Gaussian Laser Beams and Particle Acceleration

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We review a recent argument by Sprangle *et al.* [1] that acceleration of charged particles is not possible by the use of Gaussian laser beams far from and walls, lenses, mirrors, *etc.*.

1 Gaussian Laser Beams That Satisfy $\nabla \cdot \mathbf{E} = 0$

The Gaussian approximation to a laser beam focus is given in many standard references as,

$$\mathbf{E} = \frac{E_0 \hat{\mathbf{x}}}{\sqrt{1+\varsigma^2}} e^{-i\tan^{-1}\varsigma} e^{i\rho^2\varsigma/(1+\varsigma^2)} e^{-\rho^2/(1+\varsigma^2)} e^{i(kz-\omega t)}, \tag{1}$$

where $\rho^2 = (x^2+y^2)/w_0^2$ with w_0 being the radius of the waist, and $\varsigma = z/z_0$ with $z_0 = \pi w_0^2/\lambda = k w_0^2/2$ being the Rayleigh range. This form describes the main features of the focus, particularly diffraction, but it does not satisfy the Maxwell equation $\nabla \cdot \mathbf{E} = \mathbf{0}$. To see this, note that a divergence-free field that points only in the x direction cannot vary with x.

For many purposes the above form is a good enough approximation. However, eq. (1) predicts that charged particles passing through the laser focus undergo a net acceleration.[1] However, there is a prejudice among many accelerator physicists that this is impossible.[2]

In SLAC experiment E-144 [6], details of the electron trajectories through the laser beam will affect the strength of the nonlinear interactions, so we have a special interest in a good model of the laser focus. Several papers offer improvements to eq. (1). The one I like best is [3]. This note is largely a detailed reworking of that paper.

A key insight of [3] is that the form of eq. (1) can more properly be used for the vector potential $\mathbf{A}$ than the electric field $\mathbf{E}$. In general, the divergence of the vector potential need not be zero and there are solutions to Maxwell's equations with nonuniform vector potential that point only along the x-axis.

A second useful insight is that when the wave equation for the vector potential is written in terms of the dimensionless variables,

$$\xi = \frac{x}{w_0}, \qquad \upsilon = \frac{y}{w_0}, \qquad \rho^2 = \xi^2 + \upsilon^2, \qquad \text{and} \qquad \varsigma = \frac{z}{z_0}, \tag{2}$$

then a series expansion suggests itself. Namely, the focal region has transverse and longitudinal extent in the ratio,

$$\theta_0 = \frac{w_0}{z_0} = \frac{2}{k w_0}. \tag{3}$$

The aspect ratio θ_0 (also the diffraction angle) is typically much less than one and so can serve as the expansion parameter.

[1] This is perhaps not self evident, but can be verified numerically. See, for example, [2].

[2] This argument is particularly due to Palmer [4], and applies to "weak" laser fields. In "strong" laser fields, acceleration of electrons "in vacuum" is possible. Some discussion of this is given in [5].

We consider only harmonic fields with time dependence $e^{-i\omega t}$. Then the wave equation for the vector potential in free space becomes,

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = 0, \qquad \text{where} \qquad k = \frac{\omega}{c} = \frac{2\pi}{\lambda}. \tag{4}$$

We work in the Lorentz gauge, so the scalar potential may be written,

$$\phi = -\frac{i}{k} \boldsymbol{\nabla} \cdot \mathbf{A}. \tag{5}$$

The electric and magnetic field can be deduced from the vector potential via,

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = \frac{i}{k}\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) + ik\mathbf{A}, \qquad \text{and} \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{6}$$

We seek fields that propagate in the $+z$ direction, have limited transverse extent, and for which the vector potential has only an x component. We try,

$$\mathbf{A}(\mathbf{r}) = \hat{\mathbf{x}}\psi(\mathbf{r})e^{i(kz-\omega t)}, \tag{7}$$

where ψ varies "slowly". From eqs. (4) and (7) we find that ψ must obey,

$$\nabla^2 \psi + 2ik\frac{\partial \psi}{\partial z} = 0. \tag{8}$$

In terms of the dimensionless variables introduced in eqs. (2) and(3) this becomes,

$$\nabla_\perp^2 \psi + 4i\frac{\partial \psi}{\partial \varsigma} + \theta_0^2 \frac{\partial^2 \psi}{\partial \varsigma^2} = 0, \qquad \text{where} \qquad \nabla_\perp^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \upsilon^2}. \tag{9}$$

This form suggests the series expansion,

$$\psi = \psi_0 + \theta_0^2 \psi_2 + \theta_0^4 \psi_4 + ... \tag{10}$$

Inserting this into eq. (9) and collecting terms of order θ_0^0, and θ_0^2, we find,

$$\nabla_\perp^2 \psi_0 + 4i\frac{\partial \psi_0}{\partial \varsigma} = 0, \tag{11}$$

and,

$$\nabla_\perp^2 \psi_2 + 4i\frac{\partial \psi_2}{\partial \varsigma} = -\frac{\partial^2 \psi_0}{\partial \varsigma^2}, \tag{12}$$

respectively. Equation (11) can be recognized as the paraxial wave equation whose Gaussian solution was given in eq. (1). That is,

$$\psi_0 = f e^{-f\rho^2}, \tag{13}$$

where,

$$f = \frac{-i}{\varsigma - i} = \frac{1 - i\varsigma}{1 + \varsigma^2} = \frac{e^{-i\tan^{-1}\varsigma}}{\sqrt{1 + \varsigma^2}}. \tag{14}$$

Davis [3] cleverly guessed the solution to eq. (12),

$$\psi_2 = \left(\frac{f}{2} - \frac{f^3 \rho^4}{4} \right) \psi_0. \tag{15}$$

The electric and magnetic fields are obtained by inserting our vector potential,

$$\mathbf{A} = \hat{\mathbf{x}} e^{i(kz-\omega t)} (\psi_0 + \theta_0^2 \psi_2), \tag{16}$$

into eq. (6), with the result (to order θ_0^3 and after dividing out a factor of ik),

$$E_x = e^{i(kz-\omega t)} \left[\psi_0 + \theta_0^2 \left(\psi_2 + \frac{1}{4} \frac{\partial^2 \psi_0}{\partial \xi^2} \right) \right] = E_0 \left[1 + \theta_0^2 \left(f^2 \xi^2 - \frac{f^3 \rho^4}{4} \right) \right], \tag{17}$$

$$E_y = e^{i(kz-\omega t)} \frac{\theta_0^2}{4} \frac{\partial^2 \psi_0}{\partial \xi \partial \upsilon} = E_0 \theta_0^2 f^2 \xi \upsilon, \tag{18}$$

$$E_z = \frac{i\theta_0}{2} e^{i(kz-\omega t)} \left[\frac{\partial \psi_0}{\partial \xi} + \theta_0^2 \left(\frac{\partial \psi_2}{\partial \xi} - \frac{i}{2} \frac{\partial^2 \psi_0}{\partial \xi \partial \varsigma} \right) \right] = -i\theta_0 f \xi E_0 \left[1 + \theta_0^2 \left(f^2 \rho^2 - \frac{f}{2} - \frac{f^3 \rho^4}{4} \right) \right], \tag{19}$$

$$B_y = e^{i(kz-\omega t)} \left[\psi_0 + \theta_0^2 \left(\psi_2 - \frac{i}{2} \frac{\partial \psi_0}{\partial \varsigma} \right) \right] = E_0 \left[1 + \theta_0^2 \left(\frac{f^2 \rho^2}{2} - \frac{f^3 \rho^4}{4} \right) \right], \tag{20}$$

$$B_z = \frac{i\theta_0}{2} e^{i(kz-\omega t)} \left(\frac{\partial \psi_0}{\partial \upsilon} + \theta_0^2 \frac{\partial \psi_2}{\partial \upsilon} \right) = -i\theta_0 f \upsilon E_0 \left[1 + \theta_0^2 \left(\frac{f^2 \rho^2}{2} + \frac{f}{2} - \frac{f^3 \rho^4}{4} \right) \right], \tag{21}$$

where $E_0 = e^{i(kz-\omega t)} \psi_0$ is also given in eq. (1). As a check, I have verified that these expression satisfy $\nabla \cdot \mathbf{E} = 0 = \nabla \cdot \mathbf{B}$ plus terms of order θ_0^4. Also, eqs. (17)-(21) with terms only to order θ_0 satisfy $\nabla \cdot \mathbf{E} = 0 = \nabla \cdot \mathbf{B}$ plus terms of order θ_0^2.

Remarks:

- The fields corresponding to circular polarization are "readily" constructed from the above.

- The E-144 simulations should in principle include the transverse deflection of the electrons as they pass through the above laser fields. The transverse displacements are, however, rather small. Recall that for a circularly polarized plane wave of intensity parameter $\eta = eE/m\omega c$ the radius of the helical motion is $r^\star = \eta \lambda^\star / 2\pi$ in the average rest frame of the electron. In terms of quantities measured in the lab frame, the radius is then $r = r^\star = \eta \lambda / 2\pi\gamma$.

- To use the theories of nonlinear Compton scattering and multiphoton pair creation we will no doubt continue to insert the local value of the field from the above forms into the calculations based on uniform plane waves. The corrections to the simple Gaussian approximation remind us that this procedure may not be completely valid.

- The paper [7] gives another version of improved Gaussian beams for which the electric field remains transverse, but contains evanescent waves confined to the focal region as well as propagating waves; I don't think this approximation satisfies $\nabla \cdot \mathbf{E} = 0$.

2 Vacuum Laser Acceleration

The papers [1] used the electric fields found above to order θ_0 to show that the laser acceleration scheme [2] won't work (at that order). Earlier references to schemes of this sort include [8, 9].

2.1 Laser Beams at Small Angles to the Electron Momentum

Consider a particle of unit charge moving at velocity c along the line $x = z\theta$ where θ is a small angle. The particle passes through a laser field given by eqs. (17) and (19), where it suffices to keep terms only to order θ_0. Take $y = 0$ always, and suppose that particle passes the origin at $t = 0$. In terms of the dimensionless variables ξ, ς and ρ, the trajectory is,

$$\rho = \xi = \frac{\theta\varsigma}{\theta_0}. \tag{22}$$

The electric field component along the particle's trajectory is,

$$E_\parallel = E_x \sin\theta + E_z \cos\theta \approx E_x\theta(1 - if\varsigma) = E_x\theta f, \tag{23}$$

noting that eq. (19) implies,

$$E_z \approx -i\theta_0 f\xi E_x = -i\theta f\varsigma E_x, \tag{24}$$

and that eq. (14) leads to the identity,

$$1 - if\varsigma = f. \tag{25}$$

The particle has coordinate z at time $t = z/(c\cos\theta)$, so ,

$$kz - \omega t \approx -\frac{kz\theta^2}{2} = -\frac{\theta^2}{\theta_0^2}\varsigma. \tag{26}$$

Then,

$$E_x = E_0 f e^{-f\rho^2} e^{i(kz-\omega t)} \approx E_0 f e^{-f\varsigma^2\theta^2/\theta_0^2} e^{-i\varsigma\theta^2/\theta_0^2} = E_0 f e^{-if\varsigma\theta^2/\theta_0^2}, \tag{27}$$

using eqs. (25) and (26). Inserting this into eq. (23) we have,

$$E_\parallel = \theta f^2 E_0 e^{-if\varsigma\theta^2/\theta_0^2} = \frac{d}{d\varsigma}\left(\frac{i\theta_0^2}{\theta} E_0 e^{-if\varsigma\theta^2/\theta_0^2}\right). \tag{28}$$

That is, the force on the particle along its trajectory can be derived from a potential. The change in energy along the trajectory is then just the change in the potential. However, the potential,

$$U = -\frac{i\theta_0^2}{\theta} E_0 e^{-if\varsigma\theta^2/\theta_0^2} \tag{29}$$

has the same value at $\varsigma = -\infty$ and $+\infty$, so the particle undergoes no net acceleration as it crosses the laser beam.

2.2 Laser Beam at 90° to the Electron Momentum

The preceding argument does not hold for large angles, so we also consider the case of $\theta = 90^\circ$. Here, the particle's trajectory is $x = ct$, $y = z = 0$. Then, $kz - \omega t$ becomes $-kx = -2\xi/\theta_0$ at the particle, and so,

$$E_\parallel = E_x = E_0 e^{-\xi^2} \cos(-2\xi/\theta_0)[1 + \theta_0^2(\xi^2 - \xi^4/4)], \tag{30}$$

taking the real part of eq. (17) and noting that $f = 1$ and $\rho = \xi$ now. The energy gain (and also the momentum gain, since $v \approx c$) is proportional to $\int_{-\infty}^{\infty} E_\parallel d\xi$ which would be nonzero if we ignored the order θ_0^2 terms. In my integral tables I find,

$$\int_{-\infty}^{\infty} \xi^{2n} e^{-\xi^2} \cos(-2\xi/\theta_0) d\xi = (-1)^n \frac{\sqrt{\pi}}{2^{2n}} e^{-1/\theta_0^2} H_{2n}(1/\theta_0), \tag{31}$$

where H_n is a Hermite polynomial,

$$H_0(x) = 1, \qquad H_2(x) = 2 - 4x^2, \qquad H_4(x) = 12 - 48x^2 + 16x^4, \qquad \ldots \tag{32}$$

These forms show that our expansion of E_x to order θ_0^2 is not sufficient to settle the question since the term $\theta_0^2 \xi^4 e^{-1/\theta_0^2}$ in the energy-gain integral leads to a term proportional to $e^{-1/\theta_0^2}/\theta_0^2$. A suggestive fact is that,

$$\int_{-\infty}^{\infty} e^{-\xi^2} \cos(-2\xi/\theta_0)(1 + \theta_0^2 \xi^2) d\xi = \sqrt{\pi} \frac{\theta_0^2}{2} e^{-1/\theta_0^2}, \tag{33}$$

so it may well be that the energy-gain integral vanishes when the higher-order terms are properly summed over.

2.3 A General Argument

Sprangle *et al.* [1] then propose an acceleration scheme based on laser beams of two frequencies, claiming the ponderomotive force provides the acceleration. I very much doubt that this scheme is valid, since (in the first approximation) the ponderomotive force is proportional to the gradient of the square of the time-averaged vector potential.[3] An electron that enters and leaves a region of realistic laser fields far from walls is asymptotically in regions of vanishing vector potential, so the net change of the ponderomotive potential is zero and the electron exchanges no net energy with the laser field.

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