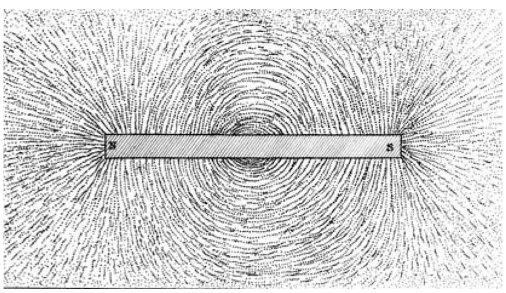
Some Comments on Field Lines

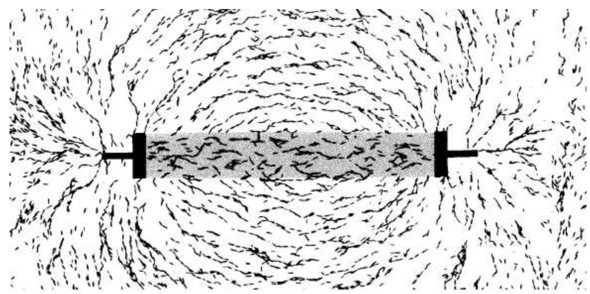
Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(August 15, 2024; updated August 29, 2024)

1 Overview

This note discusses several aspects of the concept of field lines (of a 3-vector field) that are perhaps not universally appreciated.

1. Field lines are not "physical". They are mathematical constructs (like electromagnetic potentials) that are often useful. Of course, the fact that iron filings align themselves along the local direction of the magnetic field,[1] and that some grass seeds align themselves along the local direction of the electric field,[2] as sketched below, gives the impression that "lines of force" have some kind of existence.[3]





2. In general, field lines cannot be described by a function whose argument is an arbitrary point (but see the beginning of Sec. 2 below). Rather, field lines must be constructed one by one, first choosing a point in space, at a particular time, and then defining a field line to extend continuously from that point in the direction of the local vector field (as well as in the opposite direction if the chosen point is not a source point).[4] In practice, only a finite number of field lines are so constructed, but in principle a countably infinite number of field lines could be constructed.[5] Not every conceivable vector field has field lines.

3. There is no general procedure to extrapolate the field lines identified at one time to those at a later time. In electromagnetism, in some cases one can assign a velocity to field lines of $\mathbf{v} = \mathbf{E}\times\mathbf{B}/B^2$ (in SI units) [56, 62, 65, 83, 85, 87, 88], which is proportional to the Poynting vector $\mathbf{S} = \mathbf{E}\times\mathbf{B}/\mu_0$ where μ_0 is the permeability of the vacuum.

[1] See, for example, [20]. References [1]-[111] are listed in chronological order.

[2] See, for example, [57] and Chap. II of [73].

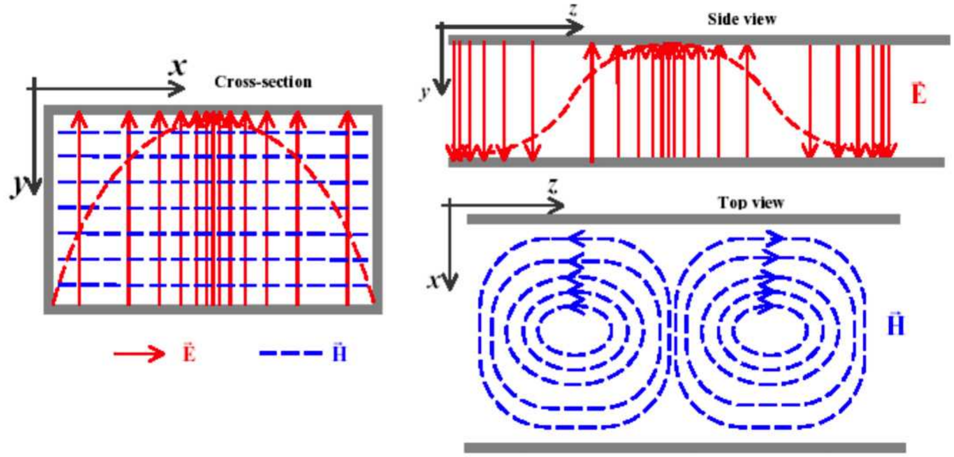
[3] Field lines are perhaps the "angels" of field theory, in the sense of the question: How many angels can dance on the head of a pin? https://en.wikipedia.org/wiki/How_many_angels_can_dance_on_the_head_of_a_pin%3F

[4] This construction cannot be made at so-called null points where the field is zero. However, if the field is nonzero in the neighborhood of a null point, it is considered that field lines can be extrapolated to the null point. See, for example, [80].

[5] Recall that a countable infinity is the lowest order of infinity, called $\aleph_0$ by Cantor [36]. See also the discussion of "surreal numbers" in [110].

This prescription is agreeable for a plane electromagnetic wave with $\mathbf{E} = c\mathbf{B} \times \hat{\mathbf{k}}$ (in SI units) where c is the speed of light in vacuum and $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$ is the unit wave vector.

However, it fails for the TE_{10} mode of a rectangular waveguide whose pattern of field lines (sketched below, from [106]) moves down the guide, unchanged in the frame of the phase velocity of the wave, which velocity exceeds the speed of light. This reinforces item 1, that field lines are not "physical". The Poynting vector in this case is not constant, and in general does not point down the guide.[6]



4. Even for a field $\mathbf{B}$ that obeys $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ everywhere, such as the magnetic field, its field lines are not necessarily closed, but can have infinite length. A field line of infinite length can extend to "infinity", but infinite field lines can also be constructed that lie entirely within a sphere of finite radius.

   An example of the latter is considered in Sec. 2 below.

5. A field line, even if infinite in length, has zero volume, and the (countably infinite) set of all field lines also has zero volume.

   As such, field lines do not "densely" fill 3-space as claimed in [40, 82, 102].

6. By construction, different field lines never intersect at points where the field strength is nonzero and finite.[7] However, field lines can intersect at "null points" (and null lines and null surfaces) where the field strength is zero.

   An example of this is discussed in Sec. 3 below.

7. Field lines are represented in drawings that illustrate only a very few lines. A useful convention in such drawings is "that the number of lines per unit area at right angles to the lines is proportional to the field strength", as noted by Feynman in Sec. 1-2 of [61]. This convention makes sense only when considering a finite number of field lines, since an infinite number of field lines could be constructed, in principle, through any unit area on which the field strength is nonzero.

[6] More details of this example are discussed in [104].

[7] Approximations to physical fields include nonphysical "singular" points where the field is infinite (such as at an idealized source point). Field lines can intersect at "singular" points.

A field line might pass many times through a small area, so the convention should be that a field line through a small area is only counted once in the relation that the (finite) number of different field lines through the small area is proportional to the local field strength.[8]

Pedagogic discussions (with drawings) of field lines include [37, 39, 40, 45, 46, 47, 48, 51, 57, 61, 63, 64, 67, 75, 76, 77, 79, 85, 102, 108, 109].

8. A field line can be regarded as ending at any point along such a line, and another field line starting at that point, as noted by Slepian on p. 88 of [48]: "As we prolong the lines of force, we may at any point end the lines and start new ones with the same density, without in any way impairing the correspondence between the assemblage of lines of force and the vector field. The assemblage of broken lines serves just as well as the assemblage of continuous lines of force."[9]

   In general, this concept of lines ending and starting at an arbitrary point has little impact on our physics understanding, but an example where it is helpful is considered in Sec. 2 below.

9. In plasma astrophysics the concept of "breaking and reconnecting" of field lines at null points[10] (or null lines or null surfaces) has been greatly developed, starting with Giovanelli [42, 43], Hoyle [44], and Dungey [50], then elaborated upon by Sweet [52, 53] and Parker [54, 60].[11] More recent reviews are given in [68, 80, 84, 92, 93, 100, 101]. The earliest example of "breaking and reconnecting of field lines" was given by Hertz in Fig. 4 of [35], without explicit recognition of this phenomenon.

   See also Sec. 3 below. Additional comments by the author on "breaking and reconnection" of field lines are given in [98].

   The Appendix below reviews the history of field lines up to 1880.

2 Chaotic Field Lines

As reviewed by Maxwell in Art. 183 of [32], any analytic function $f(z) = u(x, y) + i\,v(x, y)$ of a complex variable $z = x + i\,y$ where $i = \sqrt{-1}$ obeys

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\,, \tag{1}$$

away from possible poles and branch cuts. That is, the (conjugate) functions u and v obey Laplace's equation for the potentials of two-dimensional static electric fields. Furthermore,

[8] This caveat was implied in item 2), Sec. IV of [102].

[9] Slepian's statement would be better phrased as applying to any individual line. His implication that many different lines passes through a single "point" is misleading. He no doubt meant a small area rather than a mathematical point.

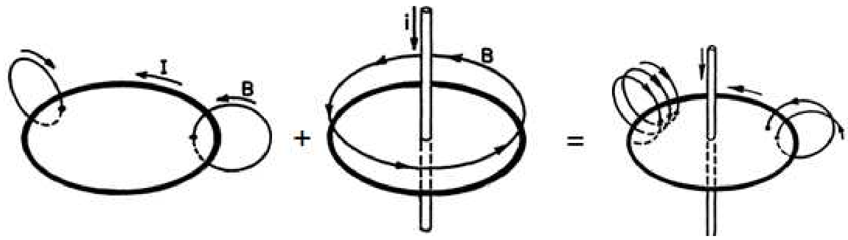
[10] An animation of "breaking and reconnection" of field lines is given at
https://en.wikipedia.org/wiki/Magnetic_reconnection

[11] It seems to me that the "three-dimensional magnetic reconnection without null points" considered in Fig. 9 of [78] occurs along a null line, which is a collection of null points.

equipotentials of u corresponds to lines of the electric field associated with potential v, and *vice versa*. This happy mathematics can give the impression that general methods also exist for finding the field lines of three-dimensional fields, but such optimism is misplaced.

One of the earliest discussions of the greater complexity of three-dimensional field lines was in the 1929 Russian edition of the electrodynamics textbook by Tamm [40]; see §34. Perhaps the earliest discussion in English was in the second [45] of a series of "electrical essays" by Slepian. This concerned the magnetic field (and field lines) of a long solenoid with a helical current winding. Slepian noted that this winding is equivalent to the sum of an axial current together with a set of circular loops of currents perpendicular to the solenoid axis.

In a subsequent essay, Slepian considered a single circular loop of current together with an axial current, as sketched below (from [46][12]).[13,14]



Slepian asked the reader whether this process "intensifies" the magnetic field, since the helical character of the field lines in the rightmost figure gives them a very high density. His answer (delayed by one issue of the journal) was `no`. We have argued in item 7 of Sec. 1 that a field line should be counted only once when relating the field strength to the (finite) number of (different) field lines that pass through a small area perpendicular to that line.

The closed, finite-length, magnetic-field lines of the left figure above morph into the infinite-length magnetic-field lines of the right figure above with even the slightest addition of the axial current shown in the middle figure. We can say that the field lines in this example are "chaotic", meaning that small changes in one parameter of a system can lead to large changes in other parameters [95];[15] see also, for example, [82, 89, 90, 91, 94, 97, 99].

For a closed field line to morph into an "unclosed" one of infinite length, the line must "break" at some point.[16] However, there are no null points along a closed field line in Slepian's example, so this cannot be a case of "breaking and reconnecting". Rather, we consider it to be a case where Slepian's comment in [48] is relevant, that any field line can be considered to be "broken" at any point along the line.

If the magnetic field in the left figure above is established before the field of the middle figure is turned on, then the transient fields due to the turn-on of the axial current i move

[12]The axial current i in Slepian's figures should be "up", not "down".

[13]Slepian's next essay [47] posed another "puzzler" arising from taking field lines too seriously.

[14]This example was considered by Tamm [40].

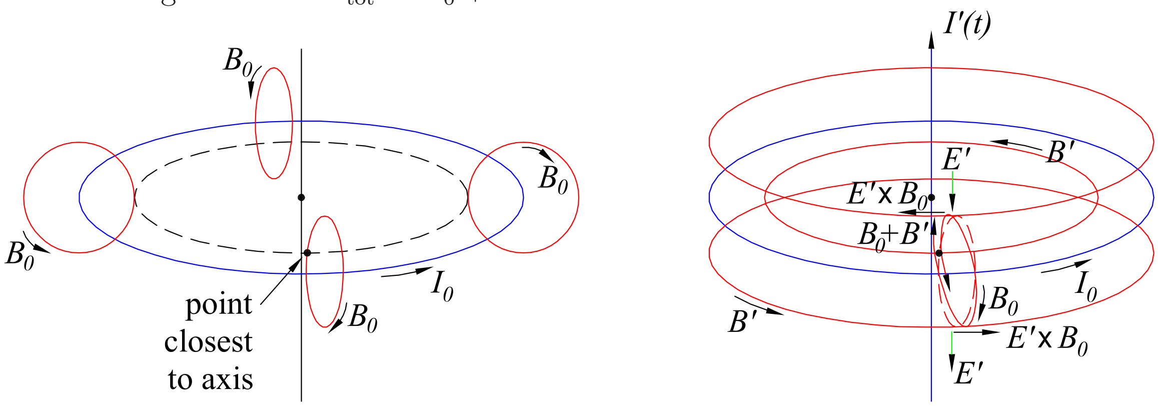
[15]Modern "chaos theory" was initiated by Lorenz [58], who popularized it as the "butterfly effect" [66].

[16]Aug. 22, 2004. David Griffiths (private communication) suggests an alternative view that avoids use of Slepian's notion by supposing that any "closed" field line be regarded a involving an infinite number of turns of the line (which is not then "closed"). This infinite coil of lines could then later deform, without "breaking", into an infinite helical field line. In this view (which would be excluded by Occam's razor), there are no "closed" field lines of finite length. More extensive comments by Griffiths are at [111].

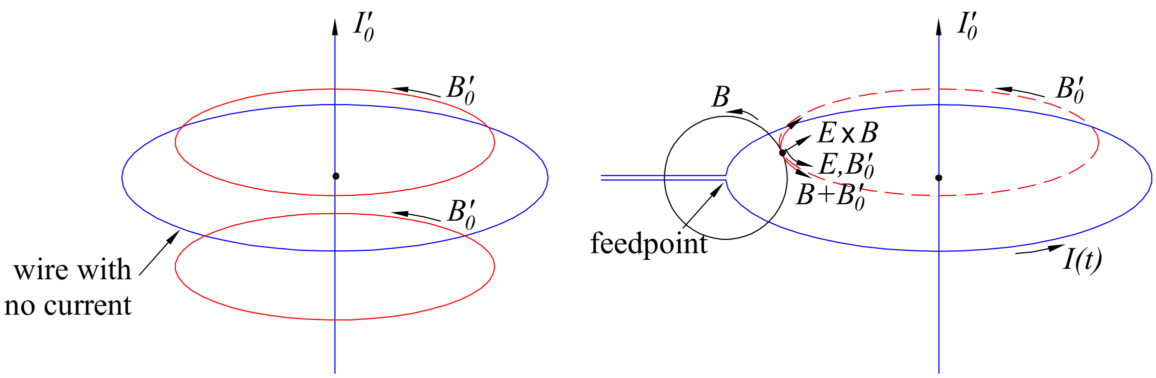
away from the axis at the speed of light[17] and first encounter an originally closed field line (labeled $\mathbf{B}_0$ in the left figure below) at the point on the closed line closest to the axis.

This point is where the originally closed field line of $\mathbf{B}_0$ "breaks" and transforms (instantaneously) into an infinite-length helix (of total field $\mathbf{B}_{\rm tot} = \mathbf{B}_0 + \mathbf{B}'$, with very long initial pitch). Such a transformation is mathematically possible, but could not occur for a "physical object", which reinforces item 1 of Sec. 1.

The transformation is illustrated in the figures below; the left figure is "before", and the right figure "after", the turn-on of the axial current I'. It is noteworthy that $\mathbf{E} \times \mathbf{B}_{\rm tot}$ of the "after" fields points in the direction of the deflection of the "broken" end of an originally closed line of field $\mathbf{B}_0$ due to current I_0 in the circular loop, as it becomes a helical field line of the total magnetic field $\mathbf{B}_{\rm tot} = \mathbf{B}_0 + \mathbf{B}'$.



Similarly, we can consider the case where the axial current I'_0 (and its magnetic field $\mathbf{B}'_0$) is established first, and the circular loop of current I (and its magnetic field $\mathbf{B}$) is turned on at a later time. This is illustrated in the "before" and "after" figures below. We identify the relevant "break" point in a circular line of magnetic field $\mathbf{B}'_0$ due to axial current I'_0 as the point closest to the feedpoint of the circular loop of current I. Again, $\mathbf{E} \times \mathbf{B}_{\rm tot}$ of the "after" fields points in the direction of the deflection of the "broken" end of an originally closed line of field $\mathbf{B}'_0$, as it becomes a helical field line of $\mathbf{B}_{\rm tot} = \mathbf{B} + \mathbf{B}'_0$.



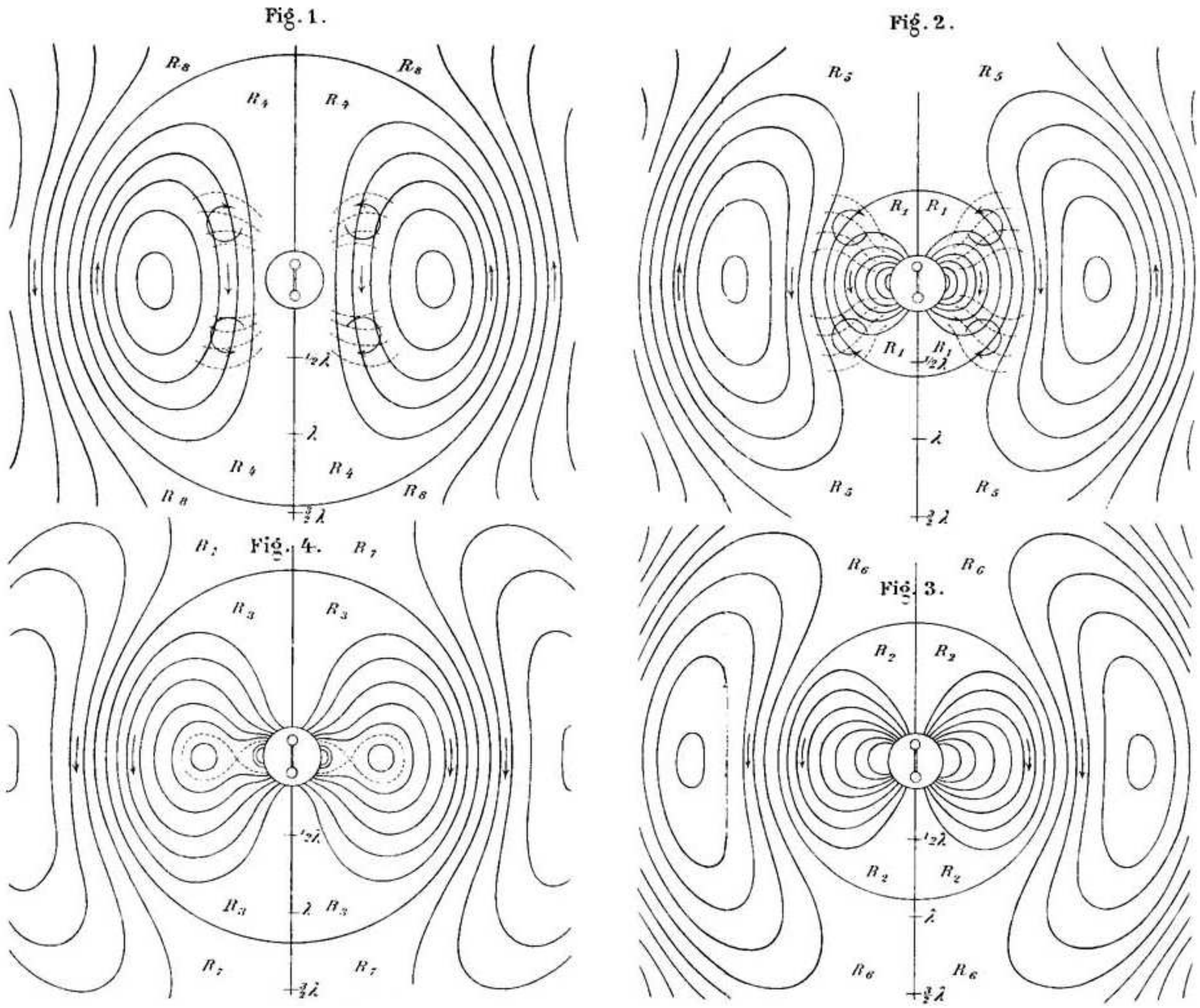
Slepian's example was mentioned, without figures, in [48, 49], and then with figures in [51, 86, 90, 94, 95, 102, 108, 109]. As with most discussions of field lines, not everything said in these papers is in agreement with the views expressed here.

[17] See [81] for a discussion of the fields of an axial current that rises linearly with time.

3 Electric Dipole Radiation

Hertz' great paper of 1889 [35], in which he gave the first theoretical analysis of electric dipole radiation (after having previously demonstrated it experimentally; see Chaps. II-VIII of [38]), contains two features (not noted by him) of interest to the present discussion. His Fig. 4, shown below, has the first depiction of a null point where "breaking and reconnection" of an (electric) field line occurs. Also, some parts of the electric field lines in his Figs. 1-4 move outward faster than the speed of light, which illustrates that field lines are not "physical" objects.

Two null points are shown in Fig. 4 at the intersections of the dashed electric field lines. Here, an electric field line, that previously terminated on the positive and negative electric charges of the dipole, "pinches" together resulting in a closed field line which thereatfer propagates away from the dipole, as well as a shorter field line that remains connected to the two charges.



Figures 1-4 (arranged clockwise) are at times T, $T/4$, $T/2$ and $3T/4$, where $T = 2\pi\lambda/c$ is the period of oscillation of the system and λ is the wavelength of the radiation far from the dipole (taken to be in vacuum). "Breaking and reconnection" of electric field lines occurs throughout the interval $T/4 < t < 3T/4$ of each period. Meanwhile, magnetic field lines, which are circular closed loops perpendicular to, and centered on, the symmetry axis of the system, are being continually created at points along that axis, which is a null line of the magnetic field.[18]

Just after time $t = 0$, as in Fig. 1, all of the electric field lines attached to the electric charges of the system lie within a small sphere contained the charges. In principle, the radius of this sphere can be arbitrarily small for a "point" electric dipole. By time $t = 3T/4 = 3\pi\lambda/2c$, as in Fig. 4, these field lines have moved outward, with some nearly touching the sphere shown in the figure, whose radius r is greater that the wavelength λ of the (far-field) radiation. As such, the (average) radial speed of the outermost field lines within the sphere of radius r at, say, its "equator" is $r/t > \lambda/t = 2\pi c/3 > 2c$. In other terms, the phase velocity of the electromagnetic waves exceeds the speed of light close to the oscillating dipole.

A Appendix: Some History of Field Lines from Perigrinus to Maxwell

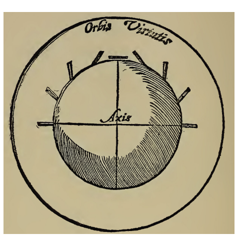
A.1 Peregrinus

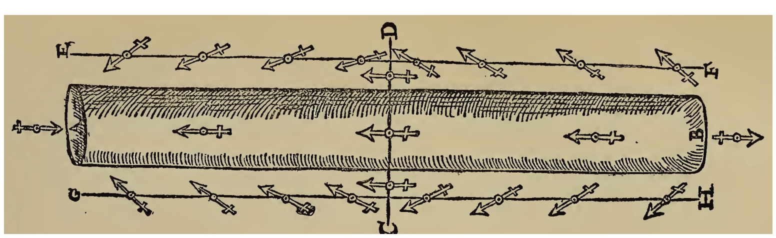
In 1269, Peregrinus [1] concluded from experiment that a (permanent) magnet has two "poles", and that like poles of different magnets repel, while unlike poles attract. Peregrinus' methods were later notably extended by Gilbert.

A.2 Gilbert

In 1600, Gilbert published a treatise [2], which included qualitative notions of magnetic energy and lines of force.[19] These seem to have been inspired by experiments (suggested by Peregrinus [1]) in which magnetized needles (or steel filings, p. 162 of [2]) were used to probe the space around larger magnets, particularly spherical magnets called terrellas. The observed directions of the needle (or filings) suggested the existence of lines of force throughout space, and the ability of the magnet to deflect the needle into alignment with them suggested (to Gilbert) that some kind of magnetic energy existed outside the magnet itself.

The figures below are from pp. 122 and 247 of [2].



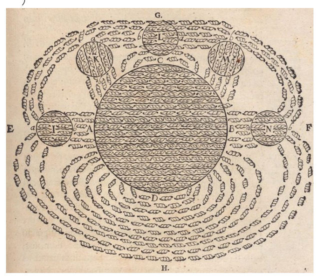


[18] Additional discussion of "breaking and reconnection" in electric dipole radiation is given is Sec. 20.5.2 of [96] and in [103].

[19] Then, as now, magnetism seems to have inspired claims of questionable merit, which led Gilbert to pronounce on p. 166: *May the gods damn all such sham, pilfered, distorted works, which do but muddle the minds of students!*

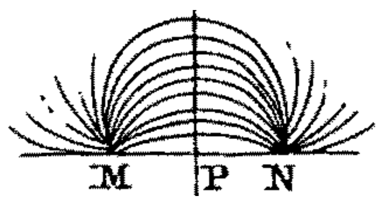
A.3 Descartes

In 1644, Descartes published a qualitative treatise on physical science [3]. Among its notable features is perhaps the earliest conception of momentum (mass times velocity, p. 59 ff), and that light could be due to static pressure in a kind of elastic medium that fills all space, later called the æther (pp. 94-104, part III). And, on p. 271, part IV, he presented a figure based on use of iron filings near a magnet, which illustrates Gilbert's lines of magnetic force.



A.4 Aepinus

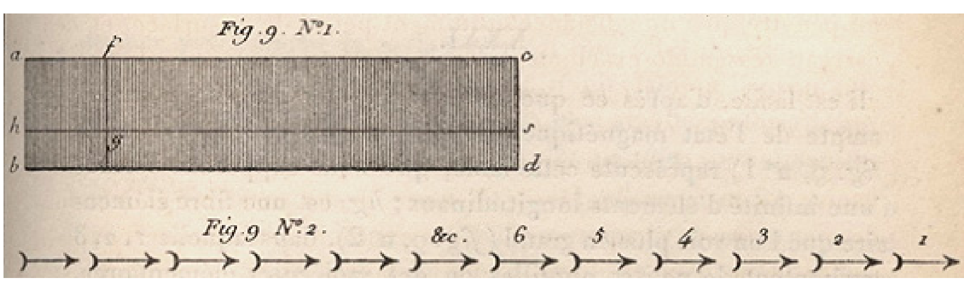
In 1759, Aepinus [4] published a long essay on electricity and magnetism that included a sketch (shown on the next page) of lines of iron filings on a plane above a horseshoe magnet. He speculated that these line formed circles, and that the magnetic force due to a single magnetic pole varied as $1/r^2$.



A.5 Coulomb

In 1785, Coulomb confirmed (and made widely known) that the static force between pairs of electric charges q_1 and q_2 varies as $q_1 q_2/r^2$ [6], and that the force between idealized magnetic poles p_1 and p_2 at the ends of long, thin magnets varies as $p_1 p_2/r^2$ [7]. The electric and magnetic forces were considered to be unrelated, except that they obeyed the same functional form.

Coulomb also noted that magnetic poles appear not to be isolatable, conjecturing (p. 306 of [8]) that the fundamental constituent of magnetism, a *molécule de fluide magnétique*, is a dipole, such that effective poles appear at the ends of a long, thin magnet.



Coulomb's argument is much superior to that by Gilbert, p. 247 of [2].

A.6 Faraday

A.6.1 Evolution of Electricity from Magnetism

Faraday's first report, Arts. 27-28 of [11] (1831), of an effect of magnetic induction was via an iron ring with two coils wound upon it, as in Fig. 1 below. On connecting or disconnecting one coil to/from a battery, a transient current was observed in the other coil.

Fig. 1.

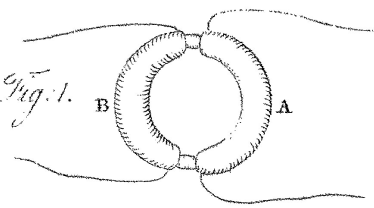
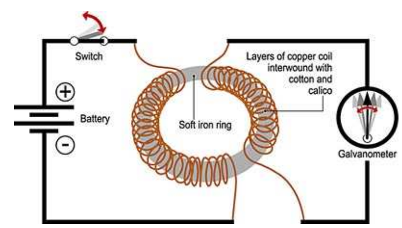


Photo of Faraday's original transformer (courtesy MIT Burndy Library).



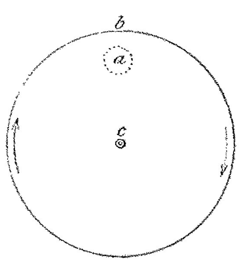
This was a "transformer" effect, and involved no motion of conductors relative to magnetic fields. In 1831, he gave no explanation of transformer action as an effect of time-dependent magnetic flux (or number of magnetic field lines). Art. 1108 of [13] may be as close as Faraday came to relating electromagnetic induction in fixed circuits to a variation in the magnetic flux (number of magnetic field lines) through a circuit:
From the facility of transference to neighbouring wires, and from the effects generally, the inductive forces appear to be lateral, i.e. exerted in a direction perpendicular to the direction of the originating and produced currents: and they also appear to be accurately represented by the magnetic curves, and closely related to, if not identical with, magnetic forces.

However, Art. 1114 of [13] indicates that Faraday's views on this were not very clear.

A.6.2 Anticipation of the Biot-Savart/Lorentz Force Law

In Art. 99 of [11], Faraday gave a first interpretation of the behavior he had observed using a galvanometer connected to a rotating copper disk via sliding contacts:

Fig. 15.



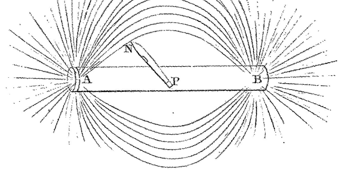
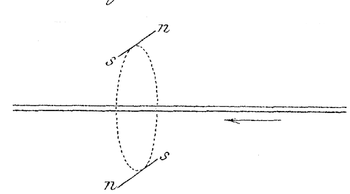


Fig. 25.

Fig. 40.



The relation of the current of electricity produced, to the magnetic pole, to the direction of rotation of the plate, &c. &c., may be expressed by saying, that when the unmarked (south) *pole is beneath the edge of the plate, and the latter revolves horizontally, screw-fashion, the electricity which can be collected at the edge of the plate nearest to the pole is positive. As the pole of the earth may mentally be considered the unmarked pole, this relation of the rotation, the pole, and the electricity evolved, is not difficult to remember. Or if, in fig. 15* (above), *the circle represent the copper disk revolving in the direction of the arrows, and a the outline of the unmarked pole placed beneath the plate, then the electricity* *collected at b and the neighbouring parts is positive, whilst that collected at the centre c and other parts is negative. The currents in the plate are therefore from the centre by the magnetic poles towards the circumference.*

We recognize this as a version of the Biot-Savart-Lorentz law that the force $d\mathbf{F}$ on an electric current element $I\,d\mathbf{l}$ in a magnetic field $\mathbf{B}$ is given by vector relation,[20]

$$d\mathbf{F} = I\,d\mathbf{l} \times \mathbf{B}. \tag{2}$$

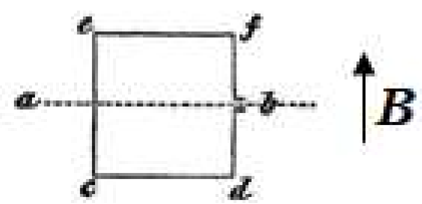
A.6.3 *Magnetic Curves* aka Field Lines

Faraday continued his discussion of the generation of electric currents in Arts. 114-116 of [11], referring to the *magnetic curves* in Fig. 25 above:
By magnetic curves, I mean the lines of magnetic forces, however modified by the juxtaposition of poles, which would be depicted by iron filings; or those to which a very small magnetic needle would form a tangent.

This is Faraday's first representation of magnetic field lines.[21] In Fig. 25, A is the north pole of the magnet, and B is the south. When, the tip of a knife blade is rotated up/out of the page, with its base remaining on the magnet, Faraday noted that an electric current flows from the tip to the base, *i.e.*, from N to P, in agreement with eq. (2) and the verbal statement thereof in Art. 99.

A.6.4 Effect of "Cutting" the Magnetic Curves

In Art. 114 Faraday spoke of such action as involving a conductor *cutting* the *magnetic curves*, which notion has come to be regarded as a central feature of Faraday's vision of magnetic induction of electric currents. Now, it seems better to de-emphasize the (appealing) notion of "cutting of field lines", and rather to emphasize the interpretation of a motional $\mathcal{EMF}$ due to the Lorentz force on moving conduction electrons in a magnetic field.

Fig. 3.


That Faraday's view was close to that of motional $\mathcal{EMF}$ is illustrated in Art. 3192 of [21]. In the case of a rectangular wire loop, *c-d-e-f* rotated about a median line *a-b* that is perpendicular to a uniform, constant magnetic field, as sketched in Fig. 3 above, Faraday remarked:
In the first 180° of revolution round the axis a-b, the contrary direction in which the two parts c-d and e-f intersect the lines of magnetic force within the area c-e-d-f, will cause

[20]The "Biot-Savart-Lorentz" force law was first stated by Maxwell, in many different versions over the years, as reviewed in Appendix A.24 of [105].

[21]Faraday showed that the *magnetic curves* associated with a current-carrying wire are circles, Arts. 232-233 of [12] and Fig. 40 on the previous page. In this, he was recalling his discussion of *magnetic curves* in [10] (1822), and Davy's use of iron filings in studies of magnetic effects [9] (1820), following in the footsteps of Gilbert (1600), Appendix A.2, and Descartes (1644), Appendix A.3.

them to conspire in producing one current, tending to run round the rectangle. The parts c-e and d-f of the rectangle may be looked upon simply as conductors; for as they do not in their motion intersect any of the lines of force, so they do not tend to produce any current.

A delicacy is in the interpretation of the term *intersect*, as wire segments *c-e* and *d-f* do touch lines of force, and the number of lines they touch varies with time. One might well say that these moving wires do "cut" lines of force. However, the effect of the $\mathbf{v} \times \mathbf{B}$ is transverse to the wires, and does not drive any current. This was noted by Faraday, who must have had a good intuition as to the vector-cross-product character of the cause of the induced current.

Thus, Art. 3192 of [21] provided a clarification to earlier statements by Faraday, such as that in Art. 256 of [12]: *If a terminated wire move so as to cut a magnetic curve, a power is called into action which tends to urge an electric current through it*, which downplays the cross-product character of the "urge".

A.6.5 Electric Lines of Force

The notion of electric lines of force, with tension along them and repulsion between them, appears in Art. 1297 of [14]:
The direct inductive force, which may be conceived to be exerted in lines between the two limiting and charged conducting surfaces, is accompanied by a lateral or transverse force equivalent to a dilatation or repulsion of these representative lines (Art. 122.); or the attractive force which exists amongst the particles of the dielectric in the direction of the induction is accompanied by a repulsive or a diverging force in the transverse direction (Art. 1304).

His summary in Art. 1304 of [14] includes the statements:
I have used the phrases lines of inductive force and curved lines of force (Arts. 1231, 1297, 1298, 1302) in a general sense only, just as we speak of the lines of magnetic-force. The lines are imaginary, and the force in any part of them is of course the resultant of compound forces, every molecule being related to every other molecule in all directions by the tension and reaction of those which are contiguous.

A.6.6 The Magnetic Field

We have already noted that Faraday used the term *lines of magnetic force* in a footnote to Art. 114 of [11] (1831).

In 1845, Art. 2247 of [16], the term *magnetic field* appears for the first time in print:
The ends of these bars form the opposite poles of contrary name; the magnetic field between them can be made of greater or smaller extent, and the intensity of the lines of magnetic force be proportionately varied.

A.6.7 Magnetic Power

In 1850, Art. 2806 of [18], Faraday wrote:
Any portion of space traversed by lines of magnetic power, may be taken as such a (magnetic) field, and there is probably no space without them. The condition of the field may vary in intensity of power, from place to place, either along the lines or across them...

Art. 2807. When a paramagnetic conductor, as for instance, a sphere of oxygen, is introduced into such a magnetic field, considered previously as free from matter, it will cause a concentration of the lines of force on and through it, so that the space occupied by it transmits more magnetic power than before (fig. 1). If, on the other hand, a sphere of diamagnetic matter be placed in a similar field, it will cause a divergence or opening out of the lines in the equatorial direction (fig. 2); and less magnetic power will be transmitted through the space it occupies than if it were away.

Fig. 1.

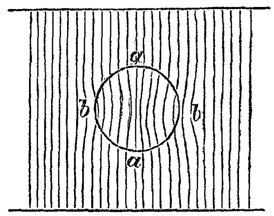
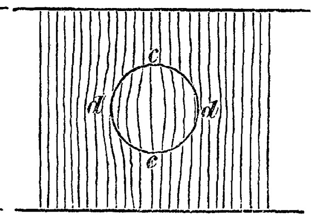


Fig. 2.



Here, one can identify Faraday's usage of the term *magnetic power* with the magnetic flux $\Phi_{\mathbf{B}} = \int \mathbf{B} \cdot d\mathbf{Area}$.

A further consequence of his interaction with Thomson appears to be that in 1852, beginning in Art. 3070 of [19], Faraday wrote about *lines of force* more abstractly, but without full commitment to their physical existence independent of matter. Thus, in Art. 3075 he stated:

I desire to restrict the meaning of the term line of force, so that it shall imply no more than the condition of the force in any given place, as to strength and direction; and not to include (at present) any idea of the nature of the physical cause of the phenomena...

A few sentences later he continued:

...for my own part, considering the relation of a vacuum to the magnetic force and the general character of magnetic phenomena external to the magnet, I am more inclined to the notion that in the transmission of the force there is such an action, external to the magnet, than that the effects are merely attraction and repulsion at a distance. Such an action may be a function of the ether; for it is not at all unlikely that, if there be an ether, it should have other uses than simply the conveyance of radiations (Arts. 2591, 2787).

In Art. 3175, at the end of [19], he added:

...wherever the expression line of force is taken simply to represent the disposition of the forces, it shall have the fullness of that meaning; but that wherever it may seem to represent the idea of the physical mode of transmission of the force, it expresses in that respect the opinion to which I incline at present.

This has led many to infer that Faraday then believed in the physical existence of the lines of force even though he could not "prove" that.

Faraday's famous notion, that induced electrical currents are associated with wires "cutting" lines of magnetic force, is presented in Art. 3104, and a version of what is now called Faraday's law,

$$\mathcal{EMF} = -\frac{d}{dt} \int \mathbf{B} \cdot \text{Area}, \tag{3}$$

is given verbally in Art. 3115,[22]

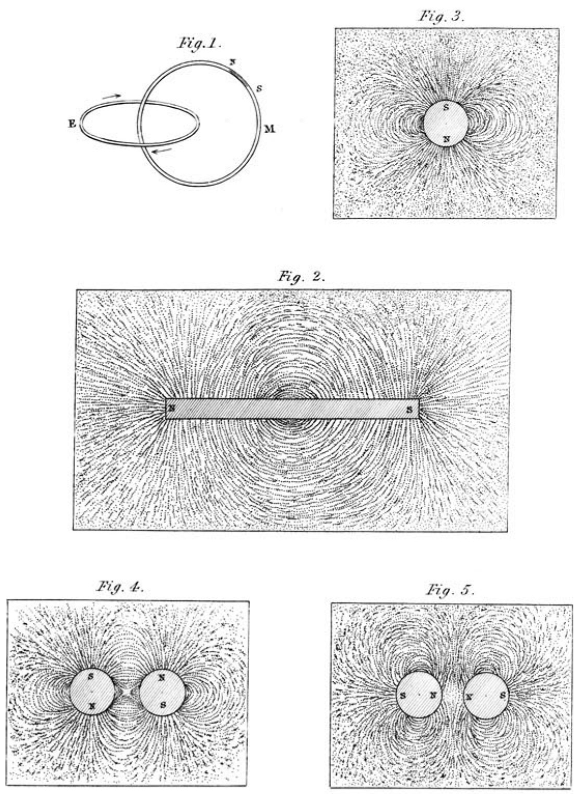
[22] One should not infer from this that Faraday had an explicit notion of the magnetic field $\mathbf{B}$ as a measure

The quantity of electricity thrown into a current is directly as the amount of curves intersected.

In Art. 3117 Faraday noted that magnetic lines of force form closed circuits:
Every line of force therefore, at whatever distance it may be taken from the magnet, must be considered as a closed circuit, passing in some part of its course through the magnet, and having an equal amount of force in every part of its course.
However, the last phrase indicates that Faraday did not have a clear view of what we call the strength of a magnetic field.

In Art. 3118 Faraday (re)affirmed that magnetic field lines do not rotate with a rotating magnet, and performs various experiments with what is now called a unipolar (or homopolar) generator to demonstrate this, which experiments are an early investigation of the relativity of rotating frames.

In 1852, Faraday also published a set of more speculative comments [20] in the Phil. Mag. (rather than Phil. Trans. Roy. Soc. London, the usual venue for his *Experimental Researches*), arguing more strongly for the physical reality of the lines of force.



In Art. 3258 of [20] he considered the effect of a magnet in vacuum, concluding (perhaps for the first time) that the lines of force have existence independent of a material medium:
A magnet placed in the middle of the best vacuum we can produce...acts as well upon a needle as if it were surrounded by air, water or glass; and therefore these lines exist in such a vacuum as well as where there is matter.

Faraday used examples of magnets and iron filings in various configurations to reinforce

of the density of lines of magnetic force. Rather, he emphasized the total number of lines within some area (the magnetic flux) as the *amount of magnetic force* (Art. 3109).

his vision of a tension along the lines of forces, and in Art. 3295 added the insight that there is a lateral repulsion between adjacent lines, referring to Fig. 5 on the previous page.

Faraday's last published comments on lines of force are in [23] (1854).[23]

A.7 W. Thomson (Lord Kelvin)

In 1850, Thomson was the first to use the term "magnetic field" in the contemporary sense, on p. 179 of [17]:

Definition.—Any space at every point of which there is a finite magnetic force is called "a field of magnetic force"; or, magnetic being understood, simply "a field of force"; or, sometimes, "a magnetic field".

Definition.—A "line of force" is a line drawn through a magnetic field in the direction of the force at each point through which it passes; or a line touched at each point of itself by the direction of the magnetic force.

However, Thomson made no drawings of field lines until 1872; see pp. 488-491 of [24].

A.8 Maxwell

Maxwell published his developments of the theory of electrodynamics in four steps: *On Faraday's Lines of Force* [25] (read 1856), *On Physical Lines of Force* [26, 27, 28, 29] (1861-62), *A Dynamical Theory of the Electromagnetic Field* [31] (1864), and in his *Treatise on Electricity and Magnetism* [32, 33] (1873). An *Elementary Treatise on Electricity* [34] was published posthumously (1881). While his dynamical theory was extrapolated from Faraday's concept of field lines, Maxwell emphasized the electromagnetic fields rather than lines thereof.[24] Drawings of static field lines appeared in [26, 27, 32, 33, 34], while Maxwell never discussed dynamical field lines.[25]

In Art. 293 of his Treatise [32], Maxwell discussed lines and tubes of flow of a hydrodynamic velocity field, using the convention mentioned in item 7 of the Overview above.

A.9 Streamlines

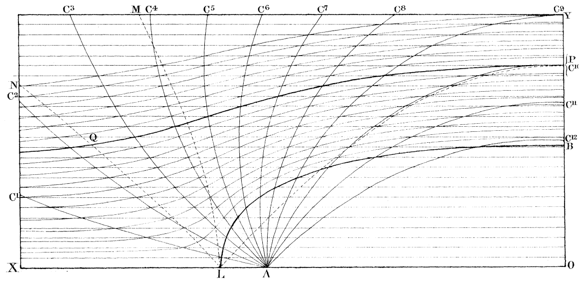
Independent of the development of the concept of magnetic-field lines outlined above, the idea of "streamlines" emerged in fluid dynamics. In 1781, Lagrange [5] introduced the notion of a scalar "stream function" ψ, that can be regarded as a "velocity potential", such that the velocity of a two-dimensional, planar, incompressible flow fluid is related by $\mathbf{v} = -\boldsymbol{\nabla}\psi$ when also $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$. The concept of "streamlines" of the velocity field $\mathbf{v}$ was discussed by Stokes (1842) [15],[26] who, like Lagrange, included no diagrams. The first published representation of streamlines in a figure may be due to Rankine (1863) [30], as shown below.

[23] van Rees [22] objected to Faraday's concept of closed-loop magnetic field lines, arguing that all field lines must have a source.

[24] In downplaying field lines, Maxwell may have been acting on his statement, p. 47 of [34]: "Imaginary quantities, such as are mentioned in treatises on analytical geometry, have no place in physical science."

[25] Maxwell's famously arcane mechanical model of the magnetic field in [27] was not a model of field lines.

[26] Stokes also discussed two-dimensional, axisymmetric, incompressible flow via a stream function $\psi(r, z) = \psi(r, \theta)$ in cylindrical coordinates (r, ϕ, z) and spherical coordinates (r, θ, ϕ), where the z-axis is the symmetry axis. Here, the fluid velocity is related by $\mathbf{v} = \boldsymbol{\nabla} \times \boldsymbol{\Psi}$ with $\boldsymbol{\Psi} = \psi\, \hat{\boldsymbol{\phi}}/r \sin\theta$. Again, $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$.



The lore of fluid streamlines is highly mathematical, and did not lead to an impression that the streamlines were "physical". Indeed, visualizations of fluid flow variously involve "streamlines", "streaklines" and "pathlines", which differ in non-steady flow. See, for example, https://en.wikipedia.org/wiki/Streamlines,_streaklines,_and_pathlines.

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Sec. 1-2: "We adopt the convention that the number of lines per unit area at right angles to the lines is proportional to the field strength. This is, of course, only an approximation, and it will require, in general, that new lines sometimes start up in order to keep the number up to the strength of the field." Sec. 1-5: "The field lines, however, are only a crude way of describing a field, and it is very difficult to give the correct, quantitative laws directly in terms of field lines. Also, the ideas of the field lines do not contain the deepest principle of electrodynamics, which is the superposition principle. Even though we know how the field lines look for one set of charges and what the field lines look like for another set of charges, we don't get any idea about what the field line patterns will look like when both sets are present together." Also: "Not only is it not possible to say whether the field lines move or do not move with chargesthey may disappear completely in certain coordinate frames."

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