Electric Flux of an Accelerated Charge

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(November 14, 2020)

1 Problem

Maxwell's first equation, $\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\rho$ (in Gaussian units), implies that the flux of electric field $\mathbf{E}$ across any surface enclosing electric charge q at some time t is $4\pi q$, independent of the motion of the charge. Show that in case of an accelerated point charge this relation holds for the flux across the spherical shell whose center is the retarded position of the charge defined by an arbitrary observation point (at time t) and whose radius is the distance from the retarded position to the observation point.

This problem was suggested by Javier Castro Paredes, and the solution follows Alfonso Fondado.

2 Solution

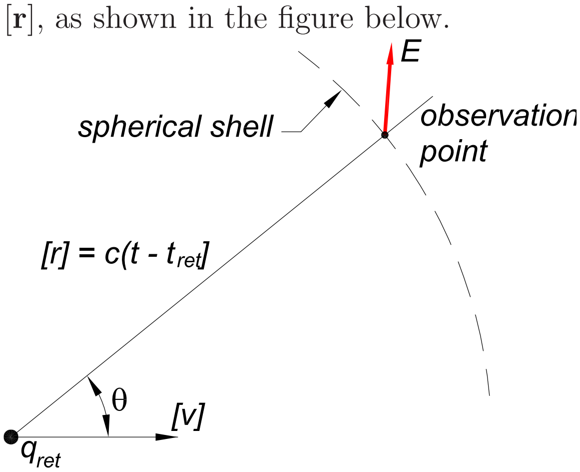
Since the velocity of the accelerated charge is less than the speed of light c in vacuum, the charge is inside the sphere under consideration, and hence the electric flux across the sphere must be $4\pi q$ (assuming there is no other charge inside the sphere).

We can verify this in more detail using the fields of an accelerated charge q with position $\mathbf{x}_q(t)$ as deduced by Liénard (1898) [1] and by Wiechert (1900) [2] (in Gaussian units),

$$\mathbf{E}(\mathbf{x}, t) = q \left[\frac{\hat{\mathbf{r}} - \boldsymbol{\beta}}{\gamma^2 r^2 (1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^3} \right] + \frac{q}{c} \left[\frac{\hat{\mathbf{r}} \times ((\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}})}{r(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^2} \right], \qquad \mathbf{B} = [\hat{\mathbf{r}}] \times \mathbf{E}, \tag{1}$$

where $\boldsymbol{\beta} = \mathbf{v}/c = d\mathbf{x}_q/d\,ct = \dot{\mathbf{x}}_q/c$, $\gamma = 1/\sqrt{1-\beta^2}$, quantities inside brackets [ ] are evaluated at the retarded time, $t_{\rm ret} = t - [r]/c$, $[\mathbf{r}] = \mathbf{x}_q(t) - \mathbf{x}_q(t_{\rm ret})$, and $[\hat{\mathbf{r}}] = [\mathbf{r}]/[r]$.

We consider some observation point $\mathbf{x}$ at time t, for which the distance to the retarded position of the charge is $[\mathbf{r}]$, as shown in the figure below.



Next, we consider the spherical shell of radius $[r]$ centered on the retarded position of the charge, and evaluate the flux of the electric field across this shell at time t. Only the first term of $\mathbf{E}$ in eq. (1) contributes to this flux,

$$\Phi = 2\pi[r]^2 \int_{-1}^{1} d\cos\theta\, \mathbf{E}\cdot[\hat{\mathbf{r}}] = 2\pi[r]^2 \int_{-1}^{1} d\cos\theta \left[\frac{q}{\gamma^2 r^2 (1-\boldsymbol{\beta}\cdot\hat{\mathbf{r}})^2}\right]$$
$$= \frac{2\pi q}{\gamma^2}\int_{-1}^{1} d\cos\theta\, \frac{1}{(1-\beta\cos\theta)^2} = \frac{2\pi q}{\gamma^2}\frac{1}{\beta}\left(\frac{1}{1-\beta} - \frac{1}{1+\beta}\right) = 4\pi q, \qquad (2)$$

taking θ to be the angle between the retarded distance $[\mathbf{r}]$ and the retarded velocity $[\mathbf{v}]$, and noting that the first term of $\mathbf{E}$ is independent of the azimuth around the direction of $[\mathbf{v}]$.

This result is a partial check that the Liénard-Wiechert fields (1) satisfy Maxwell's equations.

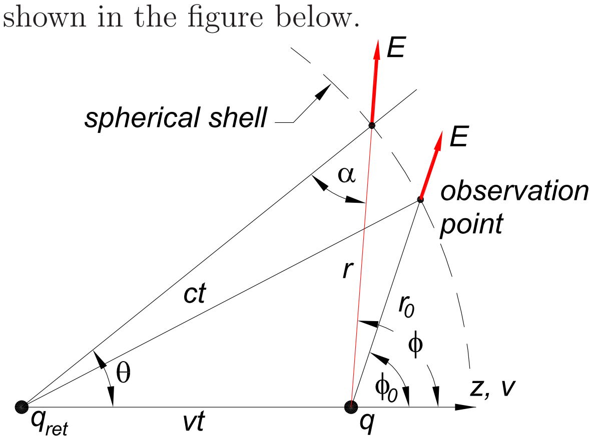
A Appendix: Uniformly Moving Charge

We can also confirm (somewhat laboriously) that the flux is $4\pi q$ for the case of a charge with uniform velocity $\mathbf{v}$, using its electromagnetic fields as deduced in 1888 by Heaviside [3] and by Thomson (1889) [4],[1] prior the results of Liénard and Wiechert,

$$\mathbf{E} = \frac{q}{\gamma^2 r^2 (1-\beta^2\sin^2\phi)^{3/2}}\,\hat{\mathbf{r}}, \qquad \mathbf{B} = \boldsymbol{\beta}\times\mathbf{E}, \qquad \boldsymbol{\beta} = \frac{\mathbf{v}}{c}, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad (3)$$

where $\mathbf{r}$ is the distance from the present position of the charge to the observer, ϕ is the angle between $\mathbf{r}$ and $\mathbf{v}$, and c is the speed of light in vacuum.

We consider an observation point with coordinates (r_0, ϕ_0) with respect to the present position of charge q, as shown in the figure below.



We now adopt a spherical coordinate system with its origin at the retarded position of the charge at the present time t, and the polar axis to lie along the line from the retarder position to the present position. We also define the retarded time of this configuration to be $t_{\rm ret} = 0$.

The distance from the retarded position to the present position is vt, and the distance from the retarded position to the present position is ct, where (although this is not needed

[1]For commentary on their deductions, see the end of [5].

below) the latter is related by,

$$(ct)^2 = r_0^2 + (vt)^2 + 2r_0 vt \cos\phi_0, \qquad ct = \gamma^2 r_0 \left(\beta\cos\phi_0 + \sqrt{1-\beta^2\sin^2\phi_0}\right), \tag{4}$$

such that if $\phi_0 = 0$ then $ct = r_0/(1-\beta)$.

The present position of the charge is inside the spherical shell of radius $[r] = ct$, so the flux Φ of electric field across this shell must be $4\pi q$,

$$\begin{aligned}
\Phi = 2\pi(ct)^2 \int_{-1}^{1} d\cos\theta\, E\cos\alpha = 2\pi q \int_{-1}^{1} d\cos\theta\, \frac{\cos\alpha}{\gamma^2(1+\beta^2-2\beta\cos\theta)(1-\beta^2\sin^2\phi)^{3/2}} \\
= \frac{2\pi q}{\gamma^2}\int_{-1}^{1} d\cos\theta\, \frac{1-\beta\cos\theta}{(1+\beta^2-2\beta\cos\theta)^{3/2}(1-\beta^2+\beta^2 cos^2\phi)^{3/2}} \\
= \frac{2\pi q}{\gamma^2}\int_{-1}^{1} d\cos\theta\, \frac{1-\beta\cos\theta}{\{(1-\beta^2)(1+\beta^2-2\beta\cos\theta)+\beta^2(\beta-\cos\theta)^2\}^{3/2}} \\
= \frac{2\pi q}{\gamma^2}\int_{-1}^{1} d\cos\theta\, \frac{1-\beta\cos\theta}{\{1-\beta^4-2(1-\beta^2)\beta\cos\theta+\beta^4-2\beta^3\cos\theta+\beta^2\cos^2\theta\}^{3/2}} \\
= \frac{2\pi q}{\gamma^2}\int_{-1}^{1} d\cos\theta\, \frac{1-\beta\cos\theta}{(1-2\beta\cos\theta+\beta^2\cos^2\theta)^{3/2}} = \frac{2\pi q}{\gamma^2}\int_{-1}^{1} d\cos\theta\, \frac{1-\beta\cos\theta}{(1-\beta\cos\theta)^3} \\
= \frac{2\pi q}{\gamma^2}\int_{-1}^{1} d\cos\theta\, \frac{1}{(1-\beta\cos\theta)^2} = \frac{2\pi q}{\gamma^2}\frac{1}{\beta}\left(\frac{1}{1-\beta} - \frac{1}{1+\beta}\right) = 4\pi q,
\end{aligned} \tag{5}$$

noting that,

$$r^2 = (ct)^2 + (vt)^2 - 2(ct)(vt)\cos\theta = (ct)^2(1+\beta^2-2\beta\cos\theta), \tag{6}$$

$$\frac{\sin\alpha}{vt} = \frac{\sin\phi}{ct} = \frac{\sin\theta}{r}, \tag{7}$$

$$\cos\alpha = \frac{\sqrt{r^2-(vt)^2+(vt)^2\cos^2\theta}}{r} = \frac{\sqrt{1-2\beta\cos\theta+\beta^2\cos^2\theta}}{\sqrt{1+\beta^2-2\beta\cos\theta}} = \frac{1-\beta\cos\theta}{\sqrt{1+\beta^2-2\beta\cos\theta}}, \tag{8}$$

$$\cos\phi = \frac{\sqrt{r^2-(ct)^2+(ct)^2\cos^2\theta}}{r} = \frac{\sqrt{\beta^2-2\beta\cos\theta+\cos^2\theta}}{\sqrt{1+\beta^2-2\beta\cos\theta}} = \frac{\beta-\cos\theta}{\sqrt{1+\beta^2-2\beta\cos\theta}}. \tag{9}$$

For another example of Gauss' law in relation to a uniformly moving charge, see the Appendix to [6].

References

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[3] O. Heaviside, *The Electromagnetic Effects of a Moving Charge—II*, Electrician **22**, 147 (1888), http://kirkmcd.princeton.edu/examples/EM/heaviside_electrician_22_147_88.pdf
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[4] J.J. Thomson, *On the Magnetic Effects produced by Motion in the Electric Field*, Phil. Mag. **28**, 1 (1889), http://kirkmcd.princeton.edu/examples/EM/thomson_pm_28_1_89.pdf

[5] K.T. McDonald, *Special Relativity*, Princeton PHY206/501 lecture notes, http://kirkmcd.princeton.edu/examples/ph501/ph501lecture18.pdf

[6] K.T. McDonald, *Classical Electromagnetic Fields of a Pair of Annihilating Charges* (June 11, 2021), http://kirkmcd.princeton.edu/examples/annihilate.pdf