such as the Earth) supposing the dumbell consists of two masses $m \ll M$ joined by a massless, rigid rod of length $2a$, where a is not necessarily small compared to the distance r of the center of the dumbell from the force center. It suffices to consider only motion in a plane.



2 Solution

For steady motion in a plane, and coordinates r, θ and ϕ as in the figure above, there exist equilibrium configurations with angle $\phi = 0$ and $\pi/2$, with equilibrium values r_0 and $\dot\theta_0$ related by Kepler's 3rd law (at least for $a \ll r_0$).[1] It seems unlikely that the equilibrium with $\phi = \pi/2$ is stable, while for $a \ll r_0$ we anticipate that the equilibrium with $\phi_0 = 0$ is stable. However, the case of a comparable to r_0 and $\phi_0 = 0$ is not immediately evident.

We would like to develop an effective potential $V_{\rm eff}$ for the general, planar motion, such that the sign of $\partial^2 V_{\rm eff}(r_0)/\partial r^2$ determines the stability of small oscillations about the equilibria. However, the angular momentum of the dumbell about the force center is not a conserved quantity, so we cannot use the form of the effective potential for orbits of spherical/point masses.

We note that the kinetic energy of the dumbell is,

$$T = m(\dot r^2 + r^2\dot\theta^2) + ma^2(\dot\theta + \dot\phi)^2, \tag{1}$$

and its potential energy is,

$$V(r,\phi) = -GMm\left(\frac{1}{r_1} + \frac{1}{r_2}\right), \qquad r_{1,2}^2(r,\phi) = r^2 \mp 2ar\cos\phi + a^2. \tag{2}$$

The Lagrangian $\mathcal{L} = T - V$ does not depend explicitly on time, nor on coordinate θ, so the Hamiltonian/energy of the system is constant, as is the generalized momentum,

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot\theta} = 2m(r^2 + a^2)\,\dot\theta, \qquad \dot\theta = \frac{p_\theta}{2m(r^2 + a^2)}\,. \tag{3}$$

[1] There also exists an equilibrium for the dumbell perpendicular to the plane of the orbit of the center of mass, and we anticipate that this is unstable.

With this, we can write the conserved energy as,

$$
\begin{aligned}
E = T + V &= m\dot{r}^2 + \frac{mr^2\, p_\theta^2}{4m^2(r^2+a^2)^2} + ma^2\left(\frac{p_\theta^2}{4m^2(r^2+a^2)^2} + \dot{\phi}^2\right) + V(r,\phi) \\
&\equiv m\dot{r}^2 + ma^2\dot{\phi}^2 + V_{\rm eff}(r,\phi), \qquad V_{\rm eff} = \frac{p_\theta^2}{4m(r^2+a^2)} + V, \qquad (4)
\end{aligned}
$$

$$
\frac{\partial V_{\rm eff}}{\partial r} = -\frac{rp_\theta^2}{2m(r^2+a^2)^2} + GMm\left(\frac{r - a\cos\phi}{r_1^3} + \frac{r + a\cos\phi}{r_2^3}\right), \qquad (5)
$$

$$
\begin{aligned}
\frac{\partial^2 V_{\rm eff}}{\partial r^2} &= -\frac{p_\theta^2}{2m(r^2+a^2)^2} + \frac{2r^2p_\theta^2}{m(r^2+a^2)^3} \\
&\quad + GMm\left(\frac{1}{r_1^3} + \frac{1}{r_2^3} - \frac{3(r - a\cos\phi)^2}{r_1^5} + \frac{3(r + a\cos\phi)^2}{r_2^5}\right), \qquad (6)
\end{aligned}
$$

$$
\frac{\partial V_{\rm eff}}{\partial \phi} = GMm\left(\frac{ar\sin\phi}{r_1^3} - \frac{ar\sin\phi}{r_2^3}\right), \qquad (7)
$$

$$
\frac{\partial^2 V_{\rm eff}}{\partial \phi^2} = GMm\left(\frac{ar\cos\phi}{r_1^3} - \frac{ar\cos\phi}{r_2^3} - \frac{3a^2r^2\sin^2\phi}{r_1^5} - \frac{3a^2r^2\sin^2\phi}{r_2^5}\right). \qquad (8)
$$

The equations of motion for r and ϕ are,

$$
\frac{\partial E}{\partial r} = 0 = 2m\ddot{r} + \frac{\partial V_{\rm eff}}{\partial r} \qquad \frac{\partial E}{\partial \phi} = 0 = 2ma^2\ddot{\phi} + \frac{\partial V_{\rm eff}}{\partial \phi}. \qquad (9)
$$

The equilibria are related by $\ddot{r} = 0 = \ddot{\phi} = 0$, and hence,

$$
\frac{\partial V_{\rm eff}(r_0,\phi_0)}{\partial r} = 0 = \frac{\partial V_{\rm eff}(r_0,\phi_0)}{\partial \phi}. \qquad (10)
$$

Also, the motion for small departures from equilibrium is springlike (stable) if,

$$
\frac{\partial^2 V_{\rm eff}(r_0,\phi_0)}{\partial r^2} > 0, \qquad \frac{\partial^2 V_{\rm eff}(r_0,\phi_0)}{\partial \phi^2} > 0. \qquad (11)
$$

From eqs. (2) and (10) we have, noting that the equilibrium angular velocity of the center of mass of the dumbell is,

$$
\dot{\theta}_0 \equiv \Omega = p_\theta/2m(r_0^2+a^2), \qquad p_\theta = 2m\Omega(r_0^2+a^2), \qquad (12)
$$

$$
\frac{\partial V_{\rm eff}(r_0,\phi_0)}{\partial r} = 0 = -2mr_0\Omega^2 + GMm\left(\frac{r_0 - a\cos\theta_0}{r_{1,0}^3} + \frac{r_0 + a\cos\theta_0}{r_{2,0}^3}\right), \qquad (13)
$$

$$
\frac{\partial V_{\rm eff}(r_0,\phi_0)}{\partial \phi} = 0 = GMm\left(\frac{a\sin\theta_0}{r_{1,0}^3} - \frac{a\sin\theta_0}{r_{2,0}^3}\right). \qquad (14)
$$

Equation (14) indicates that equilibria exist for $\theta_0 = 0$ with $r_{1,0} = r_0 - a$ and $r_{2,0} = r_0 + a$, and $\pi/2$ with $r_{1,0} = r_{2,0} = \sqrt{r_0^2 + a^2}$. For $\theta_0 = 0$, eq. (13) tells us that,

$$
r_0 = \frac{GM}{2\Omega^2}\left(\frac{1}{(r_0-a)^2} + \frac{1}{(r_0+a)^2}\right) = \frac{GM}{\Omega^2}\frac{r_0^2+a^2}{(r_0^2-a^2)^2} \qquad (\theta_0 = 0), \qquad (15)
$$

and for $\theta_0 = \pi/2$,

$$r_0 = \frac{GM}{2\Omega^2} \frac{2r_0}{(r_0^2 + a^2)^{3/2}}, \qquad GM = \Omega^2 (r_0^2 + a^2)^{3/2}, \qquad (\theta_0 = \pi/2). \tag{16}$$

As to stability of the equilibria, we see from eq. (8) that $\partial^2 V_{\rm eff}(r_0, \phi_0)/\partial\phi^2 > 0$ for $\phi_0 = 0$, but is negative for $\phi_0 = \pi/2$. Hence, the equilibrium at $\phi_0 = \pi/2$ is not stable.

For what it's worth, eqs. (6), (12) and (16) tell us that,

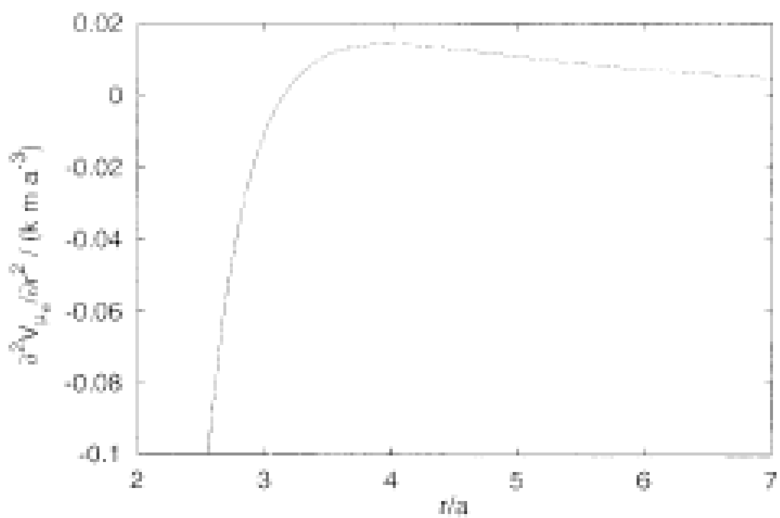
$$\frac{\partial^2 V_{\rm eff}(r_0, \pi/2)}{\partial r^2} = -\frac{p_\theta^2}{2m(r_0^2 + a^2)^2} + \frac{2r_0^2 p_\theta^2}{m(r_0^2 + a^2)^3} + \frac{2GMm}{(r_0^2 + a^2)^{3/2}}$$

$$= -\frac{2m\Omega^2 (r_0^2 + a^2)^2}{(r_0^2 + a^2)^2} + \frac{8m\Omega^2 r_0^2}{r_0^2 + a^2} + 2m\Omega^2 = \frac{8m\Omega^2 r_0^2}{r_0^2 + a^2} > 0 \qquad (\phi_0 = \pi/2). \tag{17}$$

Finally, for $\phi_0 = 0$, using eq, (15),

$$\begin{aligned} \frac{\partial^2 V_{\rm eff}(r_0, 0)}{\partial r^2} &= -\frac{p_\theta^2}{2m(r_0^2 + a^2)^2} + \frac{2r_0^2 p_\theta^2}{m(r_0^2 + a^2)^3} \\ &\qquad + GMm \left(\frac{1}{(r_0 - a)^3} + \frac{1}{(r_0 + a)^3} - \frac{3(r_0 - a)^2}{(r_0 - a)^5} + \frac{3(r_0 + a)^2}{(r_0 + a)^5} \right), \\ &= -2m\Omega^2 + \frac{8mr_0^2\Omega^2}{r_0^2 + a^2} - 2GMm \left(\frac{1}{(r_0 - a)^3} + \frac{1}{(r_0 + a)^3} \right) \\ &= -2m\Omega^2 + \frac{8mr_0^2\Omega^2}{r_0^2 + a^2} - \frac{2m\Omega^2 r_0 (r_0^2 - a^2)^2}{r_0^2 + a^2} \left(\frac{1}{(r_0 - a)^3} + \frac{1}{(r_0 + a)^3} \right) \qquad (\phi_0 = 0). \end{aligned} \tag{18}$$

For $a \ll r$, $\partial^2 V_{\rm eff}(r_0, 0)/\partial r^2 \approx 2m\Omega^2 > 0$, and the equilibrium at $\theta_0 = 0$ is stable. However, for a large enough, the third, negative term in the last form of eq. (18) dominates, and the equilibrium at $\phi_0 = 0$ is unstable. A numerical computation indicates that this equilibrium is unstable for $a > r_0/3$, as shown in the figure below, from [1].



The third equilibrium, with the dumbell perpendicular to the plane of the orbit of its center of mass, is discussed in [1] and shown to be unstable.

2.1 Skyhook

A variant of a dumbell satellite is the "skyhook", a long, rigid rod in a geosynchronous orbit, with its lower end just above the surface of the Earth.[2] For this, $a \approx 12r_{\rm E}$ and $r_0 \approx 13r_E$, so we infer that the skyhook is unstable (although in principle it could be stabilized by a very large mass at its upper end, in geosynchronous orbit).

2.2 Pendulum in Orbit

Another related problem is a pendulum in orbit, connected to a much larger mass also in orbit [10]-[12].

References

[1] M. Krupa, A. Steindl and H. Troger, *Stability of Relative Equilibria. Part II: Dumbell Satellites*, Meccanica **35**, 353 (2001),
http://kirkmcd.princeton.edu/examples/mechanics/krupa_meccanica_35_353_01.pdf

[2] J.D. Isaacs *et al.*, *Satellite Elongation into a True "Sky-Hook"*, Science **151**, 682 (1966),
http://kirkmcd.princeton.edu/examples/mechanics/isaacs_science_151_682_66.pdf

[3] Y. Artsutanov, *To the Cosmos by Electric Train*, Pravda (July 31, 1960),
http://kirkmcd.princeton.edu/examples/mechanics/artsutanov_pravda_600731b.pdf

[4] J. Pearson, *The orbital tower: a spacecraft launcher using the Earth's rotational energy*, Acta Astro. **2**, 785 (1975),
http://kirkmcd.princeton.edu/examples/mechanics/pearson_aa_2_785_75.pdf

[5] J. Gersten, H. Soodak and M. Tiersten, *Jack and the skyhook: The beanstalk revisited*, Am. J. Phys. **49**, 118 (1981),
http://kirkmcd.princeton.edu/examples/mechanics/gersten_ajp_49_118_81.pdf

[6] K.T. McDonald. *Phy 205 Problem Set 6* (1988), Prob. 3(a),
http://kirkmcd.princeton.edu/examples/ph205set6.pdf

[7] P.K. Aravind, *The physics of the space elevator*, Am. J. Phys. **75**, 125 (2007),
http://kirkmcd.princeton.edu/examples/mechanics/aravind_ajp_75_125_07.pdf

[8] M. van Pelt, *Space Tethers and Space Elevators* (Copernicus Books, 2009),
http://kirkmcd.princeton.edu/examples/mechanics/vanpelt_09.pdf

[9] Y. Chen *et al.*, *History of the Tether Concept and Tether Missions: A Review*, IRSN A.A. **2013**, 502973, http://kirkmcd.princeton.edu/examples/mechanics/chen_irsnaa_2013_502973.pdf

[10] J.L. Synge, *On the Behaviour, According to Newtonian Theory, of a Plumb Line or Pendulum Attached to an Artificial Satellite*, Proc. Roy. Irish Acad. A **60**, 1 (1959),
http://kirkmcd.princeton.edu/examples/mechanics/synge_priaa_60_1_59.pdf

[2]The concept of the skyhook became well known following the publication of [2]. See also [3]-[9].

[11] L. Blitzer, *Equilibrium and stability of a pendulum in an orbiting spaceship*, Am. J. Phys. **47**, 241 (1979), http://kirkmcd.princeton.edu/examples/mechanics/blitzer_ajp_47_241_79.pdf

[12] K.T. McDonald. *Pendulum in Orbit* (Dec. 1, 2017), http://kirkmcd.princeton.edu/examples/pendulum.pdf