Force-Free Magnetic Fields aka Eigenfunctions of the Curl Operator

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1 Problem

Deduce forms of a static magnetic field $\mathbf{B}(\mathbf{x})$ such that the Lorentz force density $\mathbf{J} \times \mathbf{B}$ on the associated current density $\mathbf{J}$ is everywhere zero.[1,2]

Assuming that the medium has permeability μ_0 (and that any electric field is also static), the current density is proportional to $\boldsymbol{\nabla} \times \mathbf{B}$, so the Lorentz force vanishes if $(\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B} = 0$, which obtains when,

$$\boldsymbol{\nabla} \times \mathbf{B} = f(\mathbf{x})\mathbf{B} \qquad (1)$$

for any scalar function $f(\mathbf{x})$, noting that $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$. In particular, the function f can be a constant k, such that any (vector) eigenfunction of the curl operator is a possible force-free magnetic field.[3]

2 Solution

2.1 Cowling's Theorem

Force-free magnetic fields are a possible model of the magnetic fields of planets, stars and other astrophysical regions, which fields are observed to be quasistatic. The question of static, force-free magnetic fields seems to have been first considered by Cowling [5, 6], who concluded that they cannot exist if they are to be axially symmetric. This result is sometimes called Cowling's Theorem. A corollary is that the Earth's magnetic field is dynamic and/or nonaxisymmetric.

However, it appears that this theorem holds only with the additional assumption that the magnetic field has no azimuthal component B_ϕ [7], contrary to the claim of Cowling.

A static, force-free magnetic field has $\mathbf{J} \propto \boldsymbol{\nabla} \times \mathbf{B} \propto \mathbf{B}$, so the magnetic field exists only where the current density $\mathbf{J}$ is nonzero. Thus, there is no force-free magnetic field external to the current distribution, and such a field cannot apply to astrophysical objects such as the

[1] There is no such thing as a force-free electric field, since force density $\varrho\mathbf{E}$ on charge density ϱ can be zero only if $\mathbf{E} = 0$ wherever $\varrho \neq 0$, but the first Maxwell equation $\boldsymbol{\nabla} \cdot \mathbf{E} = \varrho/\epsilon_0$ implies that $\mathbf{E}$ is nonzero wherever the volume charge density ϱ is nonzero.

[2] The conducting medium is subject to internal stresses described by the Maxwell stress tensor, $(1/\mu_0)(B_iB_j - \delta_{ij}B^2/2)$, which are always nonzero for nonzero $\mathbf{B}$ and can lead to deformations of the medium even if the Lorentz force is small/zero [1].

[3] If the vector $\mathbf{B}$ represents the velocity $\mathbf{v}$ of an incompressible fluid, then condition (1) corresponds to so-called Beltrami flow (1889). Vectors that obey eq. (1) are sometimes called Trkalian (1919). See, for example, [2, 3, 4].

Earth and Sun that have external magnetic fields. Thus, the corollary of Cowling's theorem that the Earth's magnetic field is dynamic and/or nonaxisymmetric appears to be basically correct.[4] *However, the concept of a static, force-free magnetic field remains interesting in principle.*

2.2 Lundquist's Solution

The first demonstration of a static, force-free magnetic field is due to Lundquist [9, 10],[5] who considered eq. (1) with $f = k$ in cylindrical coordinates (ρ, ϕ, z) for fields with dependence only ρ,

$$\frac{\partial B_z}{\partial \rho} = -kB_\phi, \qquad \frac{1}{\rho}\frac{\partial(\rho B_\phi)}{\partial \rho} = kB_z. \tag{2}$$

A particular solution to eq. (2) is,

$$B_\rho = 0, \qquad B_\phi = J_1(k\rho), \qquad B_z = J_0(k\rho), \tag{3}$$

where J_0 and J_1 are Bessel functions. The field lines are helices [9], and since the Bessel functions are oscillatory in ρ there are both left- and righthanded helices, and ones with both positive and negative B_z. Such a complex field pattern seems somewhat unlikely to occur in Nature, but it is suggestive that other force-free forms exist as well.

2.3 Other Simple Force-Free Magnetic Fields

In rectangular coordinates a force-free field that depends only on z obeys,

$$\frac{\partial B_y}{\partial z} = -kB_x, \qquad \frac{\partial B_x}{\partial z} = kB_y. \tag{4}$$

A particular solution to eq. (4) is,

$$B_x = \cos kz, \qquad B_y = -\sin kz, \qquad B_z = 0, \tag{5}$$

for which $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$. The lines of $\mathbf{B}$ are straight in any plane of constant z, making angle $\phi = kz$ to the x-axis. As with the example in sec. 2.2, this is not a physically plausible field configuration.

A force-free field that depends only on z in cylindrical coordinates must obey,

$$\frac{\partial B_\phi}{\partial z} = -kB_\rho, \qquad \frac{\partial B_\rho}{\partial z} = kB_\phi, \qquad \frac{B_\phi}{\rho} = kB_z. \tag{6}$$

A particular solution to eq. (6) is,

$$B_\rho = B_0, \qquad B_\phi = 0, \qquad B_z = 0. \tag{7}$$

[4]For a simplified discussion, see pp. 6-7 of [8].

[5]Equation (3) with $\mathbf{B}$ interpreted as fluid velocity $\mathbf{v}$ dates back to [11].

However, $\boldsymbol{\nabla} \cdot \mathbf{B} = B - 0/\rho$, so eq. (7) cannot represent a magnetic field (contrary to a claim in sec. II(a) of [12]).

In spherical coordinates (r, θ, ϕ) a force free field that depends only on r obeys,

$$B_\phi = kr \tan\theta B_r, \qquad \frac{\partial(rB_\phi)}{\partial r} = -krB_\theta, \qquad \frac{\partial(rB_\theta)}{\partial r} = krB_\phi, \tag{8}$$

for which there is no nontrivial solution, contrary to a claim in sec. III(a) of [12].

It appears that a more general method is needed to deduce the forms of additional force-free magnetic fields.

2.4 A General Solution

Considerations [13] subsequent to Lundquist's [9, 10] soon led to a general solution for force-free magnetic fields [14, 15, 16, 17].[6] Taking the curl of eq. (1) with $f = k$, we have that,

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{B}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = k^2 \mathbf{B}, \tag{9}$$

and hence, force-free magnetic fields are a subset of solutions to the vector Helmholtz equation,

$$(\nabla^2 + k^2)\mathbf{B} = 0. \tag{10}$$

A useful decomposition of solutions to the vector Helmholtz equation is due to Hansen [18] (see also sec. 7.1 of [19]), in which we write the field $\mathbf{B}$ as a linear sum of three fields,

$$\mathbf{S} = \boldsymbol{\nabla}\psi, \qquad \mathbf{T} = \boldsymbol{\nabla} \times \psi\, \mathbf{a} = \boldsymbol{\nabla}\psi \times \mathbf{a}, \qquad \text{and} \qquad \mathbf{P} = \frac{1}{k}\boldsymbol{\nabla} \times \mathbf{T}, \tag{11}$$

for any function ψ that obeys the scalar Helmholtz equation,

$$(\nabla^2 + k^2)\psi = 0, \tag{12}$$

where $\mathbf{a}$ is either a constant vector or the position vector $\mathbf{x}$ ($= r\,\hat{\mathbf{r}}$ in spherical coordinates (r, θ, ϕ)). The three fields $\mathbf{S}$, $\mathbf{T}$ and $\mathbf{P}$ have been named scaloidal, toroidal and poloidal, respectively, by Elasser [20].[7] The scaloidal/irrotational term $\mathbf{S}$ does not contribute to magnetic fields, which obey $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, and we have that,

$$\mathbf{B} = \mathbf{P} + \mathbf{T}. \tag{13}$$

Since $\mathbf{T}$ obeys eq. (10), and $\boldsymbol{\nabla} \cdot \mathbf{T} = 0$, it follows from eq. (11) that,

$$\boldsymbol{\nabla} \times \mathbf{P} = \frac{1}{k}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{T}) = -\frac{1}{k}\nabla^2 \mathbf{T} = k\mathbf{T}, \qquad \text{and} \qquad \mathbf{T} = \frac{1}{k}\boldsymbol{\nabla} \times \mathbf{P}, \tag{14}$$

and hence,

$$\boldsymbol{\nabla} \times \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{P} + \boldsymbol{\nabla} \times \mathbf{T} = k\mathbf{T} + k\mathbf{P} = k\mathbf{B}. \tag{15}$$

[6] Independently, general solutions to eq. (1) with $\mathbf{B}$ interpreted as fluid velocity $\mathbf{v}$ have been developed by several authors, as summarized in [2, 3].

[7] Equation (11) is a variant on the Helmholtz decomposition of any vector field (see, for example, [21]), in which $\mathbf{S}$ corresponds to the irrotational part, and $\mathbf{P} + \mathbf{T}$ to the rotational part, of $\mathbf{B}$.

Thus, the form (13) is an eigenfunction of the curl operator, and is a force-free magnetic field.[8]

It remains to consider a general set of solutions ψ to the scalar Helmholtz wave equation (12), which has separable solutions in 11 coordinate systems [23]. Here, we consider the basic three.[9,10]

2.4.1 Solution in Rectangular Coordinates

Solutions to the scalar Helmholtz wave equation (12) in rectangular coordinates have the form of plane waves,

$$\psi = e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{17}$$

where the wave vector $\mathbf{k} = (k_x, k_y, k_z)$ can have complex components, so long as $k^2 = k_x^2 + k_y^2 + k_z^2$. Then,

$$\boldsymbol{\nabla}\psi = i\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{18}$$

and the toroidal component of the force-free magnetic field can be taken as,

$$\mathbf{T} = \boldsymbol{\nabla}\psi \times \mathbf{x} = i\mathbf{k} \times \mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{19}$$

from which we obtain the poloidal component as,

$$\begin{aligned}
\mathbf{P} &= \frac{1}{k}\boldsymbol{\nabla} \times \mathbf{T} = \boldsymbol{\nabla} \times (i\hat{\mathbf{k}} \times \mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}}) = i\hat{\mathbf{k}}\boldsymbol{\nabla} \cdot (\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}}) - i(\hat{\mathbf{k}} \cdot \boldsymbol{\nabla})\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}} \\
&= 3i\hat{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}} - \hat{\mathbf{k}}(\mathbf{k} \cdot \mathbf{x})\, e^{i\mathbf{k}\cdot\mathbf{x}} - i\hat{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}} + k\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}}.
\end{aligned} \tag{20}$$

Thus, a force-free magnetic field can be written as,

$$\mathbf{B} = \mathbf{P} + \mathbf{T} = [2i\hat{\mathbf{k}} - (\mathbf{k} \cdot \mathbf{x})\hat{\mathbf{k}} + k\mathbf{x} + i\mathbf{k} \times \mathbf{x}]\, e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{21}$$

For example, if $\mathbf{k} = (0, 0, k)$, then,

$$\mathbf{B} = [2i\hat{\mathbf{z}} - kz\,\hat{\mathbf{z}} + k\mathbf{x} - iky\,\hat{\mathbf{x}} + ikx\,\hat{\mathbf{y}}]\, e^{ikz} = [k(x - iy)\,\hat{\mathbf{x}} + k(y + ix)\,\hat{\mathbf{y}} + 2i\hat{\mathbf{z}}]\, e^{ikz}. \tag{22}$$

Alternatively, the toroidal component of the force-free magnetic field can be taken as,

$$\mathbf{T} = \boldsymbol{\nabla}\psi \times \mathbf{a} = i\mathbf{k} \times \mathbf{a}\, e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{23}$$

for any constant vector $\mathbf{a}$. In this case the poloidal component is,

$$\mathbf{P} = \frac{1}{k}\boldsymbol{\nabla} \times \mathbf{T} = \boldsymbol{\nabla} \times (i\hat{\mathbf{k}} \times \mathbf{a}\, e^{i\mathbf{k}\cdot\mathbf{x}}) = -\mathbf{k} \times (\hat{\mathbf{k}} \times \mathbf{a})\, e^{i\mathbf{k}\cdot\mathbf{x}} = [k\,\mathbf{a} - (\mathbf{k} \cdot \mathbf{a})\,\hat{\mathbf{k}}]\, e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{24}$$

[8] A variant on the above is that for any magnetic field $\mathbf{B}'$ that satisfies the vector Helmholtz equation (10), the field,

$$\mathbf{B} = \mathbf{B}' + \frac{1}{k}\boldsymbol{\nabla} \times \mathbf{B}' \tag{16}$$

is force free [22], which can be used to deduce time-dependent forms.

[9] For a solution in toroidal coordinates, see [24].

[10] For a different characterization of eigenfunctions of the curl operator, see [25].

Thus, a force-free magnetic field can also be written as,

$$\mathbf{B} = \mathbf{P} + \mathbf{T} = [k\,\mathbf{a} - (\mathbf{k}\cdot\mathbf{a})\,\hat{\mathbf{k}} + i\mathbf{k}\times\mathbf{a}]\,e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{25}$$

For example, if $\mathbf{k} = (0, 0, k)$, then,

$$\mathbf{B} = [k\mathbf{a} - ka_z\,\hat{\mathbf{z}} + ik\,\hat{\mathbf{z}}\times\mathbf{a}]\,e^{ikz}. \tag{26}$$

With $\mathbf{a} = \hat{\mathbf{x}}/k$ we obtain,

$$\mathbf{B} = (\hat{\mathbf{x}} + i\hat{\mathbf{y}})\,e^{ikz}, \tag{27}$$

whose real part is the form (5).

2.4.2 Solution in Cylindrical Coordinates

In cylindrical coordinates (ρ, ϕ, z), solutions to the Helmholtz equation (12) that are finite on the z-axis can be written (see, for example, sec. 7.1 of [19]),

$$\psi_n = J_n(k_\rho\rho)\,e^{i(k_z z + n\phi)}, \tag{28}$$

where n is a non-negative integer, J_n is a Bessel function and $k_\rho^2 + k_z^2 = k^2$. Then,

$$\boldsymbol{\nabla}\psi_n = \frac{dJ_n(k_\rho\rho)}{d\rho}\,e^{i(k_z z+n\phi)}\,\hat{\boldsymbol{\rho}} + \frac{in}{\rho}J_n(k_\rho\rho)\,e^{i(k_z z+n\phi)}\,\hat{\boldsymbol{\phi}} + ik_z J_n(k_\rho\rho)\,e^{i(k_z z+n\phi)}\,\hat{\mathbf{z}}. \tag{29}$$

We consider only the choice of $\mathbf{a} = \hat{\mathbf{z}}/k$ in eq. (11), such that,

$$\mathbf{T}_n = -\frac{in}{k\rho}J_n(k_\rho\rho)\,e^{i(k_z z+n\phi)}\,\hat{\boldsymbol{\rho}} + \frac{dJ_n(k_\rho\rho)}{kd\rho}\,e^{i(k_z z+n\phi)}\,\hat{\boldsymbol{\phi}}, \tag{30}$$

and,

$$\mathbf{P}_n = -\frac{ik_z}{k^2}\frac{dJ_n(k_\rho\rho)}{d\rho}\,e^{i(k_z z+n\phi)}\,\hat{\boldsymbol{\rho}} + \frac{k_z n}{k^2\rho}J_n(k_\rho\rho)\,e^{i(k_z z+n\phi)}\,\hat{\boldsymbol{\phi}} - \frac{k_\rho^2}{k^2}J_n(k_\rho\rho)\,e^{i(k_z z+n\phi)}\,\hat{\mathbf{z}}, \tag{31}$$

noting that Bessel's equation has the form,

$$\frac{d}{d\rho}\left[\rho\frac{dJ_n(k_\rho\rho)}{d\rho}\right] = \left(\frac{n^2}{\rho} - k_\rho^2\rho\right)J_n(k_\rho\rho). \tag{32}$$

Of, course, the force-free magnetic field has the form,[11]

$$\mathbf{B}_n = \mathbf{P}_n + \mathbf{T}_n. \tag{33}$$

For example,

$$\begin{aligned} \psi_0 &= J_0(k_\rho\rho)\,e^{ik_z z}, && (34)\\ \mathbf{B}_0 &= \frac{ik_\rho k_z}{k^2}J_1(k_\rho\rho)\,e^{ik_z z}\,\hat{\boldsymbol{\rho}} - \frac{k_\rho}{k}J_1(k_\rho\rho)\,e^{ik_z z}\,\hat{\boldsymbol{\phi}} - \frac{k_\rho^2}{k^2}J_0(k_\rho\rho)\,e^{ik_z z}\,\hat{\mathbf{z}}. && (35) \end{aligned}$$

In particular, if $k_z = 0$ then $k_\rho = k$ and we obtain (to within a minus sign) the form (3),

$$\mathbf{B}_0(k_z = 0) = J_1(k\rho)\,\hat{\boldsymbol{\phi}} + J_0(k\rho)\,\hat{\mathbf{z}}, \tag{36}$$

as found by Lundquist [9].

[11] The forms (30)-(31) and (33) are often called the Chandrasekhar-Kendall eigenfunctions, although they were not explicitly displayed in [16]. They form a complete set of eigenfunctions of the curl operator [26].

2.4.3 Solution in Spherical Coordinates

In spherical coordinates (r, θ, ϕ), solutions to the scalar Helmholtz equation (12) can be written in various ways, as discussed in sec. 7.3 of [19], sec. 9.6 of [27], *etc.* A form that is finite at the origin and on the z-axis is,

$$\psi_n^m = j_n(kr) P_n^m(\cos\theta)\, e^{im\phi}, \tag{37}$$

m and n are integers, $n \ge 0$, $|m| \le n$, j_n is a so-called spherical Bessel function,

$$j_0(x) = \frac{\sin x}{x}, \quad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \quad j_2(x) = \left(\frac{3}{x^2} - \frac{1}{x}\right)\sin x - \frac{3\cos x}{x^2}, \quad \cdots, \tag{38}$$

and $P_n^m(y)$ is an associated Legendre function,

$$P_0^0(y) = 1, \qquad P_1^0(y) = y, \qquad P_1^{\pm 1}(y) = \pm\sqrt{1 - y^2}, \qquad P_2^0 = \frac{3y^2 - 1}{2}, \qquad \cdots \tag{39}$$

Then,

$$\begin{aligned} \boldsymbol{\nabla}\psi_n^m &= \frac{\partial \psi_n^m}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial \psi_n^m}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial \psi_n^m}{\partial \phi}\,\hat{\boldsymbol{\phi}} \qquad (40)\\ &= \frac{dj_n(kr)}{dr} P_n^m(\cos\theta)\, e^{im\phi}\,\hat{\mathbf{r}} + \frac{j_n(kr)}{r}\frac{dP_n^m(\cos\theta)}{d\theta}\, e^{im\phi}\,\hat{\boldsymbol{\theta}} + \frac{im}{r\sin\theta} j_n(kr) P_n^m(\cos\theta)\, e^{im\phi}\,\hat{\boldsymbol{\phi}}. \end{aligned}$$

$\mathbf{a} = r\,\hat{\mathbf{r}}$

We consider first the choice of $\mathbf{a} = \mathbf{x} = r\,\hat{\mathbf{r}}$ in eq. (11), such that [16, 28, 29],

$$\mathbf{T}_n^m = \frac{im}{\sin\theta} j_n(kr) P_n^m(\cos\theta)\, e^{im\phi}\,\hat{\boldsymbol{\theta}} - j_n(kr)\frac{dP_n^m(\cos\theta)}{d\theta}\, e^{im\phi}\,\hat{\boldsymbol{\phi}}, \tag{41}$$

and,

$$\begin{aligned} \mathbf{P}_n^m = {} & \frac{n(n+1)}{kr} j_n(kr) P_n^m(\cos\theta)\, e^{im\phi}\,\hat{\mathbf{r}} + \frac{1}{kr}\frac{d[r j_n(kr)]}{dr}\frac{dP_n^m(\cos\theta)}{d\theta}\, e^{im\phi}\,\hat{\boldsymbol{\theta}} \\ & + \frac{im}{kr\sin\theta}\frac{d[r j_n(kr)]}{dr} P_n^m(\cos\theta)\, e^{im\phi}\,\hat{\boldsymbol{\phi}}, \end{aligned} \tag{42}$$

noting that the associated Legendre functions obey the differential equation,

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dP_n^m(\cos\theta)}{d\theta}\right) = \left(\frac{m^2}{\sin^2\theta} - n(n+1)\right) P_n^m(\cos\theta). \tag{43}$$

Of course, the force-free magnetic fields are,

$$\mathbf{B}_n^m = \mathbf{P}_n^m + \mathbf{T}_n^m. \tag{44}$$

For example,

$$\begin{aligned}
\psi_0^0 &= \frac{\sin kr}{kr}, \qquad \mathbf{B}_0^0 = 0, && (45)\\
\psi_1^0 &= \left(\frac{\sin kr}{k^2r^2} - \frac{\cos kr}{kr}\right)\cos\theta\,, && (46)\\
\mathbf{B}_1^0 &= 2\left(\frac{\sin kr}{k^3r^3} - \frac{\cos kr}{k^2r^2}\right)\cos\theta\,\hat{\mathbf{r}} - \left[\frac{\sin kr}{kr}\left(1 - \frac{1}{k^2r^2}\right) + \frac{\cos kr}{k^2r^2}\right]\sin\theta\,\hat{\boldsymbol{\theta}}\\
&\quad + \left(\frac{\sin kr}{k^2r^2} - \frac{\cos kr}{kr}\right)\sin\theta\,\hat{\boldsymbol{\phi}}. && (47)
\end{aligned}$$

For small r, such that $kr \ll 1$,

$$\mathbf{B}_1^0(kr \ll 1) \approx \frac{2}{3}(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) - \frac{kr\sin\theta}{3}\,\hat{\boldsymbol{\phi}} = \frac{2}{3}\,\hat{\mathbf{z}} - \frac{kr\sin\theta}{3}\,\hat{\boldsymbol{\phi}}. \tag{48}$$

$\mathbf{a} = \hat{\mathbf{z}}$

We can also consider that $\mathbf{a} = \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$ in eq. (11) [3], for which,

$$\mathbf{T} = \frac{1}{r}\frac{\partial\psi}{\partial\phi}\,\hat{\mathbf{r}} + \frac{\cot\theta}{r}\frac{\partial\psi}{\partial\phi}\,\hat{\boldsymbol{\theta}} - \left(\sin\theta\frac{\partial\psi}{\partial r} + \frac{\cos\theta}{r}\frac{\partial\psi}{\partial\theta}\right)\hat{\boldsymbol{\phi}}, \tag{49}$$

and,

$$\begin{aligned}
\mathbf{P} &= -\frac{1}{kr\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin 2\theta\frac{\partial\psi}{\partial r} + \frac{\sin\theta\cos\theta}{r}\frac{\partial\psi}{\partial\theta}\right) + \frac{\cot\theta}{r}\frac{\partial^2\psi}{\partial\phi^2}\right]\hat{\mathbf{r}}\\
&\quad + \frac{1}{kr}\left[\frac{1}{r\sin\theta}\frac{\partial^2\psi}{\partial\phi^2} + \sin\theta\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \cos\theta\frac{\partial^2\psi}{\partial r\partial\theta}\right]\hat{\boldsymbol{\theta}}\\
&\quad + \frac{1}{kr}\left[\cot\theta\frac{\partial^2\psi}{\partial r\partial\phi} - \frac{1}{r}\frac{\partial^2\psi}{\partial\theta\partial\phi}\right]\hat{\boldsymbol{\phi}}. && (50)
\end{aligned}$$

For the case of no azimuthal dependence, $\partial\psi/\partial\phi = 0$, the force-free magnetic field has the form,

$$\mathbf{B} = \mathbf{P} + \mathbf{T} = \frac{1}{kr^2\sin\theta}\frac{\partial\Psi}{\partial\theta}\,\hat{\mathbf{r}} - \frac{1}{kr\sin\theta}\frac{\partial\Psi}{\partial r}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\Psi\,\hat{\boldsymbol{\phi}}, \tag{51}$$

where,[12]

$$\Psi = -\left(r\sin^2\theta\frac{\partial\psi}{\partial r} + \sin\theta\cos\theta\frac{\partial\psi}{\partial\theta}\right) = -\rho\frac{\partial\psi}{\partial\rho}\,, \tag{53}$$

[12] The function Ψ is akin to a stream function in fluid dynamics, as discussed in secs. 4.5 and 5.1 of [2]. Of course, $\Psi = -\rho\,\partial\psi/\partial\rho$ can also be introduced in cylindrical coordinates (sec. 2.4.2) in case of azimuthal symmetry, for which,

$$\mathbf{B} = \mathbf{P} + \mathbf{T} = \frac{1}{k\rho}\frac{\partial\Psi}{\partial z}\,\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\Psi\,\hat{\boldsymbol{\phi}} + \frac{1}{k\rho}\frac{\partial\Psi}{\partial\rho}\,\hat{\mathbf{z}}. \tag{52}$$

with $\rho = r\sin\theta$. Then, since $(\boldsymbol{\nabla}\times\mathbf{B})_\phi = kB_\phi$, the auxiliary function Ψ obeys the differential equation,

$$\frac{\partial^2\Psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\theta}\right) + k^2\Psi = 0. \tag{54}$$

For example,

$$\Psi_0 = \frac{\sin kr}{k}, \qquad \mathbf{B}_0 = -\frac{\cos kr}{kr\sin\theta}\,\hat{\boldsymbol{\theta}} + \frac{\sin kr}{kr\sin\theta}\,\hat{\boldsymbol{\phi}}, \tag{55}$$

$$\Psi_1 = \frac{\sin kr}{k}\cos\theta, \qquad \mathbf{B}_1 = -\frac{\sin kr}{k^2r^2}\,\hat{\mathbf{r}} - \frac{\cos kr}{kr}\cot\theta\,\hat{\boldsymbol{\theta}} + \frac{\sin kr}{kr}\cot\theta\,\hat{\boldsymbol{\phi}}, \tag{56}$$

$$\Psi_2 = \left(\frac{\sin kr}{k^2r^2} - \frac{\cos kr}{kr}\right)\sin^2\theta\,, \tag{57}$$

$$\begin{aligned}\mathbf{B}_2 &= 2\left(\frac{\sin kr}{k^3r^3} - \frac{\cos kr}{k^2r^2}\right)\cos\theta\,\hat{\mathbf{r}} - \left[\frac{\sin kr}{kr}\left(1 - \frac{1}{k^2r^2}\right) + \frac{\cos kr}{k^2r^2}\right]\sin\theta\,\hat{\boldsymbol{\theta}} \\ &\quad + \left(\frac{\sin kr}{k^2r^2} - \frac{\cos kr}{kr}\right)\sin\theta\,\hat{\boldsymbol{\phi}} = \mathbf{B}_1^0.\end{aligned} \tag{58}$$

Note that $\mathbf{B}_0$ and $\mathbf{B}_1$ are infinite on the z-axis, which reminds us that the P_n^m in eq. (37) could also be the associated Legendre functions of the second kind, Q_n^m.[13]

The fields obtained using $\mathbf{a} = r\,\hat{\mathbf{r}}$ are not independent of those found using $\mathbf{a} = \hat{\mathbf{z}}$. It is shown in [29] that the former set of fields is complete.

2.5 Exponential Decay of a Force-Free Magnetic Field

The fourth Maxwell equation relates the curl of the magnetic field to the conduction current $\mathbf{J}$ and the so-called displacement current $\epsilon_0\,\partial\mathbf{E}/\partial t$,

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\left(\mathbf{J} + \epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right). \tag{59}$$

In astrophysical situations the time dependence of the currents and fields may be sufficiently slow that the displacement-current term in eq. (59) can be neglected. In this case we can write,

$$\mathbf{J}(t) \approx \frac{1}{\mu_0}\boldsymbol{\nabla}\times\mathbf{B}(t). \tag{60}$$

If the currents flow in a medium of electrical conductivity σ, they are related to the electric field by $\mathbf{J} = \sigma\mathbf{E}$, and eq. (60) tells us that,

$$\mathbf{E}(t) \approx \frac{1}{\mu_0\sigma}\boldsymbol{\nabla}\times\mathbf{B}(t). \tag{61}$$

[13] The form $\mathbf{B}_0$ is probably what was meant to have been found in sec. III(a) of [12].

Faraday's law then gives,

$$\frac{\partial \mathbf{B}}{\partial t} = -\boldsymbol{\nabla} \times \mathbf{E} \approx -\frac{1}{\mu_0 \sigma} \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{B}) = \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B}. \tag{62}$$

If the quasistatic magnetic field is force-free, then from eq. (10) we have,

$$\frac{\partial \mathbf{B}}{\partial t} \approx -\frac{k^2}{\mu_0 \sigma} \mathbf{B}, \tag{63}$$

such that [9],

$$\mathbf{B}(\mathbf{x}, t) \approx \mathbf{B}_0(\mathbf{x})\, e^{-k^2 t/\mu_0 \sigma}, \tag{64}$$

where $\mathbf{B}_0(\mathbf{x})$ is a static, force-free magnetic field. Hence, if a force-free magnetic field could be established in a (poorly) conducting medium, it would decay away slowly without change to its spatial configuration [9].

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