Fluid Flow Velocity at a Wall

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1 Problem

Discuss the conditions for which the usual approximation holds that the velocity of fluid flow relative to a boundary is zero.

To be specific, consider the flow of a fluid of viscosity μ in a cylindrical tube of radius r_0 and length L when pressure ΔP is applied between the ends of the tube. Deduce the drag force per unit area on the boundary wall, and compare this with a simple model for the binding of fluid molecules to that wall.

2 Solution

An argument related to the following has been given in sec. 331 of H. Lamb, Hydrodynamics, $4^{\rm th}$ *ed. (Cambridge U. Press, 1916),* http://kirkmcd.princeton.edu/examples/fluids/lamb_4th_ed.pdf

The viscous drag force per unit area on the wall is given by,

$$\frac{F}{A} = \mu \frac{dv(r_0)}{dr}\,, \tag{1}$$

where μ is the viscosity, and the velocity profile in laminar flow is parabolic,

$$v(r) = v_0 \left(1 - \frac{r^2}{r_0^2}\right)\,. \tag{2}$$

Taking the derivative, we find the magnitude of the force to be,

$$\frac{F}{A} = 2\mu \frac{v_0}{r_0}\,. \tag{3}$$

For laminar fluid flow, Poiseuille's equation relates the flow rate to the pressure gradient,[1]

$$Q = \pi r_0^2 \bar{v} = \frac{\pi r_0^4 \Delta P}{8\mu L}\,. \tag{4}$$

For the parabolic velocity profile (2), the average velocity is given by,

$$\bar{v} = \frac{\int_0^{r_0} v_0 \left(1 - \frac{r^2}{r_0^2}\right) 2\pi r\ dr}{\pi r_0^2} = \frac{v_0}{2}\,. \tag{5}$$

[1]J.L.M. Poiseuille, *Researches expérimentales sur le mouvement des liquides dans les tubès de tres petite diamètres*, Comptes Rendus Acad. Sci. **12**, 112 (1941), http://kirkmcd.princeton.edu/examples/fluids/poiseuille_cras_12_112_41.pdf

Combining eqs. (3)-(5), we have

$$\frac{F}{A} = \frac{r_0 \Delta P}{2L}\,, \tag{6}$$

independent of the fluid viscosity!

We now estimate the sticking pressure F/A of molecules to the wall by supposing that the molecules have characteristic length of 1 Angstrom = 10^{-10} m and binding energy to the wall of, say, 0.1 eV $\approx 10^{-20}$ J. We suppose that the molecule must be displaced by ≈ 1 Angstrom to break the bond, so,

$$\frac{F}{A} \approx \frac{10^{-20}\ \mathrm{J/1\ Angstrom}}{(1\ \mathrm{Angstrom})^2} \approx 10^{10}\ \mathrm{N/m^2}. \tag{7}$$

For pressure gradients of order 1 atmosphere = 10^5 N/m^2, the viscous drag force would be sufficient to keep molecules from sticking to the wall only if,

$$\frac{r_0}{L} \approx 10^5. \tag{8}$$

However, in almost all mechanical or biological systems r_0/L is less than one. Hence, we conclude that viscous drag is far from being able to dislodge those fluid molecules that stick to the wall, and the usual assumption of zero relative velocity of a fluid next to a wall is valid.

Since the sticking pressure between adjacent molecule of the fluid is similar to that between fluid molecules and the wall, we also conclude that viscous drag forces during ordinary fluid flow will not result in tearing or cavitation. However, the latter can and does occur when the fluid velocity is high enough that the pressure in the fluid goes negative, according to Bernoulli's law,

$$P + \rho g h + \frac{1}{2}\rho v^2 = \text{ constant}. \tag{9}$$

In particular, a moving wall, such as a ship's propellor, with velocity v will cause the local fluid velocity to be v also, with the consequence that the local pressure is reduced. In water, we see that cavitation can occur when $v > \sqrt{2P/\rho} \approx 15$ m/s near the surface, or $v \gtrsim 20$ m/s at depth $h = 10$ m below the surface, *etc.* This argument is oversimplified in that a fluid can sustain a negative pressure without tearing, particularly if the fluid is static. For example, very pure water in a centrifuge with highly polished walls can sustain negative pressures up to several hundred atmospheres.[2] But in turbulent flow, as near a ship's propellor, the simple model above gives reasonable guidance.

[2]L.J. Briggs, *Limiting Negative Pressure of Water*, J. Appl. Phys. **21**, 721 (1950), http://kirkmcd.princeton.edu/examples/fluids/briggs_jap_21_721_50.pdf