Helmholtz' Vector Potentials

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1 Problem

In 1870, Helmholtz made a review of electrodynamics, and in eq. (1), p. 76 of [8],[1] he deduced that a general form for the magnetic interaction energy (his P, but our U_M) of two current elements (at $\mathbf{x}_1$ and $\mathbf{x}_2$), which are parts of closed circuits with steady currents, could be written as a combination of forms he attributed to Neumann [1, 3] and to Weber [2, 4],[2]

$$d^2U_{M,12} = \frac{\mu_0}{4\pi}\left(\frac{1+k}{2}\frac{I_1\,d\mathbf{l}_1 \cdot I_2\,d\mathbf{l}_2)}{r} + \frac{1-k}{2}\frac{(I_1\,d\mathbf{l}_1 \cdot \hat{\mathbf{r}})(I_2\,d\mathbf{l}_2 \cdot \hat{\mathbf{r}})}{r}\right), \qquad r = |\mathbf{r}| = |\mathbf{x}_1 - \mathbf{x}_2|\,, \quad (1)$$

where $k = 1$ for Neumann's form and $k = -1$ for Weber's. Then, in eq. (1^a) he argued that the scalar U_M is related to a vector potential (his (U, V, W) but our $\mathbf{A}$) as,[3]

$$U_M = \frac{1}{2}\int \mathbf{J}\cdot\mathbf{A}\,d\mathrm{Vol}, \qquad (2)$$

noting that $I\,d\mathbf{l} \leftrightarrow \mathbf{J}\,d\mathrm{Vol}$ where $\mathbf{J}$ is the current density, and the vector potential is,

$$\mathbf{A} = \frac{1+k}{2}\frac{\mu_0}{4\pi}\int \frac{\mathbf{J}}{r}\,d\mathrm{Vol} + \frac{1-k}{2}\frac{\mu_0}{4\pi}\int \frac{(\mathbf{J}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}}{r}\,d\mathrm{Vol} = \frac{1+k}{2}\mathbf{A}_\mathrm{N} + \frac{1-k}{2}\mathbf{A}_\mathrm{W}, \qquad (3)$$

$$\mathbf{A}_\mathrm{N} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}}{r}\,d\mathrm{Vol}, \qquad (4)$$

$$\mathbf{A}_\mathrm{W} = \frac{\mu_0}{4\pi}\int \frac{(\mathbf{J}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}}{r}\,d\mathrm{Vol}, \qquad (5)$$

although Neumann never wrote the form called $\mathbf{A}_\mathrm{N}$ here. Kirchhoff, p. 530 of [5], attributed $\mathbf{A}_\mathrm{W}$ to Weber, who later transcribed Kirchhoff's paper into sec. I.1. of [6], with $\mathbf{A}_\mathrm{W}$ appearing on p. 578.[4,5]

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[1]For comments by the author on this paper, see [18]. See also commentaries in [11]-[15].

[2]See also sec. IIB of [16], and [17]. The potential that Helmholtz associated with Weber was never actually advocated by the latter, who had a somewhat different vision of magnetic energy, as discussed, for example, in sec. A.23 of [19].

[3]To go from eq. (1) to (2) requires the assumption that $\boldsymbol{\nabla}\cdot\mathbf{J} = 0$, *i.e.*, that the current density $\mathbf{J}$ flows in closed loops. Hence, if one considers isolated current elements, the form (3) does not follow from (1).

[4]Both $\mathbf{A}_\mathrm{N}$ and $\mathbf{A}_\mathrm{W}$ lead to the same magnetic field, $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}_\mathrm{N} = \boldsymbol{\nabla}\times\mathbf{A}_\mathrm{W}$, which is an early example of gauge invariance.

[5]Helmholtz' discussion was tacitly restricted to electro- and magnetostatics, such that his eq. (3^a), p. 80, $\boldsymbol{\nabla}\cdot\mathbf{A} = k\,dV/dt$, where V is the instantaneous electric scalar potential, led him to identify $k = 0$ with Maxwell's theory [7] with its emphasis on $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$. Maxwell was more interested in electrodynamics than electro/magnetostatics, such that his only mention of the "Neumann" magnetostatic vector potential, our eq. (7), was in his eq. (9), Art. 422 of [9].

On pp. 119-120 of [8], Helmholtz considered the *magnetisierenden Kräfte* $(\mathfrak{L}, \mathfrak{M}, \mathfrak{N})$, which we would call the magnetic field $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, with vector potential $\mathbf{A}$ from our eq. (3). However, he was not able to use physical arguments to assign a value to the unknown constant k.

Why couldn't Helmholtz decide on a value for the parameter k in our eq. (3)?

2 Solution

This note was inspired by discussions with Chananya Groner and Tim Minteer.

If we take $k = 1$ in eq. (3), the magnetostatic vector potential has the now-standard form (for steady currents in closed circuits),

$$\mathbf{A}_{\rm N}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}')}{r}\, d{\rm Vol}', \qquad \text{where} \qquad r = |\mathbf{r}| = |\mathbf{x} - \mathbf{x}'|\,, \tag{6}$$

$$\mathbf{B}_{\rm N}(\mathbf{x}) = \boldsymbol{\nabla} \times \mathbf{A}_{\rm N} = \frac{\mu_0}{4\pi} \int \left(\boldsymbol{\nabla}\frac{1}{r}\right) \times \mathbf{J}(\mathbf{x}')\, d{\rm Vol}' = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}') \times \hat{\mathbf{r}}}{r^2}\, d{\rm Vol}', \tag{7}$$

where the form (7) is often called the Biot-Savart magnetic field.

If we take $k = -1$ in eq. (3), the magnetostatic vector potential is,

$$\mathbf{A}_{\rm W}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{(\mathbf{J}(\mathbf{x}') \cdot \mathbf{r})\, \mathbf{r}}{r^3}\, d{\rm Vol}', \tag{8}$$

and the corresponding magnetic field is,

$$\begin{aligned}
\mathbf{B}_{\rm W}(\mathbf{x}) = \boldsymbol{\nabla} \times \mathbf{A}_{\rm W} &= \frac{\mu_0}{4\pi} \int \left[\left(\boldsymbol{\nabla}\frac{\mathbf{J}(\mathbf{x}') \cdot \mathbf{r}}{r^3}\right) \times \mathbf{r} + \frac{\mathbf{J}(\mathbf{x}') \cdot \mathbf{r}}{r^3} \boldsymbol{\nabla} \times r\right] d{\rm Vol}' \\
&= \frac{\mu_0}{4\pi} \int \left[-\frac{3\mathbf{J}(\mathbf{x}') \cdot \mathbf{r}}{r^4} \boldsymbol{\nabla} r + \frac{\boldsymbol{\nabla}(\mathbf{J}(\mathbf{x}') \cdot \mathbf{r})}{r^3}\right] \times \mathbf{r}\, d{\rm Vol}' \\
&= \frac{\mu_0}{4\pi} \int \left[-\frac{3\mathbf{J}(\mathbf{x}') \cdot \mathbf{r}}{r^5} \mathbf{r} + \frac{(\mathbf{J}(\mathbf{x}') \cdot \boldsymbol{\nabla})\mathbf{r} + \mathbf{J}(\mathbf{x}') \times (\boldsymbol{\nabla} \times \mathbf{r})}{r^3}\right] \times \mathbf{r}\, d{\rm Vol}' \\
&= \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}') \times \hat{\mathbf{r}}}{r^2}\, d{\rm Vol}' = \mathbf{B}_{\rm N}(\mathbf{x}),
\end{aligned} \tag{9}$$

recalling that $\boldsymbol{\nabla} r = \mathbf{r}/r = \hat{\mathbf{r}}$ and $\boldsymbol{\nabla} \times \mathbf{r} = 0$.

Thus, both of Helmholtz' vector potentials $\mathbf{A}_{\rm N}$ and $\mathbf{A}_{\rm W}$ correspond to the same magnetic field $\mathbf{B}$ (and to the same magnetic interaction energy U_M), so the value of k is indeterminate.

This illustrates the concept of gauge invariance, that a physical field can correspond to an infinite set of potentials, which was not well understood in 1870.[6]

A technicality of possible interest is that both the Neumann and the Weber vector potentials (6) and (8) vanish at infinity for bounded distribution of current density, so the restriction of vector potentials to those that vanish at infinity does not lead to a unique magnetic vector potential for spatially bounded current distributions (whereas this restriction does lead to a unique electric scalar potential for spatially bounded charge distributions).

[6]Maxwell showed beginnings of an awareness of this in secs. 96-99 of [7]. For a review of this topic, see [16].

A Appendix: Interaction Energy of Two Isolated Current Elements (added May 28, 2021; inspired by T.M. Minteer)

The Maxwellian view of the energy associated with a magnetic field $\mathbf{B}$ is that it is given by,

$$U_M = \int \frac{|\mathbf{B}|^2}{2\mu_0}\, d\text{Vol}. \tag{10}$$

We consider the case of two current elements, $I_1\, d\mathbf{l}_1$ at $\mathbf{x}_1$ and $I_1\, d\mathbf{l}_2$ at $\mathbf{x}_2$.

If the observation point $\mathbf{x}$ is close enough to $\mathbf{x}_1$ and $\mathbf{x}_1$ that we can ignore effects of retardation (*i.e.*, effects of the finite speed of light), and we also ignore any radiation, we can make the quasistatic approximation, that the magnetic field at $\mathbf{x}$ follows from the Biot-Savart form,

$$d\mathbf{B}(\mathbf{x}) = d\mathbf{B}_1 + d\mathbf{B}_2 \approx \frac{\mu_0}{4\pi}\frac{I_1\, d\mathbf{l}_1 \times (\mathbf{x} - \mathbf{x}_1)}{|\mathbf{x} - \mathbf{x}_1|^3} + \frac{\mu_0}{4\pi}\frac{I_2\, d\mathbf{l}_2 \times (\mathbf{x} - \mathbf{x}_2)}{|\mathbf{x} - \mathbf{x}_2|^3}\,. \tag{11}$$

The magnetic energy associated with these two currents elements is,

$$\begin{aligned} d^2U_M &= \int \frac{|d\mathbf{B}(\mathbf{x})|^2}{2\mu_0}\, d\text{Vol} = \int \frac{|d\mathbf{B}_1|^2}{2\mu_0}\, d\text{Vol} + \int \frac{|d\mathbf{B}_2|^2}{2\mu_0}\, d\text{Vol} + \int \frac{d\mathbf{B}_1 \cdot d\mathbf{B}_2}{\mu_0}\, d\text{Vol} \\ &= d^2U_{M,1} + d^2U_{M,1} + d^2U_{M,12}, \end{aligned} \tag{12}$$

where $d^2U_{M,i}$ is the (infinite) self energy of current element i, and $d^2U_{M,12}$ is the interaction energy between the two elements (which is finite if the current elements are at different points),[7]

$$\begin{aligned} d^2U_{M,12} &\approx \frac{\mu_0}{16\pi^2}\int d\text{Vol}\frac{(I_1\, d\mathbf{l}_1 \cdot I_2\, d\mathbf{l}_2)(\mathbf{x} - \mathbf{x}_1)\cdot(\mathbf{x} - \mathbf{x}_2) - I_1\, d\mathbf{l}_1 \cdot (\mathbf{x} - \mathbf{x}_2) I_2\, d\mathbf{l}_2 \cdot (\mathbf{x} - \mathbf{x}_1)}{|\mathbf{x} - \mathbf{x}_1|^3\, |\mathbf{x} - \mathbf{x}_2|^3} \\ &= \frac{\mu_0}{8\pi}\frac{I_1\, d\mathbf{l}_1 \cdot I_2\, d\mathbf{l}_2 + (I_1\, d\mathbf{l}_1 \cdot \hat{\mathbf{r}})(I_2\, d\mathbf{l}_2 \cdot \hat{\mathbf{r}})}{r}\,, \qquad r = |\mathbf{r}| = |\mathbf{x}_1 - \mathbf{x}_2|\,, \end{aligned} \tag{13}$$

which is Helmholtz' form (1) with k = 0.

This result is intriguing, but should not be interpreted as "proving" that $k = 0$, or that the magnetic vector potential $\mathbf{A}$ is Helmholtz' form (3) with $k = 0$.[8]

Helmholtz' forms hold only for examples in which the current elements are parts of closed loops of steady currents, and do not hold for, say, pairs of moving electric charges, whose fields involve effects of retardation and radiation.

It happens that Helmholtz did associate $k = 0$ with Maxwell's electrodynamics, but for different reasons (as discussed in footnote 4 above).[9,10]

[7] Thanks to T.M. Minteer, private communication, for verification of the volume integral in eq. (13) via numerical integration with MathCad.

[8] Recall footnote 3 above.

[9] See also sec. IIB of [16].

[10] By 1870, Maxwell had used the Biot-Savart form for steady currents, but like Ampère and Helmholtz, did not speculate about isolated current elements. At this time, only Weber pursued the study of the interactions of moving electric charges, although in the late 1870's Helmholtz sponsored the experiment of Rowland [10] that first demonstrated a magnetic effect associated with moving electric charges (convection currents).

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