Electromagnetic-Field Angular Momentum of a Classical Charged Particle in a Uniform Magnetic Field

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1 Problem

What is the electromagnetic-field angular momentum of a charged particle in a circular orbit in a uniform magnetic field $\mathbf{B}_0 = B_0\,\hat{\mathbf{z}}$, supposing the velocity of the particle is small compared to the speed c of light?

This problem was inspired by [1], where it was claimed (after eq. (19)) that the electric field of the circling charge q obeys $\boldsymbol{\nabla} \times \mathbf{E}_q = 0$, which implies that the magnetic field of the charge is constant in time. This is not so,[1] *and hence a better solution for the field angular momentum is required.*[2]

2 Solution

This problem is somewhat ill posed, as a charged particle in a circular orbit radiates angular momentum, such that the angular momentum stored in the electromagnetic field depends on the past history of the particle.

To give the discussion a crisper basis, we suppose an electric charge q of rest mass m is initially at rest in the magnetic field, and is slowly accelerated to velocity $\mathbf{v}_0$ (with $v_0 \ll c$) along a straight line perpendicular to $\mathbf{B}_0$, and released at time $t = 0$. We ignore the tiny amount of radiation emitted during the initial acceleration,[3] such that lines of the magnetic field $\mathbf{B}_q(t = 0)$ of the charged particle are circles (rather than a vortex) in planes perpendicular to $\mathbf{v_0}$. Then, at time $t = 0$ the interaction field energy, $U_{\rm EM,0} = \int \mathbf{B}_q(t = 0) \cdot \mathbf{B}_0\, d{\rm Vol}/4\pi$ (in Gaussian units) is zero, and the total energy of the charged particle is just its kinetic energy $U_0 = mv_0^2/2$.

[1] In [1], "quantum" particles are considered, whose stationary states in a uniform magnetic field can be a recently discovered [2, 3] variant of Landau levels (for the latter, see, for example, §111, p. 424 of [4]) called electron-vortex states. The charge density ρ and current density $\mathbf{J}$ associated with these electron-vortex states are time independent, such that one can say that the electric and magnetic fields of the charged particle are time independent, and hence $\boldsymbol{\nabla} \times \mathbf{E}_q = 0$ while these states last. However, the vortex states (except the lowest-energy level) decay via photon emission, so that these states do have an association with time-dependent electric fields.

Here, we consider this association in a "classical" context, where circling charged particles emit radiation at all times (in contrast to the "quantum" case where radiation is emitted only during "quantum jumps").

[2] A famous example of (classical) electromagnetic-field angular momentum was given by Feynman [5]. See also [6, 7, 8] and references therein.

[3] The rate of radiation of energy (and angular momentum) varies as a^2 according to the Larmor formula (3), where the initial acceleration is $a = v_0/T$ for constant acceleration during time interval T. The total angular momentum radiated at $t < 0$ varies as $a^2T \propto 1/T$, which is negligible for large T.

At time $t = 0$ the charged particle is set free, and enters a circular orbit of radius $r(t = 0) \equiv r_0$, losing energy slowly due to electromagnetic radiation, such that $r(t)$ drops to zero. The charge eventually comes to rest at the center of its circular orbit at time $t = 0$.

The angular momentum about the center of the circular orbit of radius r_0 (at time $t = 0$) is simply the mechanical angular momentum, $L(t = 0) = mv_0 r_0$, of the charge. That is, the field angular momentum, $\mathbf{L}_{\rm EM}(t = 0) = \int \mathbf{r}' \times (\mathbf{E}_q(t = 0) \times \mathbf{B}_0)\, d{\rm Vol}'/4\pi c$, is zero, taking the self-field angular momentum of the charge to be part of its mechanical angular momentum, and noting that the electric field $\mathbf{E}_q(t = 0)$ is essentially the (spherically symmetric) static electric field of the charge at that time.

For $v_0 \ll c$, trajectory of the particle is always approximately circular, so the speed $v(t)$ is related to the radius $r(t)$ of the circular orbit in, say, the x-y plane by,

$$F = m\frac{v^2}{r} = \frac{qvB_0}{c} \qquad v = \frac{qB_0 r}{mc}, \qquad r_0 = \frac{mcv_0}{qB_0}, \qquad \omega = \frac{v}{r} = \frac{qB_0}{mc}, \tag{1}$$

where we take $\mathbf{B}_0 = B_0\,\hat{\mathbf{z}}$, $\mathbf{r}_0 = r_0\,\hat{\mathbf{x}}$ and $\mathbf{v}_0 = -v_0\,\hat{\mathbf{x}}$, such the the motion in the x-y plane is clockwise from "above" with uniform angular velocity ω (the "cyclotron" frequency). The total acceleration a is approximately the radial acceleration,

$$a \approx \frac{v^2}{r} = \omega^2 r. \tag{2}$$

The power radiated as the charge spirals inward to the center of the circular orbit follows from the Larmor formula,

$$\frac{dU{\rm rad}}{dt} = \frac{2q^2a^2}{3c^3}. \tag{3}$$

Oct. 2, 2024. Thanks to Paul Berman (private communication and [9]) for noting that the subsequent analysis was somewhat incomplete in an earlier version of this note.

The emission of radiation is associated with a radiation-reaction/damping force, given for $v \ll c$ by[4]

$$\mathbf{F}_{\rm damping} = \frac{2q^2}{3c^3}\,\ddot{\mathbf{v}}. \tag{5}$$

For the present case of approximately uniform circular motion with angular velocity ω, $\ddot{\mathbf{v}} \approx -\omega^2\mathbf{v}$, and so

$$\mathbf{F}_{\rm damping} \approx -\frac{2q^2\omega^2}{3c^3}\,\mathbf{v}. \tag{6}$$

[4]The radiation-damping force (5) was first identified by Planck [10] in 1896 by supposing that the rate of work done by the damping force equals (the negative of) the rate of radiated electromagnetic energy,

$$dW_{\rm damping}/dt = \mathbf{F}_{\rm damping} \cdot \mathbf{v} = -dU_{\rm rad}/dt. \tag{4}$$

The expression (5) had previously been identified by Lorentz as the self force of an accelerated charge of finite extent, without relating the self force to radiation. For a review, see [12].

Then, the total force on the charge q is the sum of the Lorentz force $q\mathbf{v}/c \times \mathbf{B}_0$ due to the uniform magnetic field $\mathbf{B}_0$ and the radiation-damping force, eq. (6). The equation of motion of the charge is then,

$$\mathbf{F} = m\dot{\mathbf{v}} \approx q\frac{\mathbf{v}}{c} \times \mathbf{B}_0 - \frac{2q^2\omega^2}{3c^3}\mathbf{v}, \qquad \dot{v}_x = \omega v_y - \gamma v_x, \qquad \dot{v}_y = -\omega v_x - \gamma v_y, \tag{7}$$

where the damping constant is

$$\gamma = \frac{2q^2\omega^2}{3mc^3} \ll \omega. \tag{8}$$

The solution to eq. (7) is, for $\mathbf{v}_0 = -v_0\,\hat{\mathbf{y}}$,

$$\mathbf{v} = -v_0\, e^{-\gamma t}[\sin(\omega t)\,\hat{\mathbf{x}} + \cos(\omega t)\,\hat{\mathbf{y}}], \tag{9}$$

whose time integral is, recalling that $\mathbf{r}_0 = r_0\,\hat{\mathbf{x}}$,

$$\begin{aligned} \mathbf{r}(t) = \mathbf{r}_0 + \int_0^t \mathbf{v}(t')\,dt' &= \mathbf{r}_0 - \frac{v_0}{\omega^2+\gamma^2}(\omega\,\hat{\mathbf{x}} + \gamma\,\hat{\mathbf{y}}) \\ &\quad - \frac{v_0\,e^{-\gamma t}}{\omega^2+\gamma^2}\{[-\gamma\sin(\omega t) - \omega\cos(\omega t)]\,\hat{\mathbf{x}} + [-\gamma\cos(\omega t) + \omega\sin(\omega t)]\,\hat{\mathbf{y}}\} \\ &\approx \mathbf{r}_0 - r_0\,\hat{\mathbf{x}} - \frac{r_0\gamma}{\omega}\,\hat{\mathbf{y}} + r_0\,e^{-\gamma t}[\cos(\omega t)\,\hat{\mathbf{x}} - \sin(\omega t)\,\hat{\mathbf{y}}] + \frac{r_0\gamma\,e^{-\gamma t}}{\omega}[\sin(\omega t)\,\hat{\mathbf{x}} + \cos(\omega t)\,\hat{\mathbf{y}}] \\ &\approx r_0\,e^{-\gamma t}[\cos(\omega t)\,\hat{\mathbf{x}} - \sin(\omega t)\,\hat{\mathbf{y}}] + \frac{r_0\gamma}{\omega}\left[e^{-\gamma t}\sin(\omega t)\,\hat{\mathbf{x}} - \left(1 - e^{-\gamma t}\cos(\omega t)\right)\,\hat{\mathbf{y}}\right] \\ &\approx r_0\,e^{-\gamma t}[\cos(\omega t)\,\hat{\mathbf{x}} - \sin(\omega t)\,\hat{\mathbf{y}}], \end{aligned} \tag{10}$$

using Dwight 577.1 and 577.2 [13]. The magnitudes of $\mathbf{r}(t)$ and $\mathbf{v}(t)$ obey $r \approx r_0\,e^{-\gamma t}$ and $v = v_0\,e^{-\gamma t}$. In the approximation of the next to last line of eq. (10), the instantaneous center of the spiral motion is at $(r_0\gamma/\omega)\,[e^{-\gamma t}\sin(\omega t)\,\hat{\mathbf{x}} - (1 - e^{-\gamma t}\cos(\omega t))\,\hat{\mathbf{y}}]$, which converges on $\mathbf{r}(t \to \infty) \approx -(r_0\gamma/\omega)\,\hat{\mathbf{y}}$.

The rate of change of mechanical (kinetic) energy $U_{\rm mech}$ of the charge is, noting that the Lorentz force does no work on charge q, and recalling (6),

$$\frac{dU_{\rm mech}}{dt} = \mathbf{F}_{\rm total}\cdot\mathbf{v} = \mathbf{F}_{\rm damping}\cdot\mathbf{v} = -\frac{2q^2\omega^2v^2}{3c^3} = -\frac{2q^2v^4}{3c^3r^2} = -\frac{2q^2a^2}{3c^3} = -\frac{dU_{\rm rad}}{dt}\,, \tag{11}$$

as expected from Planck's derivation (4) of the radiation-damping force.[5]

[5]The behavior of $r(t)$ can be deduced from eq. (11),

$$U_{\rm mech} = \frac{mv^2}{2} \approx \frac{m\omega^2r^2}{2}\,, \qquad \frac{dU_{\rm mech}}{dt} \approx m\omega^2 r\frac{dr}{dt} = -\frac{dU_{\rm rad}}{dt} = -\frac{2q^2\omega^4r^2}{3c^3}\,. \tag{12}$$

Then, the equation of motion of the spiral is,

$$\frac{dr}{dt} \approx -\frac{2q^2\omega^2 r}{3mc^3} = -\gamma r \quad (\ll v), \qquad r \approx r_0\,e^{-\gamma t}. \tag{13}$$

It is instructive to consider the mechanical (and electromagnetic) angular momentum of the system,

$$\mathbf{L}_{\rm mech} = \mathbf{r} \times m\mathbf{v} \approx -m\omega r^2\,\hat{\mathbf{z}} = -m\omega r_0^2\, e^{-2\gamma t}\,\hat{\mathbf{z}}, \qquad (14)$$

which decays to zero at rate[6]

$$\frac{d\mathbf{L}_{\rm mech}}{dt} \approx 2m\gamma\omega r_0^2\, e^{-2\gamma t}\,\hat{\mathbf{z}} = \frac{2}{\omega}\frac{dU_{\rm rad}}{dt}\,\hat{\mathbf{z}}, \qquad (15)$$

noting that

$$\frac{dU_{\rm rad}}{dt} = \frac{2q^2a^2}{3c^3} = \frac{2q^2(\omega^2 r)^2}{3c^3} = \frac{2q^2\omega^4 r_0^2\ e^{-2\gamma t}}{3c^3} = m\gamma\omega^2 r_0^2\ e^{-2\gamma t}. \qquad (16)$$

It is appealing to suppose that the loss of mechanical angular momentum of the system is entirely due to the radiation of angular momentum, in analogy to the fact that the loss of mechanical kinetic energy is entirely due to radiated electromagnetic energy, as noted above. However, as shown in Prob. 5 of [14], the rate of radiation of angular momentum by a charge in uniform circular motion about the z-axis is related by[7]

$$\frac{d\mathbf{L}_{\rm rad}}{dt} = \frac{1}{\omega}\frac{dU_{\rm rad}}{dt}\,\hat{\mathbf{z}} = m\gamma\omega r_0^2\ e^{-2\gamma t}. \qquad (17)$$

Thus, the radiation of angular momentum accounts for only half of the loss of mechanical angular momentum as the system decays.

The mechanical angular momentum can be changed by a torque on the system as well as by radiation of electromagnetic angular momentum. The Lorentz force, $q\mathbf{v}/c \times \mathbf{B}_0$ on charge q is not entirely in the azimuthal direction as the charge slowly spirals in to the origin, such that the associated torque about the origin is nonzero,

$$\boldsymbol{\tau}_{\rm Lorentz} = \mathbf{r} \times \left(q\frac{\mathbf{v}}{\mathbf{c}} \times \mathbf{B}_0\right) = -\frac{q}{c}(\mathbf{r}\cdot\mathbf{v})\,\mathbf{B}_0 \qquad (18)$$

Using the exact form of $\mathbf{r}(t)$ from eq. (10), the time average of $\mathbf{r}\cdot\mathbf{v}$ is

$$\langle \mathbf{r}\cdot\mathbf{v}\rangle = -\frac{v_0^2\gamma\, e^{-2\gamma t}}{\omega^2+\gamma^2} \approx -\frac{v_0^2\gamma\, e^{-2\gamma t}}{\omega^2} = -r_0^2\gamma\, e^{-2\gamma t}, \qquad (19)$$

so the time-average torque due to the Lorentz force is

$$\langle \boldsymbol{\tau}_{\rm Lorentz}\rangle \approx \frac{qB_0}{c} r_0^2\gamma\, e^{-2\gamma t}\,\hat{\mathbf{z}} = m\gamma\omega r_0^2\, e^{-2\gamma t}\,\hat{\mathbf{z}} = \frac{1}{2}\frac{d\mathbf{L}_{\rm mech}}{dt}\,, \qquad (20)$$

recalling eq. (15). In addition, the torque due to the radiation-damping force is

$$\langle \boldsymbol{\tau}_{\rm damping}\rangle = \langle \mathbf{r}\times\mathbf{F}_{\rm damping}\rangle = \left\langle \mathbf{r} \times -\frac{2q^2\omega^2}{3c^3}\mathbf{v}\right\rangle \approx \frac{2q^2\omega^2}{3c^3} r_0 v_0\, e^{-2\gamma t}\,\hat{\mathbf{z}} = m\gamma\omega r_0^2\, e^{-2\gamma t}\,\hat{\mathbf{z}} = \frac{d\mathbf{L}_{\rm rad}}{dt} = \frac{1}{2}\frac{d\mathbf{L}_{\rm mech}}{dt}\,, \qquad (21)$$

[6] $\mathbf{L}_{\rm mech}$ is initially in the negative z direction, and rises to zero with time, so that $d\mathbf{L}_{\rm mech}/dt$ is in the positive z direction.

[7] In §75, p. 204 of [15] it was argued that $d\mathbf{L}_{\rm rad}/dt = \boldsymbol{\tau}_{\rm damping}$ = the torque due to radiation damping. This is confirmed to be consistent with our eq. (17) in eq. (21) below.

recalling (17). Thus, the total torque "causes" the change in the mechanical angular momentum,[8]

$$\langle \boldsymbol{\tau}_{\rm total} \rangle = \langle \boldsymbol{\tau}_{\rm Lorentz} \rangle + \langle \boldsymbol{\tau}_{\rm damping} \rangle = \frac{d\mathbf{L}_{\rm mech}}{dt} \,. \tag{23}$$

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[8] The rates of work done by the torques $\langle \boldsymbol{\tau}_{\rm Lorentz} \rangle$ and $\langle \boldsymbol{\tau}_{\rm damping} \rangle$ are

$$\langle \boldsymbol{\tau}_{\rm Lorentz} \rangle \cdot \boldsymbol{\omega} = \frac{\boldsymbol{\omega}}{2} \cdot \frac{d\mathbf{L}_{\rm mech}}{dt} = -\frac{dU_{\rm rad}}{dt} = \frac{dU_{\rm mech}}{dt} = \langle \boldsymbol{\tau}_{\rm damping} \rangle \cdot \boldsymbol{\omega}, \tag{22}$$

recalling eqs. (11), (15), (20) and (21) and noting that $\boldsymbol{\omega} = -\omega\,\hat{\mathbf{z}}$. This illustrates that the total rate of work, $\langle \boldsymbol{\tau}_{\rm total} \rangle \cdot \boldsymbol{\omega} = \langle \boldsymbol{\tau}_{\rm Lorentz} \rangle \cdot \boldsymbol{\omega} + \langle \boldsymbol{\tau}_{\rm damping} \rangle \cdot \boldsymbol{\omega} = 2\, dU_{\rm mech}/dt$, done by the sum of torques on a system with angular velocity $\boldsymbol{\omega}$ is not necessarily the total rate, $dU_{\rm mech}/dt$, of mechanical work done on the system. Also, the Lorentz torque does nonzero work on the system, although the total work done by the Lorentz force is zero, as $\mathbf{F}_{\rm Lorentz} \cdot \mathbf{v} = 0$.

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