Comments on
Energy Flow in a Negative-Group-Velocity Wave

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This note applies the formalism of a recent paper by Peatross *et al.* [1] to the question of energy flow in a negative-group-velocity wave, such as demonstrated by Wang *et al.* [2].

1 Classical Oscillator Model of a Negative-Group-Velocity Medium

We consider a linear dielectric medium extending from $z = 0$ to $z = a$. The medium is characterized by two spectral lines, $\omega_{1,2} = \omega_0 \pm \Delta/2$, both of oscillator strength -1 to indicate that the populations of both lines are inverted, with damping constants $\gamma_1 = \gamma_2 = \gamma$. In a classical model of the (dilute) medium as consisting of electrons tied to fixed lattice points by springs of frequencies ω_1 and ω_2, the susceptibility χ is given by,

$$\begin{aligned}\chi = N\alpha &= -\frac{Ne^2}{m}\frac{(\omega_0 - \Delta/2)^2 - \omega^2 + i\gamma\omega}{((\omega_0 - \Delta/2)^2 - \omega^2)^2 + \gamma^2\omega^2} - \frac{Ne^2}{m}\frac{(\omega_0 + \Delta/2)^2 - \omega^2 + i\gamma\omega}{((\omega_0 + \Delta/2)^2 - \omega^2)^2 + \gamma^2\omega^2} \\ &\approx -\frac{\omega_p^2}{4\pi}\frac{\omega_0^2 - \Delta\omega_0 - \omega^2 + i\gamma\omega}{(\omega_0^2 - \Delta\omega_0 - \omega^2)^2 + \gamma^2\omega^2} - \frac{\omega_p^2}{4\pi}\frac{\omega_0^2 + 2\Delta\omega_0 - \omega^2 + i\gamma\omega}{(\omega_0^2 + \Delta\omega_0 - \omega^2)^2 + \gamma^2\omega^2},\end{aligned} \tag{1}$$

where the approximation is obtained by the neglect of terms in Δ^2 compared to those in $\Delta\omega_0$. In this, ω_p is the plasma frequency of the medium, given by,

$$\omega_p^2 = \frac{4\pi Ne^2}{m}, \tag{2}$$

where N is the number density of atoms, and e and m are the charge and mass of an electron. Gaussian units are employed in this note.

The index of refraction of the medium is given by,

$$\begin{aligned}n(\omega) &= \sqrt{\epsilon} = \sqrt{1 + 4\pi\chi} \approx 1 + 2\pi\chi \\ &\approx 1 - \frac{\omega_p^2}{2}\left[\frac{\omega_0^2 - \Delta\omega_0 - \omega^2 + i\gamma\omega}{(\omega_0^2 - \Delta\omega_0 - \omega^2)^2 + \gamma^2\omega^2} + \frac{\omega_0^2 + \Delta\omega_0 - \omega^2 + i\gamma\omega}{(\omega_0^2 + \Delta\omega_0 - \omega^2)^2 + \gamma^2\omega^2}\right].\end{aligned} \tag{3}$$

This illustrated in Figure 1.

The index at the central frequency ω_0 is,

$$n(\omega_0) \approx 1 - i\frac{\omega_p^2\gamma}{(\Delta^2 + \gamma^2)\omega_0} \approx 1 - i\frac{\omega_p^2}{\Delta^2}\frac{\gamma}{\omega_0}, \tag{4}$$

where the second approximation holds when $\gamma \ll \Delta$. The electric field of a continuous probe wave then propagates according to,

$$E(z,t) = e^{i(kz - \omega_0 t)} = e^{i\omega_0(n(\omega_0)z/c - t)} \approx e^{z/[\Delta^2 c/\gamma\omega_p^2]}e^{i\omega_0(z/c - t)}. \tag{5}$$

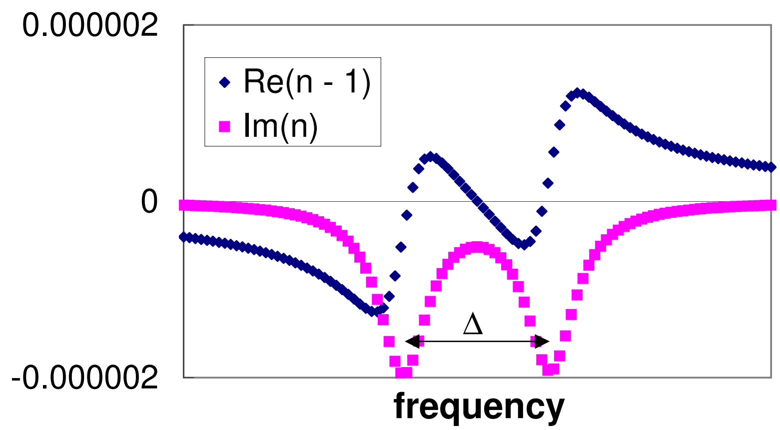

Figure 1: The real and imaginary parts of the index of refraction in a medium with two spectral lines that have been pumped to inverted populations. The lines are separated by angular frequency Δ and have widths $\gamma = 0.4\Delta$.

From this we see that at frequency ω_0 the phase velocity is c, the speed of light in vacuum, and the medium has an amplitude gain length $\Delta^2 c/\gamma\omega_p^2$.

In a medium of index of refraction $n(\omega)$, the dispersion relation can be written as,

$$k = \frac{\omega n}{c}, \tag{6}$$

where k is the wave number. The group velocity is then given by,

$$v_g = Re\left[\frac{d\omega}{dk}\right] = \frac{1}{Re[dk/d\omega]} = \frac{c}{Re[d(\omega n)/d\omega]} = \frac{c}{n + \omega Re[dn/d\omega]}. \tag{7}$$

To obtain the group velocity at frequency ω_0, we need the derivative,

$$\left.\frac{d(\omega n)}{d\omega}\right|_{\omega_0} \approx 1 - \frac{2\omega_p^2(\Delta^2 - \gamma^2)}{(\Delta^2 + \gamma^2)^2}, \tag{8}$$

where we have neglected terms in Δ and γ compared to ω_0. From eq. (7), we see that the group velocity can be negative if,

$$\gamma \ll \Delta < \sqrt{2}\omega_p, \tag{9}$$

in which case,

$$v_g \approx -\frac{c}{2}\frac{\Delta^2}{\omega_p^2}. \tag{10}$$

[For finite γ, a more detailed condition can be deduced.]

A value of $v_g \approx -c/310$ as in the experiment of Wang corresponds to $\Delta/\omega_p \approx 1/12$. In this case, the gain length $\Delta^2 c/\gamma\omega_p^2$ was approximately 40 cm.

2 Propagation of a Monochromatic Plane Wave

To illustrate the optical properties of a medium with negative group velocity, we consider the propagation of an electromagnetic wave through it. The medium extends from $z = 0$ to a, and is surrounded by vacuum. Because the index of refraction (3) is near unity in the frequency range of interest, we ignore reflections at the boundaries of the medium.

A linearly polarized monochromatic plane wave of frequency ω and incident from $z < 0$ propagates with phase velocity c in vacuum. Its electric field can be written as,

$$\mathbf{E}(z,t) = E_\omega(z,t)\hat{\mathbf{x}} = E_0 e^{i\omega z/c} e^{-i\omega t}\hat{\mathbf{x}} \qquad (z < 0). \tag{11}$$

The corresponding magnetic field is,

$$\mathbf{B}(z,t) = B_\omega(z,t)\hat{\mathbf{y}} = E_0 e^{i\omega z/c} e^{-i\omega t}\hat{\mathbf{y}} \qquad (z < 0). \tag{12}$$

Inside the medium this wave propagates with phase velocity $c/n(\omega)$ according to,

$$E_\omega(z,t) = E_0 e^{i\omega n z/c} e^{-i\omega t} \qquad (0 < z < a), \tag{13}$$

where the amplitude is unchanged since we neglect the small reflection at the boundary $z = 0$. When the wave emerges into vacuum at $z = a$, the phase velocity is again c, but it has accumulated a phase lag of $(\omega/c)(n-1)a$, and so appears as,

$$E_\omega(z,t) = E_0 e^{i\omega a(n-1)/c} e^{i\omega z/c} e^{-i\omega t} = E_0 e^{i\omega a n/c} e^{-i\omega(t-(z-a)/c)} \qquad (a < z). \tag{14}$$

It is noteworthy that a monochromatic wave for $z > a$ has the same form as that inside the medium if we make the frequency-independent substitutions,

$$z \to a, \qquad \text{and} \qquad t \to t - \frac{z-a}{c}. \tag{15}$$

Since an arbitrary waveform can be expressed in terms of monochromatic plane waves via Fourier analysis, we can use these substitutions to convert any wave in the region $0 < z < a$ to its continuation in the region $a < z$.

A general relation can be deduced in the case where the second and higher derivatives of $\omega n(\omega)$ are very small. We can then write,

$$\omega n(\omega) \approx \omega_0 n(\omega_0) + \frac{c}{v_g}(\omega - \omega_0), \tag{16}$$

where v_g is the group velocity for a pulse with central frequency ω_0. Using this in eq. (13), we have,

$$E_\omega(z,t) \approx E_0 e^{i\omega_0 z(n(\omega_0)/c - 1/v_g)} e^{i\omega z/v_g} e^{-i\omega t} \qquad (0 < z < a). \tag{17}$$

In this approximation, the Fourier component $E_\omega(z)$ at frequency ω of a wave inside the gain medium is related to that of the incident wave by replacing the frequency dependence $e^{i\omega z/c}$ by $e^{i\omega z/v_g}$, *i.e.*, by replacing z/c by z/v_g, and multiplying by the frequency-independent phase factor $e^{i\omega_0 z(n(\omega_0)/c-1/v_g)}$. Then, using transformation (15), the wave that emerges into vacuum beyond the medium is,

$$E_\omega(z,t) \approx E_0 e^{i\omega_0 a(n(\omega_0)/c-1/v_g)} e^{i\omega(z/c - a(1/c-1/v_g))} e^{-i\omega t} \qquad (a < z). \tag{18}$$

The wave beyond the medium is related to the incident wave by multiplying by a frequency-independent phase, and by replacing z/c by $z/c - a(1/c - 1/v_g)$ in the frequency-dependent part of the phase.

3 Propagation of a Pulse

The transformations between the monochromatic incident wave (11) and its continuation in and beyond the medium, (17) and (18), imply that an incident wave,

$$E(z,t) = f(z/c - t) = \int_{-\infty}^{\infty} d\omega\, E_\omega(z) e^{-i\omega t} \qquad (z < 0), \tag{19}$$

whose Fourier components are given by,

$$E_\omega(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(z,t) e^{i\omega t} dt, \tag{20}$$

propagates as,

$$E(z,t) \approx \begin{cases} f(z/c - t) & (z < 0), \\ e^{i\omega_0 z(n(\omega_0)/c - 1/v_g)} f(z/v_g - t) & (0 < z < a), \\ e^{i\omega_0 a(n(\omega_0)/c - 1/v_g)} f(z/c - t - a(1/c - 1/v_g)) & (a < z). \end{cases} \tag{21}$$

As a particular example, we consider a Gaussian pulse of temporal length τ centered on frequency ω_0 (the carrier frequency), for which the incident waveform is,

$$E(z,t) = E_0 e^{-(z/c-t)^2/2\tau^2} e^{i\omega_0 z/c} e^{-i\omega_0 t} \qquad (z < 0), \tag{22}$$

Inserting this in eq. (21) we find,

$$E(z,t) = \begin{cases} E_0 e^{-(z/c-t)^2/2\tau^2} e^{i\omega_0(z/c-t)} & (z < 0), \\ E_0 e^{-(z/v_g-t)^2/2\tau^2} e^{i\omega_0(n(\omega_0)z/c-t)} & (0 < z < a), \\ E_0 e^{i\omega_0 a(n(\omega_0)-1)/c} e^{-(z/c-a(1/c-1/v_g)-t)^2/2\tau^2} e^{i\omega_0(z/c-t)} & (a < z). \end{cases} \tag{23}$$

The factor $e^{i\omega_0 a(n(\omega_0)-1)/c}$ in eq. (23) for $a < z$ becomes $e^{\omega_p^2 \gamma a/\Delta^2 c}$ using eq. (4), and represents a small gain due to traversing the negative-group-velocity medium. In the experiment of Wang *et al.* [2] this factor was only 1.16.

According to eq. (23), the peak of the Gaussian pulse emerges from the medium at $z = a$ at time $t = a/v_g$. If the group velocity is negative, the pulse emerges from the medium before it enters at $t = 0$! Inside a negative-group-velocity medium, an (anti)pulse propagates backwards in space from $z = a$ at time $t = a/v_g < 0$ to $z = 0$ at time $t = 0$, at which point it appears to annihilate the incident pulse.

The forms (21) and (23) hold only over the frequency interval for which eq. (16) applies. For the medium described in sec. 1, the linear approximation to $\omega n(\omega)$ is only good over a frequency interval about ω_0 of order Δ, and so eq. (23) for the pulse after the gain medium applies only for pulsewidths,

$$\tau \gtrsim \frac{1}{\Delta}. \tag{24}$$

By considering the second-order term in eq. (16) it can also be shown that eq. (23) holds only for pulses with width $ct \gtrsim a$.

In vacuum, the electric and magnetic fields of a plane wave have equal strengths, but inside a dielectric medium the frequency components of the fields obey $B_\omega = n(\omega)E_\omega$. In general, this implies that the expressions for $E(z,t)$ for $0 < z < a$ in eqs. (16) and (23) do not hold for the magnetic field as well. However, for the example of a Gaussian pulse inside a negative-group-velocity medium, the index n differs from unity by at most 10^{-6} over the bandwidth of the pulse, and to good accuracy the electric and magnetic fields are equal inside the medium as well as in vacuum.

4 Energy-Flow Velocity via the Poynting Vector

A simple definition of energy-flow velocity based on the Poynting vector is[1]

$$\mathbf{v}_E = \frac{\mathbf{S}}{u_{\rm field}} = 2c\frac{\mathbf{E}\times\mathbf{B}}{E^2+B^2}\,. \tag{25}$$

Note that the magnitude of $\mathbf{v}$ is bounded by,

$$v = |\mathbf{v}| \le c\frac{2EB}{E^2+B^2} \le c\,, \tag{26}$$

and that the maximal $v = c$ only occurs when $E = B$ and $\mathbf{E}\perp\mathbf{B}$.

The energy density in the electromagnetic wave in the medium, whose permeability is taken to be unity, is,

$$u_{\rm field} = \frac{\epsilon E^2}{8\pi} + \frac{B^2}{8\pi} \approx \frac{E^2}{4\pi}, \tag{27}$$

since the dielectric constant is very close to unity over the frequency bandwidth of interest.

The Poynting vector is,

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} \approx \frac{c}{4\pi}E^2\hat{\mathbf{z}}. \tag{28}$$

Since the dielectric constant and the index of refraction are very close to unity in the negative-group-velocity medium, eqs. (27)-(25) imply that $v_E \approx c$. That is, the definition (25) seems not to provide any insight as to the apparent complexity of pulse propagation in a negative-group-velocity medium.

5 Energy-Flow Velocity via the Pulse Centroid

Peatross *et al.* proposed another definition of energy-flow velocity [1],

$$\mathbf{v}_E = \frac{\int \mathbf{S}\,d\mathrm{Vol}}{\int u_{\rm total}\,d\mathrm{Vol}} = \frac{\partial\langle\mathbf{r}\rangle}{\partial t}, \tag{29}$$

[1] The idea that an energy-flux vector is the product of energy density and energy-flow velocity seems to be due to Umov [3] (1874), based on Euler's continuity equation [4] for mass flow, $\boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = -\partial\rho/\partial t$. The energy-flow velocity (25) appeared on p. 392 of the textbook [5] and on p. 794 of [6]. See also[7]-[10]. Nonstandard definitions are considered in [11]-[13].

where,

$$\langle \mathbf{r} \rangle = \frac{\int \mathbf{r}\, u_{\rm total}\, d{\rm Vol}}{\int u_{\rm total}\, d{\rm Vol}}. \tag{30}$$

The second form of eq. (29) is obtained on integration by parts and use of total energy conservation,

$$\nabla \cdot \mathbf{S} + \frac{\partial u_{\rm total}}{\partial t} = 0. \tag{31}$$

A key feature in the above definition is the inclusion of all forms of energy, even when the only flow of energy is due to an electromagnetic wave.

This definition of energy flow derates the velocity of the pulse by including other forms of energy as well.

For example, a system containing an electromagnetic pulse of energy $u_{\rm pulse}$ moving with $v = c$ and some energy $u_{\rm static}$ that is considered to be at rest would have energy flow velocity $cu_{\rm pulse}/(u_{\rm pulse} + u_{\rm static}) < c$.

The motivation for this definition seems to have been an awareness that in some cases the pulse interacts with the rest of the system so as to rearrange energy not initially in the pulse.

However, this definition has the defect that the energy velocity is changed by the inclusion of more noninteracting energy in the system, even though this does not change the pulse propagation in any way that I would call meaningful.

Perhaps because of this, Peatross *et al.* did not seem to make any use of the definition (31), although discussion of it occupies most of their sections 2 and 3.

6 Energy Flow Velocity via the Field Energy Density

In their sec. 5, Peatross *et al.* give extensive discussion of the quantity,

$$\langle \mathbf{r} \rangle = \frac{\int \mathbf{r}\, u_{\rm field}\, d{\rm Vol}}{\int u_{\rm field}\, d{\rm Vol}}, \tag{32}$$

even though they warn us in their sec. 3 to be wary of its significance.

As an example, we apply this definition to the case of a Gaussian pulse (23) passing though a negative group velocity medium. We first make the unphysical assumption that the pulse length is much less than length a of the medium. Then, for times $t < a/v_g$ and $t > 0$ there is only one (narrow) pulse, which is in vacuum. But, for the time interval $a/v_g < t < 0$ there are three pulses present, all containing the same electromagnetic field energy; in addition to both vacuum pulses a third pulse moves inside the medium with velocity $v_g < 0$. The pulse centroid moves with velocity c according to,

$$\langle z(t) \rangle = \begin{cases} ct & (t < a/v_g), \\ \frac{2c+v_g}{3}t + \frac{a}{3}(1 - c/v_g) & (a/v_g < t < 0), \\ ct + a(1 - c/v_g) & (t > 0). \end{cases} \tag{33}$$

One might now consider the velocity,

$$\frac{d\langle z(t)\rangle}{dt} = \begin{cases} c & (t < a/v_g), \\ \frac{2c+v_g}{3} & (a/v_g < t < 0), \\ c & (t > 0). \end{cases} \tag{34}$$

For no negative value of the group velocity in the medium is this measure of pulse velocity greater than c. However, for $v_g < -2c/3$, the pulse velocity is negative during the interval $a/v_g < t < 0$, and positive at other times.

While there is nothing inconsistent in this definition, it does not seem to clarify must about the situation.

7 Exchange Energy

The energy of interaction of the wave with the medium is called the exchange energy in [1], and is defined to be the time integral of the power $\mathbf{E}\cdot\mathbf{j}_{\rm pol}$, where the polarization current density is given by,

$$\mathbf{j}_{\rm pol} = \frac{\partial \mathbf{P}}{\partial t}, \tag{35}$$

where $\mathbf{P}$ is the dielectric polarization density. Then, the exchange energy is,

$$u_{\rm exchange}(z,t) = \int_{-\infty}^{t} dt\, \mathbf{E}(z,t)\cdot\mathbf{j}_{\rm pol}(z,t) = \int_{-\infty}^{t} dt\, \mathbf{E}(z,t)\cdot\frac{\partial \mathbf{P}(z,t)}{\partial t}. \tag{36}$$

The frequency components of the polarization are related to those of the electric field by,

$$\mathbf{P}_\omega = \chi_\omega \mathbf{E}_\omega. \tag{37}$$

Peatross *et al.* argued that when describing the physical situation at time t, the Fourier analysis should involve only the history prior to that time:

$$E_\omega(z) = \frac{1}{2\pi}\int_{-\infty}^{t} E(z,t)e^{i\omega t}dt, \tag{38}$$

and hence,

$$P_\omega(z) = \frac{1}{2\pi}\int_{-\infty}^{t} P(z,t)e^{i\omega t}dt = \frac{\chi_\omega}{2\pi}\int_{-\infty}^{t} E(z,t)e^{i\omega t}dt, \tag{39}$$

Of course,

$$E(z,t) = \int_{-\infty}^{\infty} E_\omega(z)e^{-i\omega t}d\omega, \tag{40}$$

so,

$$u_{\rm exchange}(z,t) \;=\; \int_{-\infty}^{t} dt \int_{-\infty}^{\infty} E_\omega(z)e^{-i\omega t}d\omega \int_{-\infty}^{\infty} -i\omega'\chi_{\omega'}E_{\omega'}(z)e^{-i\omega' t}d\omega'$$

$$
\begin{aligned}
&= -i\int_{-\infty}^{\infty} E_\omega(z)d\omega \int_{-\infty}^{\infty} \omega' \chi_{\omega'} E_{\omega'}(z)d\omega' \int_{-\infty}^{t} e^{-i(\omega+\omega')t}dt \\
&= -i\int_{-\infty}^{\infty} E_\omega(z)d\omega \int_{-\infty}^{\infty} \omega' \chi_{\omega'} E_{\omega'}(z)d\omega' \int_{-\infty}^{t} e^{-i(\omega+\omega')t}dt \\
&= -2\pi i\int_{-\infty}^{\infty} E_\omega(z)d\omega \int_{-\infty}^{\infty} \omega' \chi_{\omega'} E_{\omega'}(z)\delta(\omega+\omega')d\omega' \\
&= 2\pi i\int_{-\infty}^{\infty} \omega\chi_\omega |E_\omega(z)|^2 d\omega \\
&= 2\pi \int_{-\infty}^{\infty} \omega Im(\chi_\omega)|E_\omega(z)|^2 d\omega, \qquad (41)
\end{aligned}
$$

where E_ω is given by eq. (38). Further justification of eq. (41) is given in [1].

WangFor our example of a negative-group-velocity medium, the imaginary part of the susceptibility (1) is negative,

$$
Im(\chi) \approx -\frac{\gamma\omega\omega_p^2}{4\pi}\left(\frac{1}{(\omega_0^2-\Delta\omega_0-\omega^2)^2+\gamma^2\omega^2} + \frac{1}{(\omega_0^2+\Delta\omega_0-\omega^2)^2+\gamma^2\omega^2}\right), \qquad (42)
$$

and hence u_{exchange} is also. This confirms the qualitative description that the pulse has extracted energy from the medium to produce the pulse in region $z > a$ earlier than would be possible in vacuum.

The phenomenon of negative-group-velocity waves as demonstrated by Wang *et al.* [2] requires that the bandwidth of the pulse be restricted such that $Re(\chi)$ varies linearly with frequency. According to eq. (1) this requires $|\omega - \omega_0| \lesssim \Delta$. Over this interval, $Im(\chi)$ can be approximated by,

$$
Im(\chi(\omega_0)) = -\frac{\gamma\omega_p^2}{2\pi\omega_0(\Delta^2+\gamma^2)} \approx -\frac{\gamma\omega_p^2}{2\pi\omega_0\Delta^2} \qquad (43)
$$

Inserting eq. (43) in eq. (41), the exchange energy is given by,

$$
\begin{aligned}
u_{\text{exchange}}(z,t) &\approx -2\pi\omega_0\frac{\gamma\omega_p^2}{2\pi\omega_0\Delta^2}\int_{-\infty}^{\infty}|E_\omega(z)|^2 d\omega \\
&= -\frac{\gamma\omega_p^2}{\Delta^2}\int_{-\infty}^{\infty}|E_\omega(z)|^2 d\omega \\
&= -\frac{\gamma\omega_p^2}{4\pi^2\Delta^2}\int_{-\infty}^{\infty}d\omega\int_{-\infty}^{t}E(z,t')e^{i\omega t'}dt'\int_{-\infty}^{t}E^\star(z,t'')e^{-i\omega t''}dt'' \\
&= -\frac{\gamma\omega_p^2}{4\pi^2\Delta^2}\int_{-\infty}^{t}E(z,t')dt'\int_{-\infty}^{t}E^\star(z,t'')dt''\int_{-\infty}^{\infty}e^{i\omega(t'-t'')}d\omega \\
&= -\frac{\gamma\omega_p^2}{2\pi\Delta^2}\int_{-\infty}^{t}E(z,t')dt'\int_{-\infty}^{t}E^\star(z,t'')\delta(t'-t'')dt'' \\
&= -\frac{\gamma\omega_p^2}{2\pi\Delta^2}\int_{-\infty}^{t}|E(z,t')|^2dt' \\
&= -\frac{\gamma\omega_p^2E_0^2}{2\pi\Delta^2}\int_{-\infty}^{t}e^{-(z/v_g-t')^2/\tau^2}dt'\,. \qquad (44)
\end{aligned}
$$

It appears that the exchange energy (44) will decrease monotonically with time.

Let us consider the exchange energy $u_{\rm exchange}(z,\infty)$ after the pulse has passed. Then,

$$u_{\rm exchange}(z,\infty) = -\frac{\gamma\omega_p^2 E_0^2}{2\pi\Delta^2}\int_{-\infty}^{\infty} e^{-(z/v_g - t')^2/\tau^2}\, dt' = -\frac{\gamma\omega_p^2 E_0^2 \tau}{2\sqrt{\pi}\Delta^2}\,. \tag{45}$$

The total exchange energy in the medium that extends from $z = 0$ to a is,

$$u_{\rm exchange}(\infty) = -\frac{\gamma\omega_p^2 E_0^2 \tau a}{2\sqrt{\pi}\Delta^2}\,. \tag{46}$$

Note that the energy of the incident pulse is,

$$u_{\rm in} = \frac{E_0^2}{4\pi}\int_{-\infty}^{\infty} e^{-(z/c - t')^2/\tau^2}\, dz = \frac{E_0^2 c\tau}{4\sqrt{\pi}}\,. \tag{47}$$

Hence,

$$u_{\rm exchange}(\infty) = \frac{2\gamma\omega_p^2 a}{c\Delta^2} u_{\rm in}\,. \tag{48}$$

In the discussion just after eq. (23) we saw that the electric field was amplified by the factor $\exp(\gamma\omega_p^2 a/c\Delta^2)$ by the gain medium. The pulse energy is therefore amplified by the factor $\exp(2\gamma\omega_p^2 a/c\Delta^2) \approx 1 + 2\gamma\omega_p^2 a/c\Delta^2$, so the energy gained is,

$$u_{\rm gain} = \frac{2\gamma\omega_p^2 a}{c\Delta^2} u_{\rm in} = -u_{\rm exchange}\,. \tag{49}$$

The result (49) is a reassuring validation of conservation of energy, yet it points out that an interesting question remains as to the interpretation of Wang's experiment. Namely, if it is really possible for three copies of a pulse to exist at certain times in and about a negative group velocity medium, where does the energy for these pulses come from? (Note that in Wang's experiment, the advance of the pulse was so slight that one would not claim that "three copies of the pulse" existed simultaneously. Such claims come from the assumption that eq. (23) is an accurate description of reality.)

Since the revival of interest in pulse propagation near absorption lines by Garret and Macumber [14], it has been common to suggest that pulses can be retarded or advanced by differential absorption or gain for the leading and trailing edges of the pulses [15, 16]. For experiments such as [17, 18] that involved strong dispersion, such a description is indeed appropriate. However, this is not the case for Wang's experiment, where the gain is actually very weak in the relevant frequency band, which lies between the two pumped frequencies. This was remarked in a qualitative way by Wang *et al.* [2], which is confirmed quantitatively by eq. (49).

So what is going on in Wang's case? I believe that answer is that Wang's experiment should be regarded as an example of quantum-mechanical barrier penetration, for which an energy uncertainty principle, $\delta E \delta t \approx \hbar$, permits an apparent violation of energy conservation for the short characteristic time $\delta t = a/|v_g|$ during which the pulse crosses the barrier.

This interpretation can be made on the basis of a classical analysis of the pulse propagation, plus one fact from quantum mechanics: the Einstein relation that $E = \hbar\omega$ for photons.

Equation (23) was deduced on the assumption that the index of refraction varied linearly with frequency. This is indeed a reasonable approximation over the central portion of the bandwidth of the pulse in Wang's experiment, as sketched in Fig. 1, but it fails for the tails of the frequency distribution that approach the pumped frequencies.

Therefore, it is relevant to consider at least the second-order term in a series expansion of the index of refraction. When this is done, one finds that a Gaussian pulse will propagate without significant distortion only if its rms width τ is greater than the time δt it takes the pulse to cross the gain medium. Also, the bandwidth $\tau \gtrsim 1/\Delta$, where Δ is frequency separation of the two pumped spectral lines. The narrowest pulse for which the effects of Wang's experiment might be clearly observed has $\delta t \approx \tau \approx 1/\Delta$.

The time interval during which there appears to be "extra" field energy in the combination of the input pulse, the output pulse, and the backwards-propagating pulse inside the gain medium is δt. The amount of "extra" energy is roughly that of the input pulse, $u_{\rm in}$ as given by eq. (47). We can also say that $u_{\rm in} = n\hbar\omega_0$, where n is the number of photons in the pulse whose central angular frequency is ω_0. During the time $\delta t \approx \tau \approx 1/\Delta$ during which these photons are crossing the gain medium, their energy is uncertain by $\delta E \approx \hbar/\delta t \approx \hbar\Delta$. The uncertainty in a coherent pulse of n photons is then $n\delta E \approx n\hbar\Delta = (\Delta/\omega_0)u_{\rm in} \ll u_{\rm in}$. That is, my claim seems to be wrong!

References

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