# The Flyball Governor

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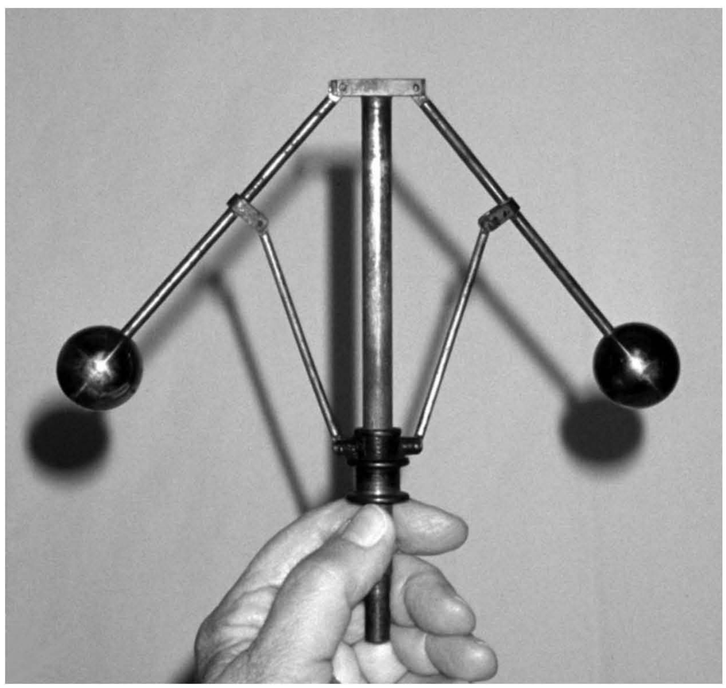
## 1 Problem

A flyball governor consists of a vertical shaft with one or more arms connected to its apex with hinges. The entire system is set in motion by turning the shaft at angular velocity $\Omega$. For large $\Omega$, the angle $\theta$ of the arms to the vertical will reach some specificed value, engaging some mechanical switch to limit further increase in the angular velocity.

For a given value of $\Omega$, deduce the equilibrium angle $\theta_0$ and the frequency $\omega$ of small oscillations about the equilbrium.

You may take the governor to have a single massless arm of length $l$ with point mass $m$ at its extremity. The hinge constrains the motion of the arm to be in a vertical plane that is fixed to (rotates with) the shaft.



Governor. The governor was invented in the second half of the eighteenth century by the Scottish engineer James Watt. The problem was to make steam engines run at a constant speed, and Watt solved it by supplying a feedback mechanism. As the jointed system rotates, the balls fly out until the inward force supplied by the struts supplies sufficient centripetal force to mak7e them travel at a constant angular speed. A lever riding against the sliding ring at the bottom of the apparatus is connected to the steam input pipe, thus supplying negative feedback. This model is in the collection of the Case Western Reserve University physics department. (Photograph and Notes by Thomas B. Greenslade, Jr., Kenyon College)

## 2 Solution

While the equilibrium condition is readliy found vis $F = ma$ in the rotating frame, the oscillation analysis is expediently accomplished via Lagrange's method.

Use angle $\theta$ as the one generalized coorindate. Then, the kinetic energy is due to motion in both vertical and horizontal planes,

$$T = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}ml^2 \sin^2\theta\Omega^2.$$

The potential energy due to gravity is just,

$$V = -mgl\cos\theta.$$

The Lagrangian is then,

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}ml^2 \sin^2\theta\Omega^2 + mgl\cos\theta.$$

While we could take derivatives to find the equation of motion, this problem is suitable for solution by the effective-potential method. To expedite the notation, I divide the Lagrangian by $ml^2$ and introduce $\omega_0 = \sqrt{g/l}$, the frequency of oscillation of a simple pendulum of length $l$. Then,

$$L = \frac{1}{2}\dot{\theta}^2 + \frac{1}{2}\Omega^2\sin^2\theta + \omega_0^2\cos\theta \equiv \frac{1}{2}\dot{\theta}^2 - V_{\rm eff},$$

with,

$$V_{\rm eff} = -\frac{1}{2}\Omega^2\sin^2\theta - \omega_0^2\cos\theta.$$

The first term in the effective potential is concave downwards, but small for small $\Omega$, while the second is concave upwards and independent of $\Omega$. For small $\Omega$, the stable equilibrium point will be at $\theta = 0$, but for large enough $\Omega$, the equilibrium point will be at some nonzero $\theta$.

The equilibrium point(s) $\theta_0$ are found by setting $dV_{\rm eff}/d\theta = 0$, and the frequencies of small oscillation about equilibrium are then given by, $\omega = \sqrt{d^2V_{\rm eff}(\theta_0)/d\theta^2}$.

$$\frac{dV_{\rm eff}}{d\theta} = -\Omega^2\sin\theta\cos\theta + \omega_0^2\sin\theta.$$

Thus, the possible equilibrium angles are

$$\theta_0 = 0 \quad \text{and} \quad \cos^{-1}\frac{\omega_0^2}{\Omega^2},$$

where the second point does not exist unless $\Omega > \omega_0$. The latter case is, however, the desired operating region of the flyball governor. (I ignore the case $\theta_0 = \pi$ as 'obviously' unstable for all $\Omega$.)

Taking the second derivative,

$$\frac{d^2V_{\rm eff}}{d\theta^2} = -\Omega^2\cos 2\theta + \omega_0^2\cos\theta.$$

For the case $\theta_0 = 0$ we then have,

$$\omega = \omega_0 \sqrt{1 - \frac{\Omega^2}{\omega_0^2}}, \qquad (\theta_0 = 0).$$

Hence, this is the stable equilibrium point until $\Omega > \omega_0$. For the latter case, we find,

$$\omega = \Omega \sqrt{1 - \frac{\omega_0^4}{\Omega^4}}, \qquad \left(\cos\theta_0 = \frac{\omega_0^2}{\Omega^2}\right).$$