# The Fractal Dimension of a Ball of Aluminum Foil

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## 1 Problem

If a sheet of aluminum foil is crumpled into a ball, the mass at radius less than $r$ can be taken as $kr^D$ where $D$ will lie between 2 and 3. We may call $D$ the fractal (Haussdorff) dimension of the crumpled aluminum ball. (In practice the above relation could only hold for $r$ larger than the thickness of the foil.)

Explain how the fractal dimension $D$ could be determined from knowledge of the velocity $v$ attained by the ball upon rolling without slipping down an incline of height $h$. Ignore air resistance, rolling friction, *etc.*

(A standard model of crumpling predicts $D = 2.5$.)

## 2 Solution

We need some other property of the ball that depends on the fractal dimension $D$. The moment of inertia $I$ about a diameter suggests itself,

$$I = 2\pi \int_0^R \int_0^\pi \rho(r) r^2 \sin\theta dr d\theta \ r^2 \sin^2\theta,$$

where the density $\rho(r)$ is related by,

$$\rho = \frac{1}{4\pi r^2}\frac{dm}{dr} = \frac{Dkr^{D-3}}{4\pi}.$$

Hence, the moment of inertia is,

$$I = \frac{2}{3}\frac{DMR^2}{D+2}.$$

For the rolling experiment, conservation of energy tells us that,

$$Mgh + \frac{1}{2}I\omega^2 + \frac{1}{2}Mv^2,$$

where the condition of rolling without slipping is $\omega = v/r$. Combining things, we find,

$$\frac{D}{D+2} = \frac{3}{2}\left(\frac{2gh}{v^2} - 1\right).$$